# Correlations among Pattern Avoiding Permutations and Their Recurrences

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**Abstract**

The Stanley-Wilf conjecture, together with Erdős–Szekeres theorem, answers the question of how avoiding a pattern $\sigma \in S_n$ correlates with the avoidance of another pattern $\pi \in S_n$. In this report we present our progress toward the same question but when we restrict our attention to only the permutations that avoid a fixed pattern $\tau \in S_n$. In particular, when $\tau = (231)$, the number of permutations avoiding both $\tau$ and another pattern must satisfy a linear recurrence [5]. The second part of this report contains our conjecture on the roots of the characteristic polynomials of these recurrences, and is dedicated to showing that the conjecture holds for some special cases.

## Preliminaries

A *permutation* $\pi$ is a bijection from $\{1, 2, ..., n\} = [n]$ to itself. In this paper, we will write permutations using one-line notation, expressing $\pi$ as $(\pi(1), \pi(2), ..., \pi(n))$. Given two permutations $\pi$ and $\sigma$, we say that $\pi$ *contains* $\sigma$ if there exists a subsequence of $\pi$ with the same relative ordering as $\sigma$. When $\pi$ does not contain $\sigma$, we say $\pi$ *avoids* $\sigma$. For example, $\pi = (523416)$ contains $\sigma = (213)$ since $(526)$ is a subsequence of $\pi$ and is order-isomorphic to $\sigma$. However, $\pi$ avoids 132.

We will write $S_n$ to denote the set of all permutations of length $n$ and $S_n(\pi_1, ..., \pi_m)$ to denote the set of permutaions of length $n$ which avoid the patterns $\pi_1, ..., \pi_m$. Similarly, $S_n(\Pi)$ will denote to the set of permutations of length $n$ which avoid $\pi$ for all $\pi \in \Pi$.

The last preliminary we will state before introducing the focus of the paper is the celebrated theorem of Erdős and Szekeres, which will help reduce the number of cases we must examine. We state it in the terms most directly applicable to our problem.

**Theorem 0.1** (Erdős-Szekeres [2])**.** *For any $k, \ell \geq 2$, $\#S_n(1...k, \ell...1) = 0$ for large enough $n$.*

In Section 1, we explore how the avoidance of one pattern correlates with the avoidance of another pattern in random, long permutations. One particular case of this problem leads into investigating a characteristic polynomial in Section 2, and cylindrical networks in Section 3.

## 1 Correlation problem

Suppose we have two patterns $u$ and $w$. Joel Lewis answers the question of how avoiding $u$ and avoiding $w$ correlate among random, long permutations [4].

In this section, we address the following variation.

**Question 1.1.** *Fix arbitrary permutations $u, v, w$. How does avoiding $u$ correlate with avoiding $w$ for random permutations in $S_n(v)$, when $n$ is large?*

*Remark.* This question can be answered trivially if $u, v$, and $w$ are not distinct or if one of $u, v$ and $w$ is $1...x$ and another is $y...1$ for some integers $x, y$ (the latter being a consequence of Theorem 0.1). We therefore proceed assuming neither of these cases hold.

Question 1.1 is easily reduced to looking at the sizes of relevant permutation classes, as we show in the following proposition.

**Proposition 1.2** (Criterion for Correlation)**.** *Avoiding* $u$ *and avoiding* $w$ *are positively correlated in* $S_n(v)$ *if and only if*

$$(\#S_n(u,v,w))(\#S_n(v)) > (\#S_n(w,v))(\#S_n(v,u)) \quad (1)$$

*Meanwhile, avoiding* $u$ *and avoiding* $w$ *negatively correlate if and only if*

$$(\#S_n(u,v,w))(\#S_n(v)) < (\#S_n(w,v))(\#S_n(v,u)) \quad (2)$$

*, while avoiding* $u$ *and avoiding* $w$ *are independent if and only if*

$$(\#S_n(u,v,w))(\#S_n(v)) = (\#S_n(w,v))(\#S_n(v,u)) \quad (3)$$

*Proof.* By definition, we have positive correlation if and only if $Pr(\pi \text{ av. } w \mid \pi \text{ av. } v) > \Pr(\pi \text{ av. } w \mid \pi \text{ av. } v, u)$. And:

$$\Pr(\pi \text{ av. } w \mid \pi \text{ av. } v) > \Pr(\pi \text{ av. } w \mid \pi \text{ av. } v, u) \iff \frac{\left(\frac{\#S_n(w,v)}{n!}\right)}{\left(\frac{\#S_n(v)}{n!}\right)} > \frac{\left(\frac{\#S_n(u,v,w)}{n!}\right)}{\left(\frac{\#S_n(v,u)}{n!}\right)}$$

$$\iff (\#S_n(w,v))(\#S_n(v,u)) > (\#S_n(u,v,w))(\#S_n(v))$$

.

The criteria for negative correlation and for independence are proved in exactly the same way. □

Another useful set of tools for us are the reversal, inverse, and complement maps, which are easy bijections from $S_n$ to itself.

**Definition 1.3.** Let $\Pi = \{\pi_1, ..., \pi_n\}$ be a set of permutations. Then $\Pi^R = \{\pi_1^R, ..., \pi_n^R\}$ is the set of the reversals of each permutation in $\Pi$. $\Pi^C$ and $\Pi^{-1}$ are defined analogously for the complement and inverse operations.

*Remark.* It is well known that for any set of permutations $\Pi$, $\#S_n(\Pi) = \#S_n(\Pi^R) = \#S_n(\Pi^C) = \#S_n(\Pi^{-1})$.

As a result of this remark, we will generally only give one representative of a set $\Pi$ and the sets which can be obtained by composing the operations referred to in the remark on $\Pi$.

## 1.1 $v$,$w$,$u$ All Length 3

By virtue of existing literature, we are able to give a complete answer to Question 1.1 in the case where $u, v, w$ are all of length 3.

According to the Correlation Criterion, it will be sufficient to find $\#S_3(\Pi)$ for all $\Pi \subset S_3$ and $\#\Pi = 1, 2, 3$.

First, it is well-known that $\#S_n(\pi) = C_n$ for all $\pi \in S_3$, where $C_n = \binom{2n}{n}/(n+1)$ is the $n^{th}$ Catalan number.

The quantities for $\#\Pi \in \{2, 3\}$ are given by [8] and are shown below in Tables 1 and 2 respectively. As mentioned previously, we omit sets which can be obtained by applying a composition of the reversal, inverse, and complement operations to an already-listed set.

| $\Pi$ | $\#S_n(\Pi)$ |
|---|---|
| $\{123, 132\}, \{132, 213\}, \{132, 231\}$ | $2^{n-1}$ |
| $\{123, 231\}$ | $\binom{n}{2} + 1$ |

Table 1: Number of permutations which avoid different pairs of length-3 patterns

The following lemmas will be helpful in applying the Correlation Criteria. (Note that since Question 1.1 pertains to large $n$, we only care about the functions asymptotically; we will write $f_1(n) \gg f_2(n)$ to indicate that $lim_{n\to\infty} f_1(n) > lim_{n\to\infty} f_2(n)$, while $f_1(n) \asymp f_2(n)$ will indicate that $lim_{n\to\infty} f_1(n) = lim_{n\to\infty} f_2(n)$.)

**Lemma 1.4.** $C_n F_{n+1} >> 2^{2n-2}$

| Sets | Number of Permutations Avoiding |
|---|---|
| $\{123, 132, 213\}$ | $F_{n+1}$ |
| $\{123, 132, 231\}$, $\{132, 213, 231\}$, $\{123, 231, 312\}$ | $n$ |
| $R \supset \{123, 321\}$ | 0 |

Table 2: Number of permutations which avoid different triples of length-3 patterns

*Proof.* Let $g(n) = C_n F_{n+1}$ and $h(n) = 2^{2n-2}$ for all positive integers $n$. In particular, $g(3) = (5)(14) = 70 > h(3) = 2^6 = 64$. We'll complete the proof by showing that $\frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}$ for all $n \geq 3$.

Clearly $\frac{h(n+1)}{h(n)} = 4$ for all $n \geq 3$.

Meanwhile, $\frac{g(n+1)}{g(n)} = \dfrac{\binom{2n+2}{n+1}/(n+2)}{\binom{2n}{n}/(n+1)} * \frac{F_{n+2}}{F_{n+1}} = \frac{4n+2}{n+2} * \frac{F_{n+2}}{F_{n+1}}$. Notice that $\frac{F_{n+2}}{F_{n+1}} > \frac{3}{2}$ for all $n \geq 3$, so we have that

$\frac{g(n+1)}{g(n)} > \frac{6n+3}{n+2} > 4$ for all $n \geq 3$, completing the proof. □

**Lemma 1.5.** $2^{n-1} >> \binom{n}{2} + 1$

*Proof.* Let $g(n) = 2^{n-1}$ and $h(n) = \binom{n}{2} + 1$. Then $g(4) = 8 > g(4) = 7$. We'll complete the proof by showing that $\frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}$ for all $n \geq 4$.

Clearly $\frac{g(n+1)}{g(n)} = 2$ for all $n \geq 4$.

Meanwhile, $\frac{h(n+1)}{h(n)} = \frac{n(n+1)/2+1}{n(n-1)/2+1} < \frac{n(n+1)/2}{n(n-1)/2} = \frac{n+1}{n-1} < 2$ for all $n \geq 4$. □

**Lemma 1.6.** $2^{2n-2} >> nC_n$

*Proof.* Let $g(n) = 2^{2n-2}$ and $h(n) = nC_n$. Then $g(3) = 16 > h(3) = 15$. We'll complete the proof by showing that $\frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}$ for all $n \geq 3$.

Clearly $\frac{g(n+1)}{g(n)} = 4$ for all $n \geq 4$.

Meanwhile, $\frac{h(n+1)}{h(n)} = \frac{(n+1)C_{n+1}}{nC_n} = \frac{(4n+2)(n+1)}{(n+2)(n)} < 4$ for all $n \geq 3$. □

**Lemma 1.7.** $nC_n >> 2^{n-1}(\binom{n}{2} + 1)$

*Proof.* Let $g(n) = nC_n$ and $h(n) = 2^{n-1}(\binom{n}{2} + 1)$. Then $g(4) = 56 = h(4)$. We'll complete the proof by showing that $\frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}$ for all $n \geq 4$.

As seen in the previous lemma, $\frac{g(n+1)}{g(n)} = \frac{(4n+2)(n+1)}{(n+2)(n)} > \frac{10}{3}$ for all $n \geq 4$.

On the other hand, $\frac{h(n+1)}{h(n)} = 2\frac{\binom{n+1}{2}+1}{\binom{n}{2}+1} < 2\frac{n+1}{n-1} \leq \frac{10}{3}$ for all $n \geq 4$. □

**Theorem 1.8.** *If $v = (132)$, then avoiding $u$ and avoiding $w$*
*(i) positively correlate in $S_n(v)$ if $(u, w) \in \{(123, 231), (123, 312), (213, 231), (213, 312), (231, 312)\}$*
*(ii) negatively correlate in $S_n(v)$ if $(u, w) \in \{(123, 213), (213, 321), (231, 321), (312, 321)\}$.*

*Proof.* Each case of (i) follows directly from Lemma 1.7.

In (ii), if $(u, w) = (123, 213)$, the result follows from Lemma 1.4; otherwise, it follows from Lemma 1.7. □

**Theorem 1.9.** *If $v = (123)$, then avoiding $u$ and avoiding $w$ negatively correlate in $S_n(v)$ if $(u, w) \in \{(132, 213), (132, 231), (132, 312), (213, 231), (213, 312), (231, 312)\}$.*

*Proof.* If $(u, w) = (132, 213)$, the result follows from Lemma 1.4. If $(u, w) = (231, 312)$, the result follows from Lemmas 1.5 and 1.7. In any other case, the result follows from Lemma 1.7 alone. □

*Remark.* Theorems 1.8 and 1.9 completely answer Question 1.1 in the case where $u, v, w \in S_3$ since any $v' \in S_3$ can be obtained by the trivial bijections (reversal, inverse, complement) from 123 or 132.

### 1.2 Avoiding 132 Given No Long Decreasing Subsequence

In this section, we investigate Question 1.1 for $v = k(k-1)...1$, $u = \ell(\ell-1)...1$ and $w = 132$, where $\ell < k$.

*Remark.* Notice that the Correlation Criteria imply that avoiding 132 and avoiding $\ell...1$ positively correlate in $S_n(k...1)$ if and only if

$$\#S_n(k...1)\#S_n(132, \ell...1) > \#S_n(\ell...1)\#S_n(132, k...1) \tag{4}$$

while there is negative correlation when the inequality in 4 goes the other way, and independence when there is equality.

*Remark.* Notice that dividing both sides of (4) by $\#S_n(k...1)\#S_n(\ell...1)$, we see that this case of Question 1.1 is equivalent to the problem of determining which of $S_n(k(k-1)...1)$ and $S_n(\ell(\ell-1)...1)$ has a higher proportion of (132)-avoiding permutations.

Fortunately, all of the quantities that appear in (4) are given (at least asymptotically) by theorems of Reifegerste and of Regev:

**Theorem 1.10** (Reifegerste [7])**.** $\#S_n(132, m(m-1)...1) = \frac{1}{n}\sum_{i=1}^{m-1}\binom{n}{i}\binom{n}{i-1}$

**Theorem 1.11** (Regev [6])**.** $\#S_n(m...1) \asymp \lambda_m \frac{(m-1)^{2n}}{n^{m(m-1)/2}}$ *for some constant* $\lambda_m$

Combining Theorems 1.10 and 1.11, we can prove positive correlation in this case of Question 1.1:

**Theorem 1.12.** *Avoiding* 132 *and avoiding* $\ell...1$ *positively correlate in* $S_n(k...1)$.

*Proof.* By Theorems 1.10 and 1.11, and by (4), it will suffice to show that

$$\lambda_k \frac{(k-1)^{2n}}{n^{k(k-2)/2}} \frac{1}{n}\sum_{i=1}^{\ell-1}\binom{n}{i}\binom{n}{i-1} \gg \lambda_\ell \frac{(\ell-1)^{2n}}{n^{\ell(\ell-2)/2}} \frac{1}{n}\sum_{i=1}^{k-1}\binom{n}{i}\binom{n}{i-1} \tag{5}$$

By induction, it will be sufficient to prove (5) in the case where $k = \ell + 1$. In particular, it will suffice to show that

$$\frac{(k-1)^{2n}}{(\ell-1)^{2n}} \gg \frac{n^{k(k-2)/2}}{n^{\ell(\ell-2)/2}} \frac{\sum_{i=1}^{k-1}\binom{n}{i}\binom{n}{i-1}}{\sum_{i=1}^{\ell-1}\binom{n}{i}\binom{n}{i-1}} \tag{6}$$

Notice that the left side of (6) is exponential in $n$; we will show that the right side can be bounded by $n^c$ for some constant $c$.

Since $k = \ell + 1$,

$$\frac{\sum_{i=1}^{k-1}\binom{n}{i}\binom{n}{i-1}}{\sum_{i=1}^{\ell-1}\binom{n}{i}\binom{n}{i-1}} = 1 + \frac{\binom{n}{\ell}\binom{n}{\ell-1}}{\sum_{i=1}^{\ell-1}\binom{n}{i}\binom{n}{i-1}} \leq 1 + \frac{\binom{n}{\ell}\binom{n}{\ell-1}}{\binom{n}{\ell-1}\binom{n}{\ell-2}} < 1 + \frac{\binom{n}{\ell}}{\binom{n}{\ell-2}} = 1 + \frac{(n-\ell+2)(n-\ell+1)}{\ell(\ell-1)} < n^3.$$

Thus we have

$$\frac{(k-1)^{2n}}{(\ell-1)^{2n}} \gg n^{k(k-2)/2-\ell(\ell-2)/2+3} \gg \frac{n^{k(k-2)/2}}{n^{\ell(\ell-2)/2}} \frac{\sum_{i=1}^{k-1}\binom{n}{i}\binom{n}{i-1}}{\sum_{i=1}^{\ell-1}\binom{n}{i}\binom{n}{i-1}},$$

and the proof is complete. □

We now want to explore Question 1.1 with $w$ being a different permutation of length 3 – that is, where $v = k...1$, $u = \ell...1$ and $w \in S_3$. It turns out there is only one other interesting case. The case where $w = 321$ is trivial and when $w = 123$, the answer follows easily from Theorem 0.1. Meanwhile, notice that $(\{m...1, 132\}^R)^C = \{m...1, 213\}$, and $(\{m...1, 231\}^R)^C = \{m...1, 312\}$ implying that $w = 213$ is the same case as $w = 132$ and $w = 231$ is the same case as $w = 312$. Thus the latter case is the only interesting remaining sub-case of the case where $v = k...1$, $u = \ell...1$, $w \in S_3$.

Unfortunately, there is no theorem analogous to Reifegerste's for the permutation 231; that is, $\#S_n(231, m...1)$ is not known in general. So we must try a different approach.

# 2 Characteristic polynomial problem

**Conjecture 2.1.** *For any 231-avoiding permutation $\pi$, $T(n) = S_n(231, \pi)$ satisfies a linear recurrence, and its characteristic polynomial has all positive real roots.*

We denote $\sigma \leq \pi$ for two permutations $\sigma, \pi$ if $\pi$ contains $\sigma$ as a pattern. A permutation class $\mathcal{C}$ is a set of permutations such that for all $\pi \in \mathcal{C}$ and for all $\sigma \leq \pi$, $\sigma$ is also in $\mathcal{C}$.

It is proven (as a corollary) in 2011 [5] that any subclass of 231-avoiding permutations $S_n(231)$ has a rational generating function, and hence it satisfies a linear recurrence. However, not much is known about the coefficients of these recurrences.

Our conjecture implies that these coefficients form a *Pólya frequency sequence*, which has a number of nice properties like log-concavity, satisfying Newton's inequalities, etc. It would also relates pattern avoiding permutations to the theory of total positivity, which is a major area in combinatorics. In this section, we will prove some special cases of this main conjecture.

## 2.1 Permutations avoiding 231 and $k(k-1)...1$

In this subsection, we will denote $T(n, k) = |S_n(231, k(k-1)...1)|$.

**Proposition 2.2.**

$$T(n+1, k) = \sum_{0 \leq i \leq n} T(i, k) T(n-i, k-1)$$

*Proof.* Let $\rho \in S_{n+1}(231)$. Consider its decomposition $\rho = \sigma(n+1)\tau$ with $|\sigma| = i$, $|\tau| = n - i$. If $\rho$ avoids $k(k-1)...1$ then it is clear that $\sigma$ avoids $231, k(k-1)...1$ and $\tau$ avoids $231, (k-1)...1$.

Now, since $\rho \in S_{n+1}(231)$, $\sigma_i < \tau_j$ for all $i, j$ (otherwise, $\sigma_i, n+1, \tau_j$ forms a 231 pattern). This then implies that $\sigma \in S_i(231, k(k-1)...1)$ and $\tau \in S_{n-i}(231, (k-1)...1)$ are also the sufficient conditions for $\rho \in S_{n+1}(231, k(k-1)...1)$.

Consequently, we have $T(i, k)$ choices for $\sigma$ and $T(n-i, k-1)$ choices for $\tau$ and thus we get each term on the right hand side. Since this holds for all $i$, we can sum all the terms up to get the desired recurrence. $\square$

This recurrence may not seem useful at first, but if we know the linear recurrence that $T(n, k-1)$ satisfies, we can substitute it in our recurrence above to deduce a linear recurrence for $T(n, k)$. For example, it is known that $|S_n(231, 321)| = 2^{n-1}$, plugging this in gives us this recurrence for $S_n(231, 4321)$: $T(n, 4) = 3T(n-1, 4) - T(n-2, 4)$.

We will generalize this process in the next theorem. Before that, we have a remark followed by a lemma which will be useful in the proof afterward,

*Remark.* For $n < k$, $T(n, k)$ counts 231-avoiding permutations, so it is just $C_n$, the Catalan number. We will make heavy use of this remark in the proof.

**Lemma 2.3.** *For any $j$ and $n$ (with the convention $\binom{a}{b} = 0$ if $a < b$),*

$$(-1)^{j+1}\binom{n-j}{j} = (-1)^{j+1}\binom{n-1-j}{j} + C_{j-1} - \sum_{i=1}^{j-1}(-1)^{i+1}C_{j-1-i}\binom{n-1-i}{i}$$

*Proof.* It is easy to check this for small cases of $n$. Suppose this is true for $n \leq k$. We will prove this for $n = k+1$.

Denote the right hand side $S(n, j)$, then $\binom{n-j}{j} = S(n, j)$ for all $n \leq k, j \leq t$. Now, $S(k+1, j) - S(k, j) = -S(k-1, j-1)$ since $\binom{n+1-j}{j} - \binom{n-j}{j} = \binom{n-j}{j-1}$. So $S(k+1, j) = (-1)^j\binom{k-j}{j} - (-1)^{j-1}\binom{k-j}{j-1} = (-1)^j\binom{k+1-j}{j}$. $\square$

**Theorem 2.4.** *Let $l = \lfloor \frac{k}{2} \rfloor$, then*

$$T(n, k) = \binom{k-1}{1}T(n-1, k) - \binom{k-2}{2}T(n-2, k) + ... + (-1)^{l+1}\binom{k-l}{l}T(n-l, k)$$

*When $k = 2l$, this result is true from $n = l$, and when $k = 2l+1$, this is true from $n = l+1$.*

*Proof.* We will prove this by induction. For $k = 2, T(n, 2) = T(n-1, 2) = 1$ for all $n \geq 1$, $k = 3$, $T(n, 3) = 2T(n-1, 3)$ and this is true only from $n = 2$, while for $k = 4$, $T(n, 4) = 3T(n-1, 4) - T(n-2, 4)$, which is true from $n = 2$.

suppose the recurrence is true for $k \leq 2l$. We will prove it for $k = 2l + 1$. Using the recurrence in Proposition 2.2, for $n \geq l + 1$,

$$\begin{aligned} T(n, 2l+1) &= \sum_{0 \leq i \leq n-1} T(i, 2l+1) T(n-1-i, 2l) \\ &= \sum_{0 \leq i \leq n-l-1} T(i, 2l+1) T(n-1-i, 2l) + \sum_{n-l \leq i \leq n-1} T(i, 2l+1) T(n-1-i, 2l) \\ &= \sum_{0 \leq i \leq n-l-1} T(i, 2l+1) T(n-1-i, 2l) + \sum_{0 \leq i \leq l-1} T(n-1-i, 2l+1) C_i \end{aligned}$$

Applying our inductive hypothesis to $T(n, 2l)$, with $a_j = (-1)^{j+1}\binom{2l-j}{j}$,

$$\begin{aligned} T(n, 2l+1) &= \sum_{0 \leq i \leq n-l-1} T(i, 2l+1) \sum_{1 \leq j \leq l} a_j T(n-1-i-j, 2l) + \sum_{0 \leq i \leq l-1} T(n-1-i, 2l+1) C_i \\ &= \sum_{1 \leq j \leq l} a_j \sum_{0 \leq i \leq n-l-1} T(i, 2l+1) T(n-1-i-j, 2l) + \sum_{0 \leq i \leq l-1} T(n-1-i, 2l+1) C_i \end{aligned}$$

Repeated use of the recurrence in proposition 2.2 gives us

$$\begin{aligned} T(n, 2l+1) &= \sum_{1 \leq j \leq l} a_j \Big( T(n-j, 2l+1) - \sum_{n-l \leq i \leq n-j-1} T(i, 2l+1) T(n-j-1-i, 2l) \Big) + \sum_{1 \leq j \leq l} T(n-j, 2l+1) C_{j-1} \\ &= \sum_{1 \leq j \leq l} a_j \Big( T(n-j, 2l+1) - \sum_{j+1 \leq i \leq l} T(n-i, 2l+1) C_{i-j-1} \Big) + \sum_{1 \leq j \leq l} T(n-j, 2l+1) C_{j-1} \\ &= \sum_{1 \leq j \leq l} a_j \Big( T(n-j, 2l+1) - \sum_{0 \leq i \leq l-j-1} T(n-i-j-1, 2l+1) C_i \Big) + \sum_{1 \leq j \leq l} T(n-j, 2l+1) C_{j-1} \end{aligned}$$

Then we can group the terms to get the coefficient of $T(n-t, 2l+1)$, which is

$$a_t + C_{t-1} - \sum_{i=1}^{t-1} a_i C_{t-1-i} = (-1)^{t+1}\binom{2l-t}{t} + C_{t-1} - \sum_{i=1}^{t-1} (-1)^{i+1} C_{t-1-i} \binom{2l-i}{i}$$

By proposition 2.4, this is $(-1)^{t+1}\binom{2l+1-t}{t}$. So our theorem is proved in this case. For the case in which the inductive hypothesis is true for $k \leq 2l-1$, proving it for $k = 2l$, we can use the exactly same proof as above to show that the recurrence is true from $n \geq l+1$. To show that it is true also for $n = l$, we need to prove that (again, recall $T(n, k) = C_n$ if $n < k$):

$$C_l = \sum_{1 \leq i \leq l} (-1)^{i+1} \binom{2l-i}{i} C_{l-i}$$

By our inductive hypothesis in the case $k = 2l - 1$,

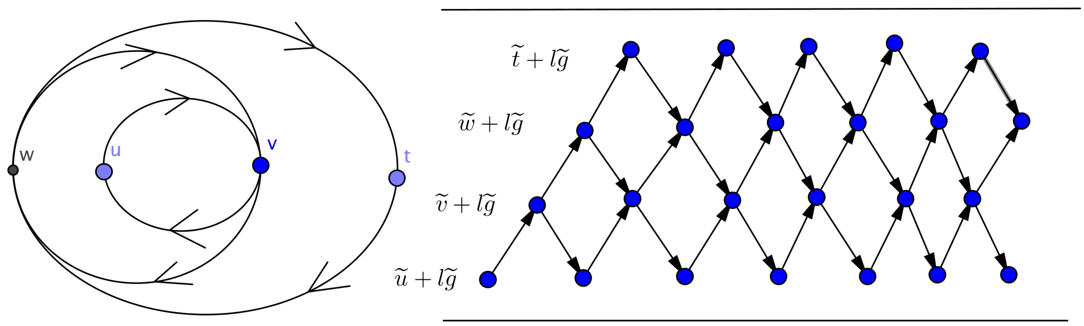
$$C_l = \sum_{1 \leq i \leq l-1} (-1)^{i+1} \binom{2l-1-i}{i} C_{l-i}$$

Then,

$$\begin{aligned}\sum_{1\le i\le l}(-1)^{i+1}\binom{2l-i}{i}C_{l-i}-C_l &= \sum_{1\le i\le l}(-1)^{i+1}\binom{2l-i}{i}C_{l-i}-\sum_{1\le i\le l-1}(-1)^{i+1}\binom{2l-1-i}{i}C_{l-i}\\ &= \sum_{1\le i\le l-1}(-1)^{i+1}\binom{2l-1-i}{i-1}C_{l-i}+(-1)^{l+1}C_0\\ &= \sum_{0\le j\le l-2}(-1)^{j}\binom{2l-2-j}{j}C_{(l-1)-j}+(-1)^{l+1}C_0\\ &= C_{l-1}-\sum_{1\le j\le l-1}(-1)^{j+1}\binom{2l-2-j}{j}C_{l-1-j}\\ &= 0\end{aligned}$$

The last equation follows from our inductive hypothesis for the case $k = 2l - 2$. □

**Note 2.1.** This induction solution has an advantage in that it can probably be generalized and used in other pattern avoiding permutations which satisfy a recurrence similar to the one in proposition 2.2, but it doesn't explain why the coefficients have such nice form. To see this, we look at a recent result on cylindrical networks. The full definitions and results can be found in Galashin and Pylyavskyy's paper earlier this year [3], here we will only use a special case, which is a planar cylindrical network with all edges weighted 1.




There is a well known bijection between $S_n(231, k..1)$ and the set of Dyck paths of length $2n$, height $\le k-1$. It's not hard (think of the first layer's $x-$coordinates as $\{0, 2, 4, ..., 2n, ...\}$, the second as $\{1, 3, 5, ..., 2n+1, ..\}$, the third $\{2, 4, 6, ...\}$ and so on) to see that there is also a bijection between the set of Dyck paths of length $2n$ and height $\le k - 1$ and the set of directed paths connecting $\widetilde{u}$ and $\widetilde{u} + n\widetilde{g}$ on the right hand side graph of our figure above (with $k$ levels of vertices instead of 4). So if we define a sequence $f(n)$ so that for each $n$, $f(n)$ equals to the number of directed paths connecting $\widetilde{u}, \widetilde{u} + n\widetilde{g}$, then $f(n) = |S_n(231, k...1)|$.

Now, the graph $\widetilde{N}$ on the right hand side can be thought of as a shift (by $\widetilde{g}$) invariant acyclic and weighted (all edges weighted 1) graph drawn in a strip $\mathcal{S}$. The graph $N$ on the left hand side can then be thought of as $\widetilde{N}$'s projection on a cylinder $\mathcal{O} = \mathcal{S}/\mathbb{Z}\widetilde{g}$. We say that a cycle in $N$ is simple if it passes each vertex of $N$ at most once.

Then, we have the following as a corollary of a main result from Galashin and Pylyavskyy's paper:

**Theorem 2.5.** *Let $f(n)$ be defined as above. For all but finitely many $n$, the sequence $f$ satisfies a linear recurrence with characteristic polynomial:*

$$Q_N(t) = \sum_{r=0}^{d}(-t)^{d-r}|\mathcal{C}^r(N)|$$

*where $d$ is the largest number of disjoint simple cycles, and $\mathcal{C}^r(N)$ is the set of all $r-$tuples of disjoint simple cycles in $N$.*

We can check that for our graph, $\mathcal{C}^r(N)$ is in bijection to the set of $r-$tuples of numbers chosen from $\{1, 2, ..., k\}$ such that no 2 adjacent numbers are in a tuple. The number of ways to choose such a tuple is $\binom{k-r}{r}$, so $|\mathcal{C}^r(N)| = \binom{k-r}{r}$. As a result the characteristic polynomial corresponds to our recurrence.

**Proposition 2.6.** *Let $P_k(x)$ be the characteristic polynomial for the linear recurrence that $T(n,k)$ satisfies. Then $P_k$ has all positive real roots.*

*Proof.* Thanks to the identity $\binom{n+1-j}{j} - \binom{n-j}{j} = \binom{n-j}{j-1}$, we have the following two equations:

$$P_{2k+1}(x) - P_{2k}(x) = -P_{2k-1}(x)$$

$$P_{2k}(x) - xP_{2k-1}(x) = -P_{2k-2}(x)$$

Using these, we will prove that each $P_k$ has all real roots, and for $P_k, P_{k+1}$ their roots are interlaced, by induction. To be more specific,

- If $P_{2k}(x)$ has $k$ real roots $\alpha_1, ..., \alpha_k$, then $P_{2k+1}(x)$ also has $k$ real roots $\beta_1, ..., \beta_k$ such that $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_k < \beta_k$ and $P_{2k}(\beta_k) > 0$.
- If $P_{2k-1}(x)$ has $k-1$ real roots $\beta_1, ..., \beta_{k-1}$ then $P_{2k}(x)$ has $k$ real roots $\alpha_1, ..., \alpha_k$ with $0 < \alpha_1 < \beta_1 < \alpha_2 < ... < \beta_{k-1} < \alpha_k$ and $P_{2k-1}(\alpha_k) > 0$.

It is easy to check for small $k$, now suppose this is true for $k \leq 2l$, we will prove it for $k = 2l+1$. Consider any two consecutive roots $\lambda_1, \lambda_2$ of $P_{2l}(x)$. Now $P_{2l+1}(\lambda_1) = -P_{2l-1}(\lambda_1), P_{2l+1}(\lambda_2) = -P_{2l-1}(\lambda_2)$. By our inductive hypothesis, there is exactly one root of $P_{2l-1}$ between $\lambda_1, \lambda_2$, so $P_{2l-1}(\lambda_1), P_{2l-1}(\lambda_2)$ have different signs, so $P_{2l+1}(\lambda_1), P_{2l+1}(\lambda_2)$ also have different signs, and it follows that there is a real root of $P_{2l+1}$ between $\lambda_1, \lambda_2$. Consider $\alpha_l$ largest root of $P_{2l}$, then,

$$P_{2l+1}(\alpha_l) = -P_{2l-1}(\alpha_l) < 0 \text{ (by inductive hypothesis)}$$

and $P_{2l+1}(x) \to \infty$ as $x \to \infty$, so there is another real root $\beta_l$ of $P_{2l+1}$ larger than $\alpha_l$. We get all $l$ real roots. Also, $P_{2l}(\beta_l) > 0$ since otherwise there is another root of $P_{2l}$ larger than or equal to $\beta_l > \alpha_l$, which is a contradiction.

Suppose our hypothesis is true for $k \leq 2l+1$, now we will prove it for $k = 2l+2$. By similar arguments as above (using the second equation instead of the first), there is a real root of $P_{2l+2}$ between any two consecutive real roots of $P_{2l+1}$. That is $l$ real roots and we are just missing 2 more. Consider $\beta_l$ largest root of $P_{2l+1}$. Then,

$$P_{2l+2}(\beta_l) = -P_{2l}(\beta_l) < 0$$

So there is a root $\alpha_{l+1}$ of $P_{2l+2}$ that is larger than $\beta_l$. Look at the smallest root $\beta_1$ of $P_{2l+1}$. Then

$$P_{2l+2}(\beta_1) = -P_{2l}(\beta_1)$$

Now, if $l$ is odd, then $P_{2l}(0) = (-1)^l \binom{2l-l}{l} < 0$. From our inductive hypothesis, there is exactly one root of $P_{2l}$ between $\beta_1$ and 0, so $P_{2l}(\beta_1) > 0$. So $P_{2l+2}(\beta_1) < 0$. Now since $l$ is odd, $P_{2l+2}(0) = (-1)^{l+1}\binom{2l+2-l-1}{l+1} > 0$, so $P_{2l+2}$ has another root $0 < \alpha_1 < \beta_1$. An analogous argument can be used in the case $l$ is even.

Now let's show $P_{2l+1}(\alpha_{l+1}) > 0$. Suppose otherwise, then $P_{2l+1}$ has a real root larger or equal to $\alpha_{l+1} > \beta_l$, which contradicts the fact that $\beta_l$ is the largest root of $P_{2l+1}$. ☐

**Conjecture 2.7.** *$P_k(4k^2/(k+1)^2) < 0$ for all $k$.*

If this is true, then the largest root $\lambda_k$ of $P_k$ is larger than $\frac{4k^2}{(k+1)^2}$. This gives us some nice insight into the growth rate of $T(n,k) \simeq C_k(\alpha_k)^{n-1}$ since it implies that $\lambda_k$ converges to 4, which seems to agree with the fact that $T(n,k) = C_n$ if $n < k$ and the growth rate of the Catalan number is $C\frac{4^n}{n^{3/2}}$.
This conjecture would also shows that for $l < k$ and large $n$,

$$\mathbb{P}(\text{av. } 231|\text{av. } l...1) > \mathbb{P}(\text{av. } 231|\text{av. } k...1)$$

which answers one of our correlation question earlier. As for why, if $\lambda_k > \frac{4k^2}{(k+1)^2}$, then for large $n$, $\frac{T(n,k)}{T(n,k+1)} = C\big(\frac{\lambda_k}{\lambda_{k+1}}\big)^{n-1} > C\big(\frac{\lambda_k}{4}\big)^{n-1}$ (since $T(n,k) < C_n$ for all $n$, the growth rate of $T(n,k)$ cannot surpass 4). The above conjecture implies that this is larger than $C\frac{k^{2n-2}}{(k+1)^{2n-2}}$. On the other hand, for large $n$,

$$\frac{|S_n(k...1)|}{|S_n((k+1)k...1)|} = C'\Big(\frac{(k-1)}{k}\Big)^{2n} n^{\frac{2k-1}{2}}$$

For large $n$, $C\frac{k^{2n-2}}{(k+1)^{2n-2}} > C'\Big(\frac{(k-1)}{k}\Big)^{2n} n^{\frac{2k-1}{2}}$ (just take their fraction, it is $> 1$ as $n \to \infty$), so $\frac{T(n,k)}{S_n(k..1)} > \frac{T(n,k+1)}{S_n((k+1)k...1)}$, and the result follows.

## 2.2 Permutations avoiding 231 and $l(l-1)...1k(k-1)...(l+1)$

**Theorem 2.8.** *For any $0 \le l < k$, $|S_n(231, l(l-1)...1k(k-1)...(l+1))| = |S_n(231, k(k-1)...1)|$.*

This result can be found in Reifegerste's paper [7]. It provides another family of pattern avoiding permutations satisfying our conjecture.

The following proposition suggests that there should be an induction proof (similar to the one in the above section) for the fact that the linear recurrence that $|S_n(231, l(l-1)...1k(k-1)...(l+1))|$ satisfies is the same as the one for $|S_n(231, k(k-1)...1)|$.

**Proposition 2.9.** *Let $\pi = l..1k...(l+1)$ and $T(n,\pi) = |S_n(231,\pi)|$ and $T(n,k) = |S_n(231, k..1)|$. Then,*

$$T(n+1,\pi) = T(n,\pi) + \sum_{0<i<n+1} \big(T(i,\pi)T(n-i,k-l-1) + T(i,l)T(n-i,\pi) - T(i,l)T(n-i,k-l-1)\big)$$

*Proof.* Again, look at a permutation $\rho \in S_{n+1}(231)$. We have the following decomposition $\rho = \sigma(n+1)\tau$. It is clear that $\sigma, \tau$ need to avoid $\{231, \pi\}$ if we want $\rho$ to avoid $\pi$. If, in addition, either $\sigma$ avoids $\{231, l(l-1)...1\}$ or $\tau$ avoids $\{231, (k-l-1)...1\}$, then $\rho$ avoids 231 and $l(l-1)...1k(k-1)...(l+1)$. Recall that since $\rho \in S_n(231)$, $\sigma(i) < \tau(j)$ for all $i, j$.

To prove that this is also the necessary condition, if $\sigma$ contains $l..1$ and $\tau$ contains $(k-l-1)...1$, then since $\sigma(i) < \tau(j)$ for all $i, j$, $\rho$ contains $\pi$.

Using inclusion-exclusion principle, we get each term on the right hand side for $|\sigma| = i$. Since this holds for all $i = 0, 1, ..., n$, we attain the desired result. □

Inparticular, for $l = 1$, the induction proof for our theorem reduces to proving the following identity, which also follows directly from the fact that $\binom{n+1-j}{j} - \binom{n-j}{j} = \binom{n-j}{j-1}$.

**Proposition 2.10.** *For any $j, n$,*

$$(-1)^{j+2}\binom{n-j-1}{j+1} = (-1)^{j+2}\binom{n-2-j-1}{j+1} - (-1)^{j+1}\binom{n-2-j}{j} + C_j - \sum_{0<i\le j}(-1)^{i+1}\binom{n-2-i}{i}C_{j-i}$$

## 2.3 Permutations avoiding 231 and $12...k$

**Theorem 2.11.** *$T_k(n) = I(n,k) = |S_n(231, 12...k)|$ has characteristic polynomial $(x-1)^{2k-3}$.*

It would then follow that $|S_n(231, 12...k)|$ satisfies our main conjectures since all the roots are 1. It is known (and shown in our first section on correlation problem) that $T_k(n)$ can be written explicitly as a sum of Narayana numbers:

$$I(n,k) = \sum_{i=1}^{k-1} \frac{1}{n}\binom{n}{i}\binom{n}{i-1} = \sum_{i=1}^{k-1} \frac{1}{i}\binom{n}{i-1}\binom{n-1}{i-1}$$

so it is natural to consider the identity below,

**Lemma 2.12.**

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i}\binom{n+i}{k}\binom{n+i-1}{k} = 0$$

If we write $S_{k+1}(n) = N(n,k+1) = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}$ for a fixed $k$, then the proposition shows that this sequence $S_{k+1}(n)$ satisfies a linear recurrence with characteristic polynomial $(x-1)^{2k+1}$. Then $S_{k+1}(n)$ also satisfies a linear recurrence with characteristic polynomial $(x-1)^{2l+1}$ for $l > k$. As a result, for $l \geq k$,

$$\sum_{i=0}^{2l+1}(-1)^i\binom{2l+1}{i}\binom{n+i}{k}\binom{n+i-1}{k} = 0$$

and hence

$$\sum_{i=0}^{2k-3}(-1)^i\binom{2k+1}{i}\binom{n+i}{t}\binom{n+i-1}{t} = 0$$

for $t \leq k-1$. Our theorem would then follow.

*Proof.* Let $f(i) = (-1)^i\binom{2k+1}{i}\binom{n+i}{k}\binom{n+i-1}{k}$, then

$$\begin{aligned}\frac{f(i+1)}{f(i)} &= (-1)\frac{\binom{2k+1}{i+1}\binom{n+i+1}{k}\binom{n+i}{k}}{\binom{2k+1}{i}\binom{n+i}{k}\binom{n+i-1}{k}} \\ &= \frac{(-1)(2k+1-i)(n+i)(n+i+1)}{(i+1)(n+i+1-k)(n+i-k)} \\ &= \frac{\big(i-(2k+1)\big)(i+n+1)(i+n)}{(i+1)(i+n+1-k)(i+n-k)}\end{aligned}$$

So our identity is equivalent to the following hypergeometric identity ${}_3F_2\left[\begin{matrix}-(2k+1) & n & n+1 \\ n-k & n+1-k\end{matrix};1\right]f(0) = 0$. Now, $f(0) = \binom{n}{k}\binom{n-1}{k}$ and will be 0 when $n \leq k$. For $n > k$, using the *Pfaff-Saalschutz identity*,

$${}_3F_2\left[\begin{matrix}-(2k+1) & n & n+1 \\ n-k & n+1-k\end{matrix};1\right] = \frac{(n-k-n)_{2k+1}(n-k-(n+1))_{2k+1})}{(n-k)_{2k+1}(n-k-n-(n+1))_{2k+1}} = \frac{(-k)_{2k+1}(-k-1)_{2k+1}}{(n-k)_{2k+1}(-k-n-1)_{2k+1}}$$

where $(a)_s = (a)(a+1)...(a+s-1)$.

It is clear that $(-k)_{2k+1} = (-k-1)_{2k+1} = 0$, and for $n > k$, all terms in $(n-k)_{2k+1}$ are larger than 0, so $(n-k)_{2k+1} > 0$. Similarly, for $n > k$, all terms in $(-k-n-1)_{2k+1}$ are smaller than 0 (since $-k-n-1 < -(2k+1)$), so $(-k-n-1)_{2k+1} < 0$. As a result, ${}_3F_2\left[\begin{matrix}-(2k+1) & n & n+1 \\ n-k & n+1-k\end{matrix};1\right] = 0$ for $n > k$. This concludes our proof for the identity in the lemma above. □

## 2.4 Method on testing for other patterns

Recall the decomposition $\rho = \sigma n \tau$ we use for $\rho \in S_n(231)$. We can use the arguments we use above to find the linear recurrence for permutations avoiding 231, and another 231-avoiding pattern $\pi = \pi_1 k \pi_2$ ($|\pi| = k$, and recall $\pi_1(i) < \pi_2(j) \forall i,j$) with $\pi_1$ being one of the nice patterns $\{l..1, \emptyset, 12\}$ and $\pi_2$ being any pattern that you already know the recurrence for $S_n(231, \pi_2)$.

The linear recurrence found might not have nice coefficients (as in having a closed, generalizable form), so this is hard to use in a proof, but convenient to use a computer program to generate the coefficients for these particular patterns.

We can also mimic the method used in Albert, Atkinson and Vatter's paper to show that if $S_n(231, \sigma)$ (with $|\sigma| = k-1$) has generating function $f_\sigma = \frac{P}{Q}$, then $S_n(231, k\sigma)$ has generating function $f$ which satisfies $f = x + \frac{f^2}{1+f} + xf_\sigma$. Solving this, we get

$$f = \frac{x(1+f_\sigma)}{1-x-xf_\sigma} = \frac{x(P+Q)}{Q-xQ-xP}$$

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