JEU DE TAQUIN AND CYCLIC SIEVING FOR FORESTS OF ROOTED TREES

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Abstract. We investigated the cyclic sieving phenomenon for linear extensions of forests of rooted trees with the cyclic action 'jeu de taquin'. We have proved many results concering the orbits of linear extensions for different types of trees. Our investigations also show that these objects just typically do not cyclicly sieve.

1. Introduction

Let F be a forest of two rooted trees, one a chain of length k and the other a chain of length $n-k$, and let X be the set of linear exentsions of F [1]. Let $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ and let C be the group generated by the 'jeu-de-taquin' (jdt) action on X. The triple $(X, X(q), C)$ exhibts the cyclic sieving phenomenon (CSP) as defined in [1]. It is worth considering whether the set of linear extensions X of a general forest F of rooted trees exhibits CSP for this cyclic action. A natural choice for $X(q)$ is

$$\frac{[n]_q!}{\prod_{x \in F} [h_x]_q}$$

where h_x is the hook length at x in F. $X(1)$ is Knuth's formula for the number of linear extensions of F.

2. Orbits

The most successful aspects of our research has been discovering many properties of the orbits of linear extensions. The following definition is invaluable when working with linear extensions of forests.

Definition 2.1. Let F be a forest of rooted trees. A n-labeling x of F is an injective map $x : F \to [n]$ such that

$$v >_F w \quad \Rightarrow \quad x(v) > x(w) \;\; \forall \;\; v, w \in F,$$

where '$>_F$' means greater than in the poset F. We also define the notation $x_v := x(v)$. x_v is commonly referred to as the *label or label value* of v in x.

Note that we do not require $n = |F|$. An $|F|$-labeling of F is just a linear extension of F. There is also natural extension of the action of C on linear extensions F of to n-labelings of F. n-labelings are useful objects to define because we can view a linear extension x of F as a set of $|F|$-labelings of all of F's component trees. In other words, finding the orbit of a linear extension can be broken up into sub problems of finding the orbit of the $|F|$-labeling for each component of F and then taking the least common multipule of these orbit sizes.

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[1]Throughout this paper we will identify F with the poset whose Hasse diagram is the graph of F. This identification blurs the distinction between F and the vertex set V of F. Unless explicitly stated otherwise, when we say "$|F|$" and "$\in F$" we mean "$|V|$" and "$\in V$".

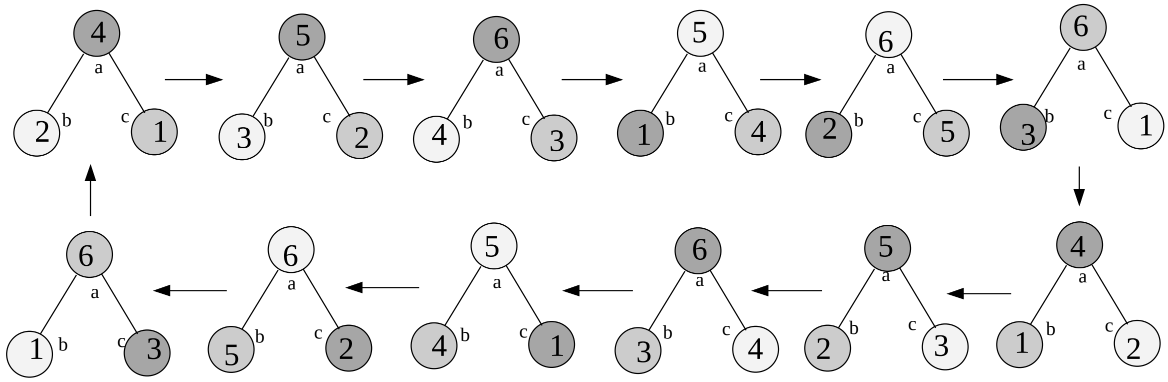

FIGURE 1. The orbit of a particular $3t$-labeling under jdt. The letters a,b,c are vertex names. The different shades are label names.

Definition 2.2. Let F be a forest of rooted trees.
$O_n(F)$ is the set of orbit sizes of the n-labelings of F under the action of C.

Theorem 2.3. *Let c_t be a chain of length t.*

$$O_n(c_t) = \{n/d : d|n, d|t\}$$

Most of our results concerning orbits of linear extensions concern special classes of trees. The following definition will be used frequently to classify trees.

Definition 2.4. Let T be a rooted tree. A branching point is any vertex of degree at least 3 or a root of degree at least 2. T is said to be stemless if its root is a branching point.

Conjecture 2.5. *Let T be a stemless single branching rooted tree with root r and let c_t be a chain of length t.*

$$O_n(c_t) = O_{|T|-1}(T - r) \cdot O_n(c_{|T|})$$

where $A \cdot B := \{ab : a \in A, b \in B\}$.

Conjecture 2.6. *Let $F := \{T_1, T_2, T_3, \dots T_s\}$ be a forest of rooted trees where each T_i is a stemless single branching rooted tree or a chain. Let $n := |F|$.*

$$O_n(F) = \{lcm(t_1, t_2, \dots t_s) : t_i \in O_n(T_i)\}$$

Note this is a very non-trivial statement about $O_n(F)$. We know given a linear extension x the orbit size of x is the least common multipule of the orbits sizes for each component's $|F|$-labeling. What this conjecture states however is that we may take the least common multipule of orbits sizes from *any* choice of $|F|$-labelings for each component to get $O_n(F)$ (even though for any two components there will be many pairs of $|F|$-labelings which cannot arise from the same linear extensions). We can show this conjecture is true in the case that each tree is a chain and in the case that there is exactly one stemless single branching tree whose size is relatively prime to n.

Theorem 2.7. *Let F be a forest of t rooted trees, each consisting of a root vertex, two leaf vertices, and no other vertices. The action of C on the set of linear extensions of F consists of $\frac{2^t t!}{2t}$ orbits*

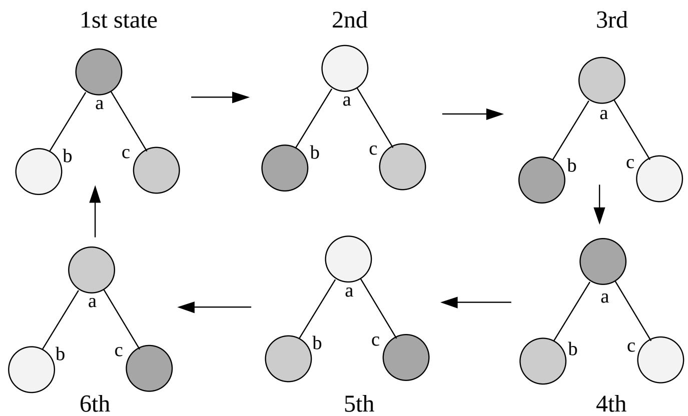

FIGURE 2. The orbit of states for a $3t$-labeling.

of size $2t$ *and* $\frac{(3t)!/3^t - 2^t t!}{6t}$ *orbits of size* $6t$.

Proof: $|F| = 3t$. Let T be one component of the forest F. We want to show each $3t$-labeling of T either has orbit size $2t$ or $6t$. This implies each linear extension of F either has orbit size $2t$ or $6t$. Knuth's formula tells us there are $(3t)!/3^t$ linear extensions of F, so to prove the theorem it will suffice to show there are $2^t t!$ linear extensions whose orbit is of size $2t$.

Let x be a $3t$-labeling of T. We can view each operation of jdt on x as a cyclic permutation of the label values followed by some reordering of the 'label names' of F (we think of the reordering of the label names as representing the sliding part of jdt). See figure 1 for an example.

In the example there are a total of 6 'states' (that is, ordering of the label names) as jdt acts on the 6-labeling 12 times. It is useful to think of the underlying orbit of states as depicted in figure 2. It is easy to see that as jdt acts on any $3t$-labeling $6t$ times the label positions will go through these 6 states in either the same order or reverse order depending on which of the two leaf vertices has a larger label.

Fix a $3t$-labeling x of T such that $x_b > x_a$. The states will then cycle as shown in figure 2. A state advances to the next state each time a label is 'demoted' from $3t$ back to 1 by jdt. Therefore if jdt is applied $6t$ times the labeling will be at its original state. Since $6t$ is divisible by $3t$, each label will have cycled back to its original value. In other words, we have returned to the original labeling:

$$jdt^{6t}(x) = x \quad \forall\ 3t\text{–}labelings\ x\ of\ T.$$

Suppose the orbit of x is not of size $6t$. i.e., let $jdt^n(x) = x$, where $0 < n < 6t$. We are interested in the state of $jdt^n(x)$. Notice that the state of x must change at least once before x can return to itself under the action of jdt. Hence $jdt^n(x)$ cannot have state 1. In the 2nd, 4th and 6th states it is not difficult to see that the bottom left label will be smaller than the bottom right label. Therefore the only possible states of $jdt^n(x)$ are the 3rd and 5th states.

Lemma 2.8. *$jdt^n(x) = x$, $0 < n < 6t$, and $jdt^n(x)$ is in the state 3 iff*

$$n = x_a - x_c \text{ and } x_a - x_b \equiv x_b - x_c \equiv x_c - x_a \text{ mod } 3t.$$

Note the last set of equalities imply $n = 2t$.

Proof of Lemma: $jdt^{x_a-x_c}$ will bring x to the 3rd state and $(jdt^{x_a-x_c}(x))_a = x_a$. If $n \neq x_a - x_c$ then $(jdt^{x_a-x_c}x)_a \neq x_a$.

$$jdt^{x_a-x_c}(x) = x \quad \Leftrightarrow \quad x_v = (jdt^{x_a-x_c}(x))_v \ \ \forall\, v \in F.$$

Hence,

$$\begin{aligned} x_b &= (jdt^{x_a-x_c}(x))_b \Leftrightarrow \\ x_b &\equiv x_a + (x_a - x_c)\ mod\ 3t \Leftrightarrow \\ x_c - x_a &\equiv x_a - x_b\ mod\ 3t, \end{aligned}$$

and,

$$\begin{aligned} x_c &= (jdt^{x_a-x_c}(x))_c \Leftrightarrow \\ x_c &\equiv x_b + (x_a - x_c)\ mod\ 3t \Leftrightarrow \\ x_c - x_a &\equiv x_b - x_c\ mod\ 3t. \end{aligned}$$

This proves the Lemma. □

If $jdt^n(x)$ is in 5th state the argument is similar and it turns out that the conditions on x are identical to the 3rd state case.

Lemma 2.9. *$jdt^n(x) = x$, $0 < n < 6t$, and $jdt^n(x)$ is in the 5th state iff*

$$n = x_a - x_b + 3t \text{ and } x_a - x_b \equiv x_b - x_c \equiv x_c - x_a\ mod\ 3t.$$

Note the last set of equalities imply $n = 4t$. In the case $x_b < x_a$, the lemmas are similar.

Corollary 2.10. *Let x be any $3t$-labeling of T. If the orbit of x is not of size $6t$ then it has size $2t$ and $x_a - x_b \equiv x_b - x_c \equiv x_c - x_a$ mod $3t$.*

The orbit of a linear extension x of F will be of size $2t$ iff the orbit of every $3t$-labeling of the component trees of F are of size $2t$. In that case there are $t!$ ways to choose x_a for each component of F. Once x_a is chosen for each component it determines x_b and x_c up to a left-right orientation. Therefore we have $2^t t!$ linear extensions of F whose orbit is of size $2t$.

This proves the Theorem. □

We wish to define a new operation, js, which acts on n-labelings and linear extensions. First we need a definition.

Definition 2.11. Let F be a forest of rooted trees, $m := |F|$. Let X be the set of n-labelings of F, and let Y be the set of linear extensions of F. Each $x \in X$ defines an m-subset of $[n]$ called $[x] := \{x_v \in [n] : v \in F\}$, i.e. $[x]$ is the image of x. Furthermore, there is a unique map $P_x : [x] \to [m]$ compatible with '$>$' (namely the map where the smallest element of $[x]$ gets sent to 1, the second smallest element of $[x]$ gets sent to 2, etc.) Define a map $R_x : X \to Y$ by

$$(R_x(x))_v = P_x(x_v) \ \ \forall\, v \in F.$$

Definition 2.12. Let F be a forest of rooted trees and let X be the set of n-labelings of F. Define a map $js : X \to X$ by:

$$(js(x))_v = R_x^{-1}(\ (jdt \cdot R_x(x_v)\ ).$$

In other words, we relabel F with $[m]$, apply jdt, then revert to the old labels to get $js(x)$. With the above definition we can define a cyclic group S genereated by js which acts on X. We think of js as a 'standardized' jeu de taquin action on X since one label gets demoted for every application of js. js is a useful tool for calculating jdt^n of n-labelings.

Theorem 2.13. *Let T be a stemless single branching rooted tree with with root r and let x be a n-labeling of T. Then*

$$jdt^n(x) = js(x).$$

Proof: The set of labels of $jdt^n(x)$ is set the of labels of x, only they will be in different positions. In the language of Theorem 2.7 let ρ_v refer to the label name[2] which has label value x_v in x. Let $\{c_1, c_2, ...\}$ be the set of components of $T - r$. Let v be any vertex of T. When jdt is applied n times the label x_v will be demoted exactly once, and hence the component c_i containing ρ_v will only change once (when a label name moves to the top position we consider it to be in its previous component, and by convention ρ_r is considered to initially be in c_1). Define "$c(v)$ *in* $jdt^m(x)$" to be the component c_i containg ρ_v in $jdt^m(x)$. The rules of jdt imply

$$\begin{aligned} c(v) \ in \ jdt^n(x) &= c(w) \ in \ x, \ where \ x_w < x_v \ and \ no \ other \ vertex \ u \\ & \quad\ satisfies \ x_w < x_u < x_v \\ &= c(w) \ in \ x, \ where \ R_x(x_w) = R_x(x_v) - 1. \end{aligned}$$

The last equation is precisely the component containg ρ_v in $js(x)$. For stemless single branching trees each c_i is a chain. There is only one way to arrange a set of labels within a chain so that the result gives a proper labeling, so in fact the position of ρ_v in $jdt^n(x)$ is the position of ρ_v in $js(x)$.

This proves the Theorem. □

For an arbitrary stemless tree T the argument of the previous theorem is valid up to the last step. In other words, $js(x)$ will give $jdt^n(x)$ up to some reordering of the vertices within each component of $T - r$. We will prove an analagous statement for general trees.

Definition 2.14. Let T be a rooted tree and let $v, w \in T$. $d(v, w) :=$ the length of the unique v, w path in T.

Definition 2.15. Let T be a rooted tree and let x be a n-labeling of T.

$$x^{<v} := \text{the restriction of } x \text{ to an } n\text{-labeling of } T^{<v} := \{w \in T : w <_T v\}.$$

We make analagous definitions for $>, \geq, \leq$.

$$x^{<v,>w} := \text{the restriction of } x \text{ to an } n\text{-labeling of } T^{<v,>w} := (T^{<v})^{>w}.$$

Again, with analagous definitions for other combinations of relation operators.

Theorem 2.16. *Let T be a single branching rooted tree with root r and branching point b. Let x be a n-labeling of T. Then*

$$jdt^n(x) = js^{d(b,r)+1}(x).$$

Proof: Trivially $(jdt^n(x))_v = (js^{d(b,r)+1}(x))_v$ when $v \in T^{\geq b}$ since there is only one way to arrange the largest $(d(r, b) + 1)$ labels within the stem. Let $v \in T^{<b}$. As in the previous theorem we argue that

$$c(v) \ in \ jdt^n(x) = c(w) \ in \ x, \ where \ R_x(x_w) = R_x(x_v) - (d(b, r) + 1),$$

since it is the location of the $(d(b, r) + 2)$th largest label which determines what component of $T^{<b}$ the top label is demoted into. For single branching trees each component of $T^{<b}$ is a chain. There is only one way to arrange the labels within each c_i, hence the arrangement of labels of c_i in $jdt^n(x)$ is the same as in $js^{d(b,r)+1}(x)$.

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[2]Just so we are clear about this object, the position of ρ_v in $jdt^m(x)$ would be the vertex of T which is labeled $(x_v + m - 1 \ mod \ n) + 1$ by $jdt^m(x)$.

This proves the Theorem. □

As before, this argument is valid for an arbitrary tree T up until the last step:

Corollary 2.17. *Let T be a rooted tree with root r. Let x be a n-labeling of T.*

If b is the first branching point of T then $js^{d(b,r)+1}(x)$ will give $jdt^n(x)$ correctly for the stem $T^{\geq b}$ and correct up to some reordering of the vertices within each component of $T^{<b}$.

3. Sieving

We have taken a lot of data concerning cyclic sieving. In a few cases described below we believe the linear extensions of forests will exhibit CSP.

Conjecture 3.1. *Let F be a forest of rooted trees, where each tree is a chain. Let X be the set of linear extensions of F. Let $X(q)$ be as defined in the introduction, then $(X, X(q), C)$ exhibits CSP.*

Theorem 3.2. *Let $F := \{T_0, T_1, T_2, \dots T_s\}$ be a forest of s rooted trees where T_0 is a stemless single branching rooted tree with root r and T_i is a chain for $i \geq 1$. Let $\{c_1, c_2, ...c_r\}$ be the components of $T_0 - r$. Define $n := |F|$, $k := |T_0| - 1$, $d := gcd(n, |T_1|, |T_2|, ...)$, $e := gcd(n, |T_0|)$, and $H :=$ the cyclic group generated by jdt^k*

If $gcd(k, |c_1|, |c_2|...|c_r|) = 1$, $either e = 1 or (d = 1 and gcd(k, n) = 1)$, and $X(q)/[n]_q$ is a polynomial in q, then the action of H on the set of linear extenesions X of F consists of only free orbits, and $(X, X(q), H)$ exhibts CSP.

Conjecture 3.3. *The converse of Theorem 3.2 is true.*

Included below is a table of data describing when cyclic sieving occurs. 'n' is the size of the forest. 'Partition of n' is the partition of n given by the component sizes of the forest F (used merely as a way to sort results). 'Shape' is a coding for describing precisely what each forest looks. Each number in 'Shape' is the number of children of each vertex (where the number of children of a vertex is the number of vertices below that vertex which it is connected to). Colons seperate vertices vertically within the same tree, commas seperate several vertices on the same vertical line in the same tree, and slashes begin a new tree. For example 1:0/1:0 consists of a two trees, each having code 1:0. 1:0 means the root has 1 child and that vertex has no children, so it is a chain of length 2. 2:1,0:0 is one tree. The root vertex has two children, one of which has no children and the other has 1 child. The child in the 3rd line has no children; it is a leaf vertex. So the tree is a chain of length 3 and a chain of length 2 fused together with their root vertices overlapping.

n	Partition of n	Shape	Number of Linear Extentions	Orbit sizes under jdt	q-Polynomial	q-Polynomial $\mathrm{mod}(q^m - 1)$	Sieving Value (m)
1	1	0	1	1	1	1	1
2	2	$1:0$	1	1	1	1	1
2	11	$0/0$	2	2	$1+q$	$1+q$	2
3	3	$1:1:0$	1	1	1	1	1
3	3	$2:0,0$	2	2	$1+q$	$1+q$	2
3	21	$1:0/0$	3	3	$1+q+q^2$	$1+q+q^2$	3
3	111	$0/0/0$	6	3^2	$1+2q+2q^2+q^3$	$2+2q+2q^2$	3
4	4	$1:1:1:0$	1	1	1	1	1
4	4	$2:1,0:0$	3	3	$1+q+q^2$	$1+q+q^2$	3
4	4	$1:2:0,0$	2	2	$1+q$	$1+q$	2
4	4	$3:0,0,0$	6	3^2	$1+2q+2q^2+q^3$	$2+2q+2q^2$	3
4	31	$1:1:0/0$	4	4	$1+q+q^2+q^3$	$1+q+q^2+q^3$	4
4	31	$2:0,0/0$	8	8	$1+2q+2q^2+2q^3+q^4$		
4	22	$1:0/1:0$	6	$4,2$	$1+q+2q^2+q^3+q^4$	$2+q+2q^2+q^3$	4
4	211	$1:0/0/0$	12	4^3	$1+2q+3q^2+3q^3+2q^4+q^5$	$3+3q+3q^2+3q^3$	4
4	1111	$0/0/0/0$	24	4^6	$1+3q+5q^2+6q^3+5q^4+3q^5+q^6$	$6+6q+6q^2+6q^3$	4
5	5	$1:1:1:1:0$	1	1	1	1	1
5	5	$2:1,0:1:0$	4	4	$1+q+q^2+q^3$	$1+q+q^2+q^3$	4
5	5	$1:2:0,1:0$	3	3	$1+q+q^2$	$1+q+q^2$	3
5	5	$1:1:2:0,0$	2	2	$1+q$	$1+q$	2
5	5	$1:3:0,0,0$	6	3^2	$1+2q+2q^2+q^3$	$2+2q+2q^2$	3
5	5	$2:1,1:0,0$	6	$4,2$	$1+q+2q^2+q^3+q^4$	$2+q+2q^2+q^3$	4
5	5	$2:0,2:0,0$	8	8	$1+2q+2q^2+2q^3+q^4$		
5	5	$3:1,0,0:0$	12	4^3	$1+2q+3q^2+3q^3+2q^4+q^5$	$3+3q+3q^2+3q^3$	4
5	5	$4:0,0,0,0$	24	4^6	$1+3q+5q^2+6q^3+5q^4+3q^5+q^6$	$6+6q+6q^2+6q^3$	4
5	41	$1:1:1:0/0$	5	5	$1+q+q^2+q^3+q^4$	$1+q+q^2+q^3+q^4$	5
5	41	$2:1,0:0/0$	15	15	$1+2q+3q^2+3q^3+3q^4+2q^5+q^6$		
5	41	$1:2:0,0/0$	10	5^2	$1+2q+2q^2+2q^3+2q^4+q^5$	$2+2q+2q^2+2q^3+2q^4$	5
5	41	$3:0,0,0/0$	30	15^2	$1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7$		
5	32	$1:1:0/1:0$	10	5^2	$1+q+2q^2+2q^3+2q^4+q^5+q^6$	$2+2q+2q^2+2q^3+2q^4$	5
5	32	$2:0,0/1:0$	20	10^2	$1+2q+3q^2+4q^3+4q^4+3q^5+2q^6+q^7$		
5	311	$1:1:0/0/0$	20	5^4	$1+2q+3q^2+4q^3+4q^4+3q^5+2q^6+q^7$	$4+4q+4q^2+4q^3+4q^4$	5
5	311	$2:0,0/0/0$	40	10^4	$1+3q+5q^2+7q^3+8q^4+7q^5+5q^6+3q^7+q^8$		
5	221	$1:0,1:0/0$	30	5^6	$1+2q+4q^2+5q^3+6q^4+5q^5+4q^6+2q^7+q^8$	$6+6q+6q^2+6q^3+6q^4$	5
5	2111	$1:0/0/0/0$	60	5^{12}	$1+3q+6q^2+9q^3+11q^4+11q^5+9q^6+6q^7+3q^8+q^9$	$12+12q+12q^2+12q^3+12q^4$	5
5	11111	$0/0/0/0/0$	120	5^24	$1+4q+9q^2+15q^3+20q^4+22q^5+20q^6+15q^7+9q^8+4q^9+q^{10}$	$24+24q+24q^2+24q^3+24q^4$	5

6	6	$2:2,1:0,0,0$	20	10^2	$1+2q+3q^2+4q^3+4q^4+3q^5+2q^6+q^7$		
6	6	$3:2,0,0:0,0$	40	10^4	$1+3q+5q^2+7q^3+8q^4+7q^5+5q^6+3q^7+q^8$		
6	6	$2:2,0:1,0:0$	15	15	$1+2q+3q^2+3q^3+3q^4+2q^5+q^6$		
6	6	$2:3,0:0,0,0$	30	15^2	$1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7$		
6	51	$1:1:1:1:0/0$	6	6	$1+q+q^2+q^3+q^4+q^5$	$1+q+q^2+q^3+q^4+q^5$	6
6	51	$1:1:2:0,0/0$	12	12	$1+2q+2q^2+2q^3+2q^4+2q^5+q^6$		
6	51	$1:2:1,0:0/0$	18	18	$1+2q+3q^2+3q^3+3q^4+3q^5+2q^6+q^7$		
6	51	$2:1,0:1:0/0$	24	24	$1+2q+3q^2+4q^3+4q^4+4q^5+3q^6+2q^7+q^8$		
6	51	$1:3:0,0,0/0$	36	18^2	$1+3q+5q^2+6q^3+6q^4+6q^5+5q^6+3q^7+q^8$		
6	51	$2:1,1:0,0/0$	36	$24,12$	$1+2q+4q^2+5q^3+6q^4+6q^5+5q^6+4q^7+2q^8+q^9$		
6	51	$3:1,0,0:0/0$	72	24^3	$1+3q+6q^2+9q^3+11q^4+12q^5+11q^6+9q^7+6q^8+3q^9+q^{10}$		
6	51	$4:0,0,0,0/0$	144	24^6	$1+4q+9q^2+15q^3+20q^4+23q^5+23q^6+20q^7+15q^8+9q^9+4q^{10}+q^{11}$		
6	51	$2:2,0:0,0/0$	48	48	$1+3q+5q^2+7q^3+8q^4+8q^5+7q^6+5q^7+3q^8+q^9$		
6	42	$1:1:1:0/1:0$	15	$6^2,3$	$1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7+q^8$	$3+2q+3q^2+2q^3+3q^4+2q^5$	6
6	42	$2:1,0:0/1:0$	45	$18^2,9$	$1+2q+4q^2+5q^3+7q^4+7q^5+7q^6+5q^7+4q^8+2q^9+q^{10}$		
6	42	$1:2:0,0/1:0$	30	$6^4,3^2$	$1+2q+3q^2+4q^3+5q^4+5q^5+4q^6+3q^7+2q^8+q^9$		
6	42	$3:0,0,0/1:0$	90	$18^4,9^2$	$1+3q+6q^2+9q^3+12q^4+14q^5+14q^6+12q^7+9q^8+6q^9+3q^{10}+q^{11}$		
6	411	$1:1:1:0/0/0$	30	6^5	$1+2q+3q^2+4q^3+5q^4+5q^5+4q^6+3q^7+2q^8+q^9$	$5+5q+5q^2+5q^3+5q^4+5q^5$	6
6	411	$2:1,0:0/0/0$	90	18^5	$1+3q+6q^2+9q^3+12q^4+14q^5+14q^6+12q^7+9q^8+6q^9+3q^{10}+q^{11}$		
6	411	$1:2:0,0/0/0$	60	6^{10}	$1+3q+5q^2+7q^3+9q^4+10q^5+9q^6+7q^7+5q^8+3q^9+q^{10}$	$10+10q+10q^2+10q^3+10q^4+10q^5$	6
6	411	$3:0,0,0/0/0$	180	18^{10}	$1+4q+9q^2+15q^3+21q^4+26q^5+28q^6+$		

4. Conclusion

Although our investigations have produced many results concerning the orbits of linear extensions it appears that cyclic sieving is not a common occurance for general forests of rooted trees. Since statements about CSP for a general forest seems unlikely, any further investigation should probably focus on special classes of forests in the hope that there are other classes of forests besides two chains which always exhibit CSP.

References

[1] V. Reiner, D. Stanton, and D. White *The cyclic sieving phenomenon*, Jounral of Combinatorial Theory, Series A**108** (2004), 17-50.

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