EXTREME RAYS OF THE (N,K)-SCHUR CONE

CHRISTIAN GAETZ, KYLE MEYER, KA YU TAM, MAX WIMBERLEY, ZIJIAN YAO,
AND HEYI ZHU

Abstract. We discuss several partial results towards proving White's conjecture on the extreme rays of the $(N, 2)$-Schur cone [6]. We are interested in which vectors are extreme in the cone generated by all products of Schur functions of partitions with k or fewer parts. For the case where $k = 2$, White conjectured that the extreme rays are obtained by excluding a certain family of "bad pairs," and proved a special case of the conjecture using Farkas' Lemma. We present an alternate proof of the special case, in addition to showing more infinite families of extreme rays and reducing White's conjecture to two simpler conjectures. We also give a superset of the extreme rays for $k = 3$, as well as experimental data for other cases.

1. Introduction

The Schur functions are a well studied basis of the ring Λ of symmetric functions. In particular, the celebrated Littlewood-Richardson rule gives an elegant combinatorial interpretation of the coefficients of a product of Schur functions in the Schur basis, in terms of semistandard Young tableaux and Yamanouchi condition on words. The Jacobi-Trudi identity expresses the Schur functions as a determinant of a matrix whose entries are complete homogeneous symmetric functions h_i's, thus writing them as a polynomial in h_i's.

Given these interesting properties of the Schur functions and their product, White, in [6], introduced the idea of the (N, k)-Schur cone $\mathcal{C}_N^k$. He defined $\mathcal{C}_N^k$ to be the cone in Λ_N generated by products of Schur functions of partitions with at most k parts. He asked for a complete characterization of the extreme rays of the Schur cone. Using the Jacobi-Trudi identity, he was able to give a necessary condition for an extreme ray (see Theorem 2.12). He further conjectured that this condition is also sufficient. Using Farkas' lemma, he reduced the problem to showing the existence of separating hyperplanes and proved the conjecture for the special case when all the partitions in A have distinct parts (see Theorem 3.1). The motivating idea of the extremal rays and White's conjecture comes from the study of certain q-log-concave sequences of polynomials [1], the detail of the motivation, however, does not find its way into the paper.

In this paper, we present more partial results towards proving White's conjecture. The paper is organized as follows. In Section 2, we recall related combinatorics background and give a brief overview of the problem. In particular, we discuss the

Date: August 25, 2014.

definitions of the Schur cone as defined in White's paper and present his conjecture.

In Section 3 we give an alternate proof of White's result on partitions of distinct parts, avoiding using Farkas' lemma and extensive notations. Furthermore, the alternate proof offers a different approach to the conjecture. Exploiting the new viewpoint, we then prove several partial results that are not covered by White. In Section 4, we show that s_A is extreme if and only if $s_{A\cup\{(p,p)\}}$ is extreme for any integer p. As a corollary, we obtain that s_A is extreme if A is nested and completely separated (see Definition 2.11 for *nested* and Definition 2.14 for *completely separated*). In Section 5, we show that $s^2_{(j,i)}$ is extreme for all $j > i > 0$.

In Section 6, we describe an induction approach and suggest as conjectures the necessary steps to complete the proof.

In Section 7, we list some infinite families of non-extreme rays for the case $k = 3$ using the Jacobi-Trudi identity and give necessary conditions for a ray to be extreme. Finally, in Section 8, we conclude our paper with a discussion of the enumeration of the number of extreme rays of $\mathcal{C}^k_N$ as k varies. We also discuss some interesting patterns and conjectures observed from computer experimentation.

2. Background and Definitions

The central object of study in this paper, the Schur cone, can be naturally realized as a set of nonnegative linear combinations of products of Schur functions, which form bases for the homogeneous subspaces of the symmetric functions as a real vector space. To this end, we review the relevant definitions and important facts about symmetric functions, particularly Schur functions, before introducing the Schur cone.

2.1. Symmetric Functions and Partitions.

The symmetric functions are formal power series which are invariant under permutation of variables. More precisely,

Definition 2.1. A formal power series $f \in \mathbb{R}[[x_1, x_2, \ldots]]$ of bounded degree is a *symmetric function* if, for any permutation $\pi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, $f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots)$.

The symmetric functions form a subring of the formal power series $\mathbb{R}[[x_1, x_2, \ldots]]$, which we will denote by Λ. Λ is naturally graded by degree with graded components Λ_n, the subspace of homogeneous symmetric functions of degree n.

Recall that a *partition* λ is a non-increasing sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of non-negative integers with finitely many nonzero terms. We say that λ is a partition of N or N is the *weight* of λ, denoted by $\lambda \vdash N$ or $|\lambda| = N$, if the finite sum $\lambda_1 + \lambda_2 + \cdots = N$. We say that λ has k *parts* if $\lambda_k > 0 = \lambda_{k+1}$. If λ has k parts, we sometimes abuse notation and write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

Let $\mathcal{P}_N$ be the set of partitions of N and $\mathcal{P}^k$ be the set of partitions with k or fewer parts. Define $\mathcal{P}^k_N$ to be the intersection of $\mathcal{P}_N$ and $\mathcal{P}^k$.

Definition 2.2. Suppose $\lambda \vdash N$ and $\mu \vdash N$. We say λ *dominates* μ if $\lambda_1+\ldots+\lambda_i \geq \mu_1+\ldots+\mu_i$ for all i and we write $\lambda \trianglerighteq \mu$ (and $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$).

It is well-known that dominance defines a partial order on $\mathcal{P}_N$.

Definition 2.3. Let λ be a partition. We define the *monomial symmetric functions* by

$$m_\lambda = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots$$

where the sum runs over all rearrangements $\alpha = (\alpha_1, \alpha_2, \ldots)$ of λ.

Definition 2.4. Let $n \geq 0$. We define the *complete homogeneous symmetric functions* by

$$h_n = \sum_{\lambda \vdash n} m_\lambda.$$

A partition λ has an associated *Young diagram*, an array of left-justified cells with λ_i cells in the i-th row counted from top. A *tableau* T is a Young diagram whose cells are filled with positive integers. The *content* of a tableau T is a vector $\rho = (\rho_1, \rho_2, \ldots)$ such that there are ρ_i occurences of the integer i in the filling. If T is a filling of the Young diagram of a partition λ, we say that T has *shape* λ, denoted by $\mathrm{sh}(T) = \lambda$.

If the filling of T is such that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom, we say that T is a *semistandard Young tableau*, abbreviated as SSYT.

If T is an SSYT such that $\mathrm{sh}(T) = \lambda \vdash n$ and T contains exactly one i for each $1 \leq i \leq n$, thne we say that T is a *standard Young tableau*, abbreviated as SYT. The hook-length formula counts the number of SYTs of a given shape.

Theorem 2.5. [5, Corollary 7.21.6] Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. For each j, let

$$\lambda'_j = \#\{k | \lambda_k \geq j\}.$$

Then, the number of SYTs of shape λ is

$$(n!) \prod_{\substack{(i,j) \\ j \leq \lambda_i}} (\lambda_i + \lambda'_j - i - j + 1)^{-1}.$$

If T is a tableau, the *reading word* $w(T)$ is the word obtained by reading the entries of T from right to left across the first row, then right to left across the second row, and so on. If α is a subset of the letters appearing in $w(T)$, we denote by $w_\alpha(T)$ the word obtained by deleting all letters in $w(T)$ not in α. We say a word is *Yamanouchi* if at any point in the word (from left to right), the number of occurrence of i's is no smaller than the number of occurrence of $(i+1)$'s.

If T is a tableau and $\rho = (\rho_1, \rho_2, \ldots)$ is its content, we define

$$x^T = x_1^{\rho_1} x_2^{\rho_2} \ldots.$$

2.2. **Schur Functions.**

Definition 2.6. If λ is a partition of N, we define the *Schur function* by

$$s_\lambda = \sum_{\substack{\text{SSYT } T \\ \text{sh}(T)=\lambda}} x^T.$$

It is known that s_λ is indeed a symmetric function, i.e. $s_\lambda \in \Lambda$. Moreover, it is known that $\{s_\lambda\}_{\lambda \vdash N}$ forms a basis of Λ_N. The Jacobi-Trudi identity expresses Schur function as a polynomial in h_n's.

Theorem 2.7. [5, Theorem 7.16.1] Let λ be a partition with k parts. Let M be the $k \times k$ matrix with $M_{ij} = h_{\lambda_i + j - i}$ (assuming $h_r = 0$ for $r < 0$). Then, $s_\lambda = \det(M)$.

Let A be a multiset of partitions from $\mathcal{P}^k$. Define

$$\mathcal{SP}_N^k = \left\{ A \subseteq \mathcal{P}^k \,\middle|\, \sum_{\lambda \in A} |\lambda| = N \right\}.$$

For $A \in \mathcal{SP}_N^k$, let $\phi(A)$ be the partition formed by concatenating the partitions in A. For example, if $A = \{(3,2),(3,1),(4)\}$, then $\phi(A) = (4,3,3,2,1)$. Define

$$\mathcal{SP}_\lambda^k = \left\{ A \in \mathcal{SP}_N^k \mid \phi(A) = \lambda \right\}.$$

We associate to A a product s_A of Schur functions:

$$s_A = \prod_{\lambda \in A} s_\lambda,$$

where $s_\lambda \in \Lambda_N$ is the Schur function associated with the partition λ.

Definition 2.8. Let A be a multiset of partitions and let λ be a partition. We define the *(generalized) Littlewood-Richardson coefficients* c_A^λ by

$$s_A = \sum_\lambda c_A^\lambda s_\lambda.$$

The generalized Littlewood-Richardson rule gives a combinatorial interpretation of these coefficients. It will be frequently referred to in our proofs.

First we fix the following notations: let $A = \{\rho^1, \rho^2, \ldots, \rho^k\}$ be a multiset of partitions where $\rho^i = (\rho_1^i, \rho_2^i, \ldots, \rho_{n_i}^i)$ and that $N = n_1 + n_2 + \ldots + n_k$. Let ρ be the composition obtained by concatenating the partitions ρ^i. Suppose $\phi(A) = \nu = (\nu_1, \nu_2, \ldots, \nu_N)$ and define the map f which sends ν_i to i for $1 \le i \le N$. Treat ρ and $\nu = (\nu_1, \nu_2, \ldots, \nu_N)$ as sequences of integers $1, 2, \ldots, N$ and suppose π is any permutation such that $\pi\rho = \nu$. Finally, let $B_i = \left\{ f \circ \pi(\rho_1^i), f \circ \pi(\rho_2^i), \ldots, f \circ \pi(\rho_{n_i}^i) \right\}$.

Theorem 2.9. [6, Section 1] With the above notations, c_A^λ is equal to the number of SSYT of shape λ and content ν such that $w_{B_i}(T)$ is a Yamanouchi word for each i.

An immediate consequence is

Corollary 2.10. Always, $c_A^\lambda \ge 0$. Moreover, if $c_A^\lambda > 0$, then $\lambda \trianglerighteq \phi(A)$. Also, $c_A^{\phi(A)} = 1$.

2.3. **The Schur Cone.** The (N,k)-*Schur cone* is

$$\mathcal{C}_N^k = \left\{ \sum_{A \in \mathcal{SP}_N^k} c_A s_A \,\middle|\, c_A \geq 0 \right\}.$$

We say A is *extreme* in $\mathcal{SP}_N^k$ or, interchangeably, s_A is *extreme* in $\mathcal{C}_N^k$, if s_A cannot be written as a positive linear combination of s_B with $B \in \mathcal{SP}_N^k$ and $B \neq A$.

We will call to the set of extreme rays the *extreme set*.

It is obvious from the definition that, when $k = 1$, every $A \in \mathcal{SP}_N^1$ is extreme. Indeed, if $\phi(A) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, then

$$s_A = \prod_{i=1}^n s_{\lambda_i} = \prod_{i=1}^n h_{\lambda_i} = h_\lambda.$$

Since the h_λ are linearly independent and $\mathcal{C}_N^1$ is just the positive span of the h_λ, s_A is extreme in $\mathcal{C}_N^1$.

Our main goal in this paper is to find the extreme set of $\mathcal{SP}_N^2$.

We consider the case where $k = 2$.

Definition 2.11. $A \in \mathcal{SP}_N^2$ is *nested* if no pair of partitions $\{\lambda, \mu\}$ in A satisfies any one of the following conditions:

(1) $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, with
$$\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2\,;$$

(2) $\lambda = (\lambda_1 > \lambda_2 > 0)$, $\mu = (\mu_1 > 0)$, with
$$\lambda_1 \geq \mu_1 \geq \lambda_2\,;$$

(3) $\lambda = (\lambda_1 > 0)$, $\mu = (\mu_1 > 0)$.

A pair satisfying one of the above conditions is called a *bad pair*.

In [6], White proved that A being extreme in $\mathcal{SP}_N^2$ implies A being nested:

Theorem 2.12. [6, Theorem 2] If A is extreme in $\mathcal{SP}_N^2$, then A is nested.

Define $\mathcal{SSP}_N$ to be the set of $A \in \mathcal{SP}_N^2$ nested and $\mathcal{SSP}_\lambda = \mathcal{SP}_\lambda^2 \cap \mathcal{SSP}_N$. Thus by Theorem 2.12 $\mathcal{SSP}_N$ is a superset of the extreme set of $\mathcal{C}_N^2$. Also, by Theorem 2.12 (3) if λ contains odd number of parts, then exactly one partitions in A has one part and if λ contains even number of parts, then every partition in A has two parts.

White conjectured in his paper that the opposite direction of Theorem 2.12 is true:

Conjecture 2.13. [6, Conjecture 1] $A \in \mathcal{SP}_N^2$ is extreme if only if A is nested.

He also showed, using Farkas' Lemma (see [2]), that if A is nested and $\phi(A)$ has distinct parts, then A is extreme in $\mathcal{SP}^2_N$ ([6], Lemma 7, Lemma 9, Theorem 15). In Section 2, we will give an alternate proof to this fact (Theorem 3.1).

In hopes of proving Conjecture 2.13, we consider two extreme cases of A:

Definition 2.14. If $\phi(A) = \lambda = \big(\lambda_1, \ldots, \lambda_{2\ell}\big) \vdash N$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2\ell} > 0$, we say A is *completely separated* if

$$A = \big\{(\lambda_1, \lambda_2), (\lambda_3, \lambda_4), \ldots, (\lambda_{2\ell-1}, \lambda_{2\ell})\big\}.$$

Definition 2.15. If $\phi(A) = \lambda = \big(\lambda_1, \ldots, \lambda_{2\ell}\big) \vdash N$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2\ell} > 0$, we say A is *completely nested* if

$$A = \big\{(\lambda_1, \lambda_{2\ell}), (\lambda_2, \lambda_{2\ell-1}), \ldots, (\lambda_\ell, \lambda_{\ell+1})\big\}.$$

Notice that if A is completely nested, then A is nested. However, if A is completely separated, it may or may not be nested. For example, $A = \{(6,5),(5,4)\}$ is not nested. In Section 4, we will show that if A is nested and completely separated, then it is indeed extreme (Corollary 4.2).

Definition 2.16. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\rho = (\rho_1, \rho_2)$ are partitions such that there exist $i < j$ with

$$\lambda_{i-1} > \lambda_i = \rho_1 \quad \text{and} \quad \lambda_j = \rho_2 > \lambda_{j+1}.$$

We define $\lambda[\rho]$ to be the partition obtained from λ by replacing λ_i with $\lambda_i + 1$ and λ_j with $\lambda_j - 1$.

Definition 2.17. For $A, B \in \mathcal{SP}^2_\lambda$, we say A and B *agree within* $\rho = (\rho_1, \rho_2)$ if whenever $\rho_1 > \mu_1 > \mu_2 > \rho_2$, then $\mu \in A$ if and only if $\mu \in B$.

The following is an important lemma in [6].

Lemma 2.18. [6, Lemma 15] Suppose A, $B \in \mathcal{SSP}_\lambda$and λ has distinct parts. Suppose $\rho = (\lambda_i, \lambda_j)$, with $\rho \in A$, $\rho \notin B$, and A and B agree within ρ. The Littlewood-Richardson coefficients satisfy the following identity:

$$c_A^{\lambda[\rho]} + 1 = c_B^{\lambda[\rho]}.$$

Furthermore, if $j = i + 1$, then $c_A^{\lambda[\rho]} = 0$ and $c_B^{\lambda[\rho]} = 1$.

3. Alternate proof when $\phi(A)$ has distinct parts

In this section we reprove the following theorem by White:

Theorem 3.1. [6, Theorem 13] If λ has distinct parts with $|\lambda| = N$, and if $A \in \mathcal{SSP}_\lambda$, then s_A is extreme in $\mathcal{C}^2_N$.

Proof. Assume by way of contradiction that A is not extreme in $\mathcal{C}^2_N$. Then there exist $c_B \geq 0$ such that

$$s_A = \sum_{\substack{B \in \mathcal{SSP}_N \\ B \neq A \\ \phi(B) \trianglerighteq \lambda}} c_B s_B. \tag{3.1}$$

Next consider the coefficient of s_λ in this equality. If $c_B^\lambda \neq 0$, then we must have $\phi(B) = \lambda$, and in this case $c_B^\lambda = 1$. Hence comparing the coefficients of s_λ gives us

$$\sum_{B \in \mathcal{SSP}_\lambda} c_B = 1. \tag{3.2}$$

Let $\rho^1, \rho^2, \ldots, \rho^\ell$ be the pairs in A in "inside-out" order, i.e., if $\phi(A \backslash \{\rho^1, \ldots, \rho^r\}) = (\mu_1, \ldots, \mu_k)$, then $\rho^{r+1} = (\mu_i, \mu_{i+1})$ for some i. Note that such a sequence exists because for each i, $A \backslash \{\rho^1, \ldots, \rho^r\}$ is nested. We will then prove the following claim.

Claim 3.2. If $B \in \mathcal{SSP}_\lambda$ satisfies $c_B > 0$, then $\rho^1, \ldots, \rho^r \in B$.

We remark that Theorem 3.1 follows from Claim 3.2. Indeed, the case $r = \ell$ for the claim implies that every $B \in \mathcal{SSP}_\lambda$ with $c_B > 0$ satisfies $B = A$, so by equation (3.2) we have $c_A = 1$. It then follows that equation (3.1) contains the term c_A on the RHS, which gives a contradiction. $\square$

It then remains to prove Claim 3.2

Proof of Claim 3.2. Proceed by induction on r. The case $r = 0$ is a tautology. Suppose $r > 0$. By induction hypothesis, $\rho^1, \ldots, \rho^{r-1} \in B$, so it suffices to show that $\rho^r \in B$. We now examine the coefficients of $s_{\lambda[\rho^r]}$ in equation (3.1). Every $B \in \mathcal{SSP}_\lambda$ for which $c_B > 0$ and A agree within ρ^{r+1} by the "inside-out" order, so by Lemma 2.18, if $\rho^r \in B$, then $c_B^{\lambda[\rho^r]} = c_A^{\lambda[\rho^r]}$; if $\rho^r \notin B$, then $c_B^{\lambda[\rho^r]} = 1 + c_A^{\lambda[\rho^r]}$. Hence by equation (3.1),

$$c_A^{\lambda[\rho^r]} = \sum_{\substack{B \in \mathcal{SSP}_\lambda \\ \rho^r \in B}} c_B c_A^{\lambda[\rho^r]} + \sum_{\substack{B \in \mathcal{SSP}_\lambda \\ \rho^r \notin B}} c_B (1 + c_A^{\lambda[\rho^r]}).$$

It then follows from equation (3.2) that

$$\sum_{\substack{B \in \mathcal{SSP}_\lambda \\ \rho^r \notin B}} c_B = 0.$$

Since all $c_B \geq 0$, we see that $c_B = 0$ whenever $\rho^r \notin B$, thus proving the desired claim. $\square$

4. A is completely separated

In this section we will prove the following theorem.

Theorem 4.1. If s_A is extreme in $\mathcal{C}_N^2$, then $s_{A \cup \{\rho\}}$ where $\rho = (p, p)$ is extreme in C_{N+2p}^2.

From Theorem 4.1 We can deduce the following special case of Conjecture 2.13.

Corollary 4.2. If $A \in \mathcal{SSP}_N$ is completely separated, then s_A is extreme.

Proof. If A is completely separated and nested, then any integer p can appear in at most one partition in A not of the form (p, p). Thus if we remove all of the partitions with repeated parts by Theorem 4.1, what is left has distinct parts and is thus extreme from Theorem 3.1. $\square$

The rest of the section is devoted to the proof of Theorem 4.1. We will fix the integer p and always write $\rho = (p, p)$. In this section all congruences are taken modulo 3.

4.1. **Proof of Theorem 4.1.** We start our proof with a generalization of the notation $\lambda[\rho]$. Roughly speaking, we define $\lambda[\rho^k]$ to be the k-fold application of $[\rho]$ on λ.

Definition 4.3. Suppose $\rho = (p, p)$ and k is a non-negative integer. We define the notation $\lambda[\rho^k]$ recursively as follows: define $\lambda[\rho^0] = \lambda$. For $k \geq 0$, we say that $\lambda[\rho^{k+1}]$ is *undefined* if $\lambda[\rho^k]$ is undefined or if $\lambda[\rho^k]$ has less than two p's. Otherwise, we define $\lambda[\rho^{k+1}] = (\lambda[\rho^k])[\rho]$.

We then introduce a relation.

Definition 4.4. Let P be a subposet of the dominance poset of partitions of N. Denote by $P_{\min}$ the set of minimal elements of P. Define a relation $\leq_p$ on $P_{\min}$ by $x \leq_p y$ if there exists k such that $y[\rho^k]$ is defined and $x \trianglelefteq y[\rho^k]$.

We remark that

Lemma 4.5. $(P_{\min}, \leq_p)$ is a poset.

The proof of Lemma 4.5 is deferred to Section 4.2.

We will now prove our theorem.

Proof of Theorem 4.1. Assume by way of contradiction that $s_A s_\rho$ is not extreme. Then there exists a positive combination

$$s_A s_\rho = \sum_{B \neq A} a_B s_B. \tag{4.1}$$

If every B for which $a_B > 0$ satisfies $\rho \in B$, then we can factor out s_ρ from both sides of equation (4.1) and write s_A as a positive combinations of some $s_{B'}$, contradicting the assumption that s_A is extreme. Thus we may assume that $a_B > 0$ for some B with $\rho \notin B$. We can then rearrange equation (4.1) to get

$$s_A s_\rho - \sum_{\rho \in B} a_B s_B = \sum_{\rho \notin B} a_B s_B. \tag{4.2}$$

Factoring out s_ρ from the LHS of equation (4.2) we get

$$\sum_{\nu \vdash N} a_\nu s_\rho s_\nu = \sum_{\rho \notin B} a_B s_B. \tag{4.3}$$

Let $P = \{\nu : a_\nu \neq 0\}$. Pick a $\leq_p$-minimal element μ in $P_{\min}$, and let $\lambda = \phi(\{\mu, \rho\})$. Let n be the number of p's in λ, and let $m = \lfloor n/2 \rfloor$. Note that μ has $(n-2)$ p's. Define

$$c_i = a_{\mu[\rho^i]} \quad \text{and} \quad d_i = \sum_{\substack{\rho \notin B \\ \phi(B) = \lambda[\rho^i]}} a_B.$$

Note that $d_0 > 0$ because the coefficient of s_λ on LHS of equation (4.3) is positive.

Let $V = \Lambda_{N+2p}$, and define the subspace

$$W = \operatorname{span}(s_\lambda, s_{\lambda[\rho]}, s_{\lambda[\rho^2]}, \ldots, s_{\lambda[\rho^m]}).$$

Consider the projection $\pi : V \to W$ given by $\pi(s_{\lambda[\rho^i]}) = s_{\lambda[\rho^i]}$ and $\pi(s_\nu) = 0$ for any ν not of the form $\lambda[\rho^i]$. We now remark two claims about Littlewood-Richardson coefficients.

Claim 4.6. (1) If $0 \le i \le m-2$, then $\pi(s_\rho s_{\mu[\rho^i]}) = s_{\lambda[\rho^i]} + s_{\lambda[\rho^{i+1}]} + s_{\lambda[\rho^{i+2}]}$.

(2) If n is odd, then $\pi(s_\rho s_{\mu[\rho^{m-1}]}) = s_{\lambda[\rho^{m-1}]} + s_{\lambda[\rho^m]}$. If n is even, then $\pi(s_\rho s_{\mu[\rho^{m-1}]}) = s_{\lambda[\rho^{m-1}]}$.

(3) If $\nu \vdash N$ satisfies $a_\nu \neq 0$ but ν is not of the form $\mu[\rho^i]$, then $\pi(s_\rho s_\nu) = 0$.

Claim 4.7. (1) If $\phi(B) = \lambda[\rho^i]$ and $\rho \notin B$, then

$$\pi(s_B) = \sum_{j=i}^{m} \left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1} \right) s_{\lambda[\rho^j]}.$$

(2) If $\rho \notin B$ satisfies $a_B > 0$ but $\phi(B)$ is not of the form $\lambda[\rho^i]$, then $\pi(s_B) = 0$.

The proof of the claims is deferred to Section 4.3.

Recall that in this section all congruences are taken modulo 3. We first consider the case where n is odd. Define the $\mathbb{R}$-linear map $f : W \to \mathbb{R}$ by

$$f(s_{\lambda[\rho^j]}) = \begin{cases} -1 & \text{if } j \equiv m-1. \\ 1 & \text{if } j \equiv m. \\ 0 & \text{if } j \equiv m+1. \end{cases}$$

It then follows from Claim 4.6 that $(f \circ \pi)(s_\rho s_\nu) = 0$ whenever $a_\nu \neq 0$. Therefore, applying $f \circ \pi$ on both sides of equation (4.3) gives us, by Claim 4.7,

(4.4)
$$\sum_{i=0}^{m} d_i \left(\sum_{\substack{j \le m \\ j \equiv m}} \left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1} \right) - \sum_{\substack{j \le m \\ j \equiv m-1}} \left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1} \right) \right) = 0.$$

Using the identity $\binom{\alpha}{\beta} = \binom{\alpha}{\alpha-\beta}$, since $n = 2m+1$, the coefficient of d_i in LHS of equation (4.4) is

$$\begin{aligned} &\sum_{\substack{k \le m-i \\ k \equiv m-i}} \binom{n-2i}{k} + \sum_{\substack{k \le m-i \\ k \equiv m-i+1}} \binom{n-2i}{k} - 2 \sum_{\substack{k \le m-i \\ k \equiv m-i-1}} \binom{n-2i}{k} \\ =&\frac{1}{2} \left(\sum_{k \equiv m-i} \binom{n-2i}{k} + \sum_{k \equiv m-i+1} \binom{n-2i}{k} - 2 \sum_{k \equiv m-i-1} \binom{n-2i}{k} \right) \\ =&\frac{1}{2} \left(\frac{2^{n-2i}+1}{3} + \frac{2^{n-2i}+1}{3} - 2 \cdot \frac{2^{n-2i}-2}{3} \right) = 1. \end{aligned}$$

Equation (4.4) is thus reduced to

$$\sum_{i=0}^{m} d_i = 0.$$

Since $d_i \geq 0$ for all i, $d_0 = 0$, thus leading to a contradiction.

We then consider the case where n is even. Define the $\mathbb{R}$-linear map $f : W \to \mathbb{R}$ by

$$f(s_{\lambda[\rho^j]}) = \begin{cases} 0 & \text{if } j \equiv m-1. \\ 1 & \text{if } j \equiv m. \\ -1 & \text{if } j \equiv m+1. \end{cases}$$

It then follows from Claim 4.6 that $(f \circ \pi)(s_\rho s_\nu) = 0$ whenever $a_\nu \neq 0$. Therefore, applying $f \circ \pi$ on both sides of equation (4.3) gives us, by Claim 4.7,

(4.5)

$$\sum_{i=0}^{m} d_i \left(\sum_{\substack{j \leq m \\ j \equiv m}} \left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1} \right) - \sum_{\substack{j \leq m \\ j \equiv m+1}} \left(\binom{n-2i}{j-i} - \binom{n-2i}{j-i-1} \right) \right) = 0.$$

Using the identity $\binom{\alpha}{\beta} = \binom{\alpha}{\alpha-\beta}$, since $n = 2m$, the coefficient of d_i in LHS of equation (4.5) is

$$\begin{aligned} &\binom{n-2i}{m-i} + 2 \sum_{\substack{k < m-i \\ k \equiv m-i}} \binom{n-2i}{k} - \sum_{\substack{k \leq m-i \\ k \equiv m-i+1}} \binom{n-2i}{k} - \sum_{\substack{k \leq m-i \\ k \equiv m-i-1}} \binom{n-2i}{k} \\ =&\frac{1}{2} \left(2 \sum_{k \equiv m-i} \binom{n-2i}{k} - \sum_{k \equiv m-i+1} \binom{n-2i}{k} - \sum_{k \equiv m-i-1} \binom{n-2i}{k} \right) \\ =&\frac{1}{2} \left(2 \cdot \frac{2^{n-2i}+2}{3} - \frac{2^{n-2i}-1}{3} - \frac{2^{n-2i}-1}{3} \right) = 1. \end{aligned}$$

Equation (4.5) is thus reduced to

$$\sum_{i=0}^{m} d_i = 0.$$

Since $d_i \geq 0$ for all i, $d_0 = 0$, thus leading to a contradiction. $\square$

4.2. **Proof of Lemma 4.5.** In this proof, if $x = (x_1, x_2, \ldots)$ is a partition, we write

$$S_i(x) = x_1 + x_2 + \cdots + x_i.$$

We begin with a claim.

Claim 4.8. Suppose $x \trianglelefteq y$ and x has at least two p's.

(1) If y has at least two p's, then $x[\rho] \trianglelefteq y[\rho]$.

(2) If y has fewer than two p's, then $x[\rho] \trianglelefteq y$.

Proof. Let s and t be integers such that

$$x_s > p \geq x_{s+1} \quad \text{and} \quad x_t \geq p > x_{t+1}.$$

Similarly, let s' and t' be integers such that

$$y_{s'} > p \geq y_{s'+1} \quad \text{and} \quad y_{t'} \geq p > y_{t'+1}.$$

Observe that $S_i(x) \leq S_i(x[\rho]) \leq S_i(x) + 1$.

(a) If $i \le s$ or $i \ge t$, then

$$S_i(x[\rho]) = S_i(x) \le S_i(y).$$

(b) If $s+1 \le i \le s'$, then for any $s+1 \le j \le i$, $y_j \ge p+1 = x_j+1$, so

$$S_i(y) = S_s(y) + \sum_{j=s+1}^{i} y_j \ge S_s(x) + 1 + \sum_{j=s+1}^{i} x_j = S_i(x[\rho]).$$

(c) If $t' \le i \le t-1$, then for any $i+1 \le j \le t$, $y_j \le p-1 = x_j - 1$, so

$$S_i(y) = S_t(y) - \sum_{j=i+1}^{t} y_j \ge S_t(x) + 1 - \sum_{j=i+1}^{t} x_j = S_i(x[\rho]).$$

If y has fewer then two p's, then $t' \le s'+1$, so $S_i(y) \ge S_i(x[\rho])$ for all i, i.e., $x[\rho] \trianglelefteq y$. This proves (2).

If y has at least two p's, then observe that $S_i(y[\rho]) \ge S_i(y)$ for all i. Moreover, if $s'+1 \le i \le t'-1$, then

$$S_i(y[\rho]) = S_i(y) + 1 \ge S_i(x) + 1 \ge S_i(x[\rho]).$$

It follows that $S_i(y[\rho]) \ge S_i(x[\rho])$ for all i, i.e., $x[\rho] \trianglelefteq y[\rho]$. This proves (1). $\square$

We now prove our lemma.

Proof of Lemma 4.5. For every x, $x \trianglelefteq x$ and hence $x \le_p x$, so $\le_p$ is reflexive.

If $x \le_p y \le_p x$, then there exist k and ℓ such that $x \trianglelefteq y[\rho^k]$ and $y \trianglelefteq x[\rho^\ell]$. By Claim 4.8, $x \trianglelefteq y[\rho^k] \trianglelefteq x[\rho^m]$ for some m, so $y[\rho^k] = x[\rho^i]$ for some i. Since both x and y are in $P_{\min}$, $x = y$. Therefore, $\le_p$ is anti-symmetric.

If $x \le_p y \le_p z$, then there exist k and ℓ such that $x \trianglelefteq y[\rho^k]$ and $y \trianglelefteq z[\rho^\ell]$. By Claim 4.8, $x \trianglelefteq y[\rho^k] \trianglelefteq z[\rho^m]$ for some m, so $x \le_p z$. Therefore, $\le_p$ is transitive. $\square$

4.3. **Proof of Claim 4.6 and 4.7.**

Proof of Claim 4.6(1). Suppose $\eta = \mu[\rho^i]$ with $0 \le i \le m-2$ and $\chi = \lambda[\rho^j]$. Let $B = \{\rho, \eta\}$. If $c_B^\chi > 0$, then we have $\lambda[\rho^j] \trianglerighteq \lambda[\rho^i]$, so $j \ge i$. Assume that $\phi(B) = \kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$. Let s and t be integers such that $\kappa_s > \kappa_{s+1} = \kappa_{s+2} = \cdots = \kappa_t = p > \kappa_{t+1}$. By Theorem 2.9, c_B^χ is the number of SSYT of shape χ and content κ such that the restrictions of its reading word on $\{t-1, t\}$ and $[n] \backslash \{t-1, t\}$ are Yamanouchi. Assume that T is such an SSYT.

We first show by induction on i that if $i \le t-2$, then all integers i appear on the i-th row. Indeed, since T is an SSYT, i must appear on or before the i-th row. If all the integers $(i-1)$ appear on the $(i-1)$-th row, then since the restriction of the reading word on $[n] \backslash \{t-1, t\}$ is Yamanouchi, i must appear on or after the i-th row, i.e., all integers i appear on the i-th row. The result thus follows from induction.

We then show by backward induction on i that if $i \ge t+1$, then all integers i appear on the i-th row. Indeed, if all integers $j > i$ appear after the i-th row, then the i-th row can only contain the integer i. Therefore all integers i appear in the i-th row because $\kappa_i = \chi_i$. The result follows from backward induction.

The above constraints require that $\chi_{t-2} \geq \kappa_{t-2} = p$, so it follows that $j \leq i+2$ whenever $c_B^\chi > 0$.

If $j = i$, then since $\chi = \kappa = \phi(B)$ we have $c_B^\chi = 1$. If $j \geq i+1$, then the extra box at the $(s+1)$-th row must be filled with $(t-1)$ since we want the restriction to $\{t-1, t\}$ to be Yamanouchi. We must then fill all other $(t-1)$ in the $(t-1)$-th row. Thus in both cases $j = i+1$ and $j = i+2$ we have $c_B^\chi = 1$. $\square$

Proof of Claim 4.6(2). Suppose $\eta = \mu[\rho^{m-1}]$ and $\chi = \lambda[\rho^j]$. Let $B = \{\rho, \eta\}$. If $c_B^\chi > 0$, we again have $j \geq m-1$. Assume that $\phi(B) = \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$.

We first consider the case n is odd. Let s be an integer such that $\kappa_s > \kappa_{s+1} = \kappa_{s+2} = \kappa_{s+3} = p > \kappa_{s+4}$. By Theorem 2.9, c_B^χ is the number of SSYT T of shape χ and content κ such that the restrictions of its reading word on $\{s+2, s+3\}$ and $[n]\backslash\{s+2, s+3\}$ are Yamanouchi.

With an analogous argument as in the proof of (1), if T is such an SSYT, then for $i \leq s+1$ and $i \geq s+4$, all integers i must appear in the i-th row. Also, if $j = m-1$, then we have $c_B^\chi = 1$. If $j = m$, then the extra box in the $(s+1)$-th row must be filled with $(s+2)$ since we want the restriction to $\{s+2, s+3\}$ to be Yamanouchi. We must then fill all other $(t-1)$ in the $(t-1)$-th row. It follows that $c_B^\chi = 1$. This proves the odd case.

Now consider the case n is even. Let s be an integer such that $\kappa_s > \kappa_{s+1} = \kappa_{s+2} = p > \kappa_{s+3}$. By Theorem 2.9, c_B^χ is the number of SSYT T of shape χ and content κ such that the restrictions of its reading word on $\{s+1, s+2\}$ and $[n]\backslash\{s+1, s+2\}$ are Yamanouchi.

With an analogous argument as in the proof of (1), if T is such an SSYT, then for $i \leq s+1$ and $i \geq s+3$, all integers i must appear in the i-th row. Also, if $j = m-1$, then we have $c_B^\chi = 1$. If $j = m$, then the $(s+1)$-th row contains an integer $(s+2)$, so the restriction to $\{s+1, s+2\}$ is not Yamanouchi. It follows that $c_B^\chi = 0$. This proves the even case. $\square$

Proof of Claim 4.6(3). Let $\eta = \phi(\{\rho, \nu\})$. We claim that if $a_\nu \neq 0$ and ν is not of the form $\mu[\rho^i]$, then $\eta \ntrianglelefteq \lambda[\rho^j]$, and it follows that $\pi(s_\rho s_\nu) = 0$. Indeed, suppose $\eta \trianglelefteq \lambda[\rho^j]$, we then can pick $\chi \in P_{\min}$ such that $\chi \trianglelefteq \nu$, so $\chi \trianglelefteq \mu[\rho^j]$ and hence $\chi \leq_p \mu$. Since μ is $\leq_p$-minimal, we must then have $\chi = \mu$, and it follows that $\mu \trianglelefteq \nu \trianglelefteq \mu[\rho^j]$. Therefore, ν is of the form $\mu[\rho^i]$, giving a contradiction. $\square$

Proof of Claim 4.7(1). Let $\eta = \phi(B) = \lambda[\rho^i]$ and $\chi = \lambda[\rho^j]$. If $c_B^\chi > 0$, then $\lambda[\rho^j] \trianglerighteq \lambda[\rho^i]$, so $j \geq i$. Let s and t be integers such that $\eta_s > \eta_{s+1} = \cdots = \eta_t = p > \eta_{t+1}$. Note that $t - s = n - 2i$. To compute c_B^χ, we wish to count the number of SSYT with shape χ and content η satisfying the relevant Yamanouchi conditions as outlined in Theorem 2.9. We assume that T is an SSYT of shape χ and content η *without* assuming that it satisfies the Yamanouchi conditions.

We first show by induction on i that if $i \leq s$, then all integers i appear on the i-th row. Indeed, since T is an SSYT, i must appear on or before the i-th row. If all the integers $j < i$ appear on the j-th row, then since $\chi_j = \eta_j$, the entire j-th row is filled with j. Thus all integers i must appear on or after the i-th row, i.e., all integers i must appear on the i-th row. The result thus follows from induction.

We then show by backward induction on i that if $i \geq t+1$, then all integers i appear on the i-th row. Indeed, if all integers $j > i$ appear after the i-th row, then the i-th row can only contain the integer i. Therefore all integers i appear in the i-th row because $\eta_i = \chi_i$. The result thus follows from backward induction.

We remark that every SSYT T of shape χ and content η necessarily satisfies the desired Yamanouchi conditions, so it suffices to count without thinking about the Yamanouchi conditions. Indeed, the assumption $\rho \notin B$ implies that we never have to consider the restriction on $\{\alpha, \beta\}$ for $s < \alpha, \beta \leq t$. It then follows that T satisfies the desired Yamanouchi conditions because if $i \leq s$ or $i > t$, then every integer i appears in the i-th row, and if $s < i \leq t$, then every integer i appears among the $(s+1)$-th through t-th rows.

Now we restrict to the $(s+1)$-th through t-th rows. It follows from the previous discussion that these rows are filled with $\{s+1, s+2, \ldots, t\}$. If we go through the j-th column ($j \leq p-1$), we get a strictly increasing sequence of length $(t-s)$ on $\{s+1, \ldots, t\}$, so the box at the i-th row and j-th column ($s < i \leq t$, $j \leq p-1$) is always filled with i.

The unfilled boxes form a skew shape κ, which is a translation of the Young diagram of $\sigma = (2^{j-i}, 1^{n-2j})$. Let T' be the tableau of shape σ obtained by restricting T to κ, translating to σ and then subtracting s from each entry. Note that every such T' obtained is an SYT of shape σ. Moreover, every SYT T' of shape σ corresponds to exactly one SSYT T of shape χ.

Therefore, the Littlewood-Richardson coefficient c_B^χ is precisely the number of SYTs of shape σ, which is equal to, by the hook-length formula (Theorem 2.5),

$$\frac{(n-2i)!}{(j-i)!(n-i-j+1)!} \cdot (n-2j+1) = \binom{n-2i}{j-i} - \binom{n-2i}{j-i-1}. \quad \square$$

Proof of Claim 4.7(2). Let $\eta = \phi(B)$. We claim that if $a_B > 0$ and η is not of the form $\lambda[\rho^i]$, then $\eta \not\trianglelefteq \lambda[\rho^j]$, and it follows that $c_B^{\lambda[\rho^j]} = 0$. Indeed, suppose $\eta \trianglelefteq \lambda[\rho^j]$. Since $a_B > 0$, the coefficient of s_η on RHS of equation (4.3) is positive. It follows that there exists ν with $a_\nu \neq 0$ such that the coefficient of s_η in $s_\rho s_\nu$ is non-zero. This implies that $\eta \trianglerighteq \phi(\{\rho, \nu\})$. We then can pick $\chi \in P_{\min}$ such that $\chi \trianglelefteq \nu$, so $\phi(\{\rho, \chi\}) \trianglelefteq \lambda[\rho^j]$ and hence $\chi \leq_p \mu$. Since μ is $\leq_p$-minimal, we must then have $\chi = \mu$, and it follows that $\lambda \trianglelefteq \eta \trianglelefteq \lambda[\rho^j]$. Therefore, η is of the form $\lambda[\rho^i]$, giving a contradiction. $\square$

5. $A = \{(j,i),(j,i)\}$ FOR $j > i > 0$

As promised in the introduction, we will prove the following theorem in this section.

Theorem 5.1. If $A = \{(j,i),(j,i)\}$ for $j > i > 0$ and $N = 2j + 2i$, then $A \in \mathcal{SP}_N^2$ is extreme.

In this section we will denote $\lambda = \phi(A) = (j, j, i, i)$ and

$$\begin{aligned} B_0 &= \big\{(j, j), (i, i)\big\}, \\ B_1 &= \big\{(j+1, j-1), (i, i)\big\}, \\ B_2 &= \big\{(j+1, i), (j-1, i)\big\}, \\ B_3 &= \big\{(j, j), (i+1, i-1)\big\}, \\ B_4 &= \big\{(j, i+1), (j, i-1)\big\}. \end{aligned}$$

Moreover, we will write $\rho_1 = (j, j)$, $\rho_2 = (i, i)$, so

$$\begin{aligned} \lambda[\rho_1] &= (j+1, j-1, i, i), \\ \lambda[\rho_2] &= (j, j, i+1, i-1). \end{aligned}$$

Also, let $\lambda^+ = (j+1, j, i, i-1)$.

We start with a claim about some Littlewood-Richardson coefficients.

Claim 5.2. We have

$$c_A^{\lambda[\rho_1]} = c_{B_0}^{\lambda[\rho_1]} + 1 \quad \text{and} \quad c_A^{\lambda[\rho_2]} = c_{B_0}^{\lambda[\rho_2]} + 1.$$

Moreover,

$$c_A^{\lambda^+} = 2 \quad \text{and} \quad c_{B_0}^{\lambda^+} = c_{B_1}^{\lambda^+} = c_{B_2}^{\lambda^+} = c_{B_3}^{\lambda^+} = c_{B_4}^{\lambda^+} = 1.$$

The proof of Claim 5.2 is deferred to the end of the section because this is fairly technical.

We shall exhibit how one can deduce Theorem 5.1 from Claim 5.2.

Proof of Theorem 5.1. Assume for the sake of contradiction that s_A is not extreme. Then, one can write

$$s_A = \sum_{\substack{B \in \mathcal{SSP}_N \\ B \neq A}} c_B s_B \tag{5.1}$$

where $c_B \geq 0$ for all B.

The coefficient of s_λ on LHS of equation (5.1) is 1. Since λ is minimal in the Schur support of s_A, i.e., there exists no μ such that $\mu \lhd \lambda$ and the coefficient of s_μ in s_A is positive, every B foe which $c_B > 0$ and $c_B^\lambda > 0$ (i.e., the coefficient of s_λ in $c_B s_B$ is positive) must satisfy $\phi(B) = \lambda$. A moment of thought reveals that there is only one such B, namely, $B = B_0$. Hence by comparing coefficients of s_λ we have $c_{B_0} = 1$.

Subtracting s_{B_0} from both sides of equation (5.1), we have

$$s_A - s_{B_0} = \sum_{\substack{B \in \mathcal{SSP}_N \\ B \neq A, B_0}} c_B s_B. \tag{5.2}$$

Using Claim 5.2, we can deduce that the coefficients of $s_{\lambda[\rho_1]}$ and $s_{\lambda[\rho_2]}$ on LHS of equation (5.2) are both 1. Note moreover that the coefficient of s_λ is zero on LHS of (5.2). We can then deduce that $\lambda[\rho_1]$ and $\lambda[\rho_2]$ are minimal in the Schur support of the LHS of (5.2), i.e., there exists no μ such that $\mu \lhd \lambda[\rho_1]$ or $\mu \lhd \lambda[\rho_2]$, and the coefficient of s_μ on LHS of (5.2) is positive. Thus every B with $c_B > 0$ and

$c_B^{\lambda[\rho_1]} > 0$ (i.e., the coefficient of $s_{\lambda[\rho_1]}$ in $c_B s_B$ is positive) must satisfy $\phi(B) = \lambda[\rho_1]$. A moment of thought reveals that such B's are $B = B_1$ and $B = B_2$. By comparing coefficients of $s_{\lambda[\rho_1]}$ on both sides of equation (5.2) we have $c_{B_1} + c_{B_2} = 1$. A similar argument using $\lambda[\rho_2]$ shows that $c_{B_3} + c_{B_4} = 1$.

Subtracting $c_{B_1}s_{B_1} + c_{B_2}s_{B_2} + c_{B_3}s_{B_3} + c_{B_4}s_{B_4}$ from both sides of equation (5.2) gives us

$$s_A - s_{B_0} - (c_{B_1}s_{B_1} + c_{B_2}s_{B_2} + c_{B_3}s_{B_3} + c_{B_4}s_{B_4}) = \sum_{\substack{B \in \mathcal{SSP}_N \\ B \neq A, B_0, \ldots, B_4}} c_B s_B. \tag{5.3}$$

Now we compare the coefficients of s_{λ^+} on both sides of equation (5.3). Applying Claim 5.2, the coefficient on LHS of (5.3) is

$$2 - 1 - (c_{B_1} + c_{B_2}) - (c_{B_3} + c_{B_4}) = 2 - 1 - 1 - 1 = -1.$$

However, the RHS of (5.3) is a non-negative combination of products of Schur functions, so the coefficient of s_{λ^+} is non-negative, giving the desired contradiction. □

We now prove Claim 5.2.

Proof of Claim 5.2. The first two equations follow from Lemma 2.18. To prove the last six values of Littlewood-Richardson coefficients, we count the number of tableaux with shape $\lambda^+ = (j+1, j, i, i-1)$ and content $\phi(A)$ (resp. $\phi(B_0)$, $\phi(B_1)$, $\phi(B_2)$, $\phi(B_3)$, $\phi(B_4)$) satisfying the relevant Yamanouchi conditions, as outlined in Theorem 2.9.

In all six cases, we will fill the tableaux with at least j 1's, at least $(j-1)$ 2's, at least i 3's and at least $(i-1)$ 4's. Moreover, the total number of 1's and 2's is always $2j$.

Note that we must put all 1's in the first row (for this is true for any SSYT). Moreover, since 1's, 2's and 3's must be put in the first three rows, the last row must be filled with 4's.

We must put all 1's and 2's within the first two rows. Note that since there are exactly $2j$ 1's and 2's in the first two rows and there are only $2j+1$ boxes in these two rows, there is at most one entry in the second row that is neither 1 nor 2. None of the entries in the second row can be 1. This means that there are at least $(j-1)$ 2's in the second row.

As explained in the previous paragraph, there is at most one entry in the first two rows that is neither 1 nor 2, so there is at most one 3 among the two rows. This means that there are at least $(i-1)$ 3's in the third row.

Hence, in all six cases, the tableau looks like

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & c_1 \\
\hline
2 & 2 & \ldots & 2 & 2 & \ldots & 2 & c_2 \\
\cline{1-8}
3 & 3 & \ldots & 3 & c_3 \\
\cline{1-5}
4 & 4 & \ldots & 4 \\
\cline{1-4}
\end{array}$$

We now consider the individual cases and see in how many ways we can fill in the boxes c_1, c_2 and c_3 so that the tableau satisfies the Yamanouchi conditions in Theorem 2.9.

For A, we have to fill in one 2, one 3 and one 4. Since we want the restriction to $\{2,3\}$ to be Yamanouchi, we must have $c_1 \neq 3$. Similarly, since we want the restriction to $\{1,4\}$ to be Yamanouchi, we must have $c_1 \neq 4$. Thus $c_1 = 2$. It then follows that $(c_2, c_3) = (3,4)$ or $(4,3)$ gives two ways of filling.

For B_0, we have to fill in one 2, one 3 and one 4. Since we want the restriction to $\{1,2\}$ to be Yamanouchi, we must have $c_1 \neq 2$. Similarly, since we want the restriction to $\{3,4\}$ to be Yamanouchi, we must have $c_1 \neq 4$. Thus $c_1 = 3$. We must then have $c_2 = 2$ because all 2's have to be put in the first two rows. Thus $c_3 = 4$ and there is one way of filling.

For B_1, we have to fill in one 1, one 3 and one 4. We must have $c_1 = 1$ because all 1's have to be put in the first row. Since we want the restriction to $\{3,4\}$ to be Yamanouchi, we must have $c_2 \neq 4$. Thus $c_2 = 3$, $c_3 = 4$ and there is one way of filling.

For B_2, we have to fill in one 1, one 3 and one 4. We must have $c_1 = 1$ because all 1's have to be put in the first row. Since we want the restriction to $\{2,3\}$ to be Yamanouchi, we must have $c_2 \neq 3$. Thus $c_2 = 4$, $c_3 = 3$ and there is one way of filling.

For B_3, we have to fill in one 2 and two 3's. Since we want the restriction to $\{1,2\}$ to be Yamanouchi, we must have $c_1 \neq 2$. Thus $c_1 = 3$ We must then have $c_2 = 2$ because all 2's have to be put in the first two rows. Thus $c_3 = 3$ and there is one way of filling.

For B_4, we have to fill in one 2 and two 3's. Since we want the restriction to $\{1,3\}$ to be Yamanouchi, we must have $c_1 \neq 3$. Thus $c_1 = 2$. It follows that $c_2 = c_3 = 3$ and there is one way of filling. $\square$

6. Conjecture on the induction step

Most of the progress made in [6] is in the direction of showing that s_A is extreme given certain conditions on $\phi(A)$. Having failed to generalize Theorem 3.1 to the case where $\phi(A)$ has repeated parts, we instead propose to approach the problem by considering the *pairing structure* of A.

Conjecture 6.1. Let s_A be extreme in $\mathcal{C}_N^k$ and s_B be extreme in $\mathcal{C}_M^k$. Let $\phi(A) = (\lambda_1, \ldots, \lambda_n)$ and $\phi(B) = (\mu_1, \ldots, \mu_m)$ with at most one of n and m odd. Suppose that $\lambda_n > \mu_1$. Then, $s_A s_B$ is extreme in $\mathcal{C}_{N+M}^k$.

Conjecture 6.2. Let s_A be extreme in $\mathcal{C}_N^k$, with $\phi(A) = (\lambda_1, \ldots, \lambda_n)$. Let $\rho = (\rho_1, \rho_2)$ be a partition, with $\rho_1 > \rho_2$ and $\rho_1 \geq \lambda_1 \geq \lambda_n \geq \rho_2$. Then $s_A s_\rho$ is extreme in $C_{N+|\rho|}^k$.

Lemma 6.3. Conjecture 2.13 is equivalent to Conjecture 6.1 and Conjecture 6.2.

Proof. The first direction is easy. By the hypotheses of Conjecture 6.1, it is clear that $A \cup B \in \mathcal{SSP}_{N+M}^2$, thus the failure of Conjecture 6.1 implies the failure of

Conjecture 2.13. Similarly, by the hypotheses of Conjecture 6.2, it is clear that $A \cup \{\rho\} \in \mathcal{SSP}^2_{N+|\rho|}$, so the failure of Conjecture 6.2 also implies the failure of Conjecture 2.13.

Now suppose that Conjecture 6.1 and Conjecture 6.2 hold and let $D \in \mathcal{SSP}_N$. Write $s_D = s_{D_1} \dots s_{D_r}$ where the D_i are chosen such that $\phi(D_i)$ and $\phi(D_j)$ have no parts in common for $i \neq j$. We can further divide each D_i by removing its outermost pair, and inductively repeating this process. Repeated applications of Conjectures 6.1 and 6.2 show that s_D is extreme. □

7. Preliminary results for $k = 3$

We suspect that there is a similar pairwise condition for higher k, and for the case of $k = 3$ we show certain pairs are bad.

Proposition 7.1. In addition to the bad pairs from before we have these additional bad pairs for $k = 3$:

(1) $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1)$
(2) $\lambda = (\lambda_1, \lambda_2, 1)$, $\mu = (\mu_1)$ and $\lambda_1 > \mu_1 \geq \lambda_2$
(3) $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2)$ and $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2 \geq \lambda_3$
(4) $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2 \geq \lambda_3 > \mu_3$
(5) $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$ and $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$

Remark. The proposition does not provide an exhaustive list of bad pairs. For example, the pair $\lambda = (4,3,1)$ and $\mu = (1,1)$ is a bad pair and is *not* contained in the list. We further remark that such inequalities of the parts λ_i and μ_j do not suffice to characterize the set of bad pairs. As an example, the pair $\lambda' = (5,2,1)$ and $\mu' = (1,1)$ is *not* a bad pair, that is to say, $s_{(5,2,1)} s_{(1,1)}$ is extreme in the $(10,3)$-Schur cone. Note that the parts in the pair $((4,3,1),(1,1))$ and the parts in $((5,2,1),(1,1))$ satisfy the same inequalities, namely $\lambda_1 > \lambda_2 > \lambda_3 = \mu_1 = \mu_2$.

Proof. (1) For this case we simply note when expanding out $s_\lambda s_\mu$ in the Schur basis we only get Schur functions with at most 3 parts, so it is not extreme.

(2) For this case we get from Jacobi-Trudi that

$$s_\lambda s_\mu = s_{(\lambda_1)} s_{(\mu_1, \lambda_2, 1)} + s_{(\lambda_2 - 1)} s_{(\lambda_1, \mu_1 + 1, 1)}.$$

(3) Again by Jacobi-Trudi we get

$$s_\lambda s_\mu = s_{(\lambda_2 - 1, \mu_2)} s_{(\lambda_1, \mu_1 + 1, \lambda_3)} + s_{(\mu_1, \lambda_2)} s_{(\lambda_1, \mu_2, \lambda_3)} + s_{(\lambda_1, \lambda_3 - 1)} s_{(\mu_1, \lambda_2, \mu_2 + 1)}.$$

(4) Also by Jacobi-Trudi we have

$$s_\lambda s_\mu = s_{(\lambda_1, \mu_1 + 1, \lambda_2 + 1)} s_{(\mu_2 - 1, \lambda_3 - 1, \mu_3)} + s_{(\lambda_1, \lambda_2, \mu_3)} s_{(\mu_1, \mu_2, \lambda_3)} + s_{(\lambda_1, \lambda_2, \mu_2 + 1)} s_{(\mu_1, \lambda_3 - 1, \mu_3)}.$$

(5) If $\mu_1 > \lambda_2$, then we have by Jacobi-Trudi

$$s_\lambda s_\mu = s_{(\lambda_1, \mu_1, \mu_2)} s_{(\lambda_2)} + s_{(\lambda_1 + 1, \mu_2)} s_{(\mu_1 - 1, \lambda_2)} + s_{(\lambda_1 + 1, \lambda_2 + 1)} s_{(\mu_1 - 1, \mu_2 - 1)}.$$

If $\mu_1 = \lambda_2 = \mu_2$, then

$$s_\lambda s_\mu = s_{(\lambda_1,\lambda_2)} s_{(\lambda_2,\lambda_2)} = s_{(\lambda_1,\lambda_2,\lambda_2)} s_{(\lambda_2)} + s_{(\lambda_1+1,\lambda_2+1)} s_{(\lambda_2-1,\lambda_2-1)}. \qquad \square$$

8. Enumerative Questions

Let ξ_N^k denote the number of extreme rays of the cone C_N^k. The following computer data suggests these interesting questions:

- Is the sequence $\xi_N^1, \xi_N^2, \dots, \xi_N^N$ unimodal for all N?
- Does this sequence always have a maximum when $k = 3$?

N/k	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	2								
3	3	3	3							
4	5	5	5	5						
5	7	7	7	7	7					
6	11	13	13	11	11	11				
7	15	17	18	17	15	15	15			
8	22	28	29	27	24	22	22	22		
9	30	40	47	41	36	32	30	30	30	
10	42	61	70	68	55	48	44	42	42	42

9. Weakly Separated Sets

One obstacle to extending our results and conjectures to higher values of k is the difficulty of generalizing Conjecture 2.13. Whereas nested sets in the $k = 2$ case are related to non-crossing matchings, this correspondence has no clear generalization. It has been suggested by Pavlo Pylyavskyy that the notion of *weakly separated* sets [3, Definition 2] may provide such a generalization. Specifically, if we let M be the infinite matrix with entries $m_{i,j} = h_{j-i+1}$, we can write a Schur function s_λ as the determinant of a minor of the matrix M. According to Pylyavskyy, two terms s_λ and s_μ would form a *bad pair*, in a sense generalizing Definition 2.11, if the sets of columns used in the minors expressing the two Schur functions are not weakly separated. In light of the following example, however, we need to limit which minors are allowable, or require something of the sets of rows used in the minors as well.

Example. We can write

$$s_{(1,1)} = \det \begin{pmatrix} 1 & h_2 & h_3 \\ 0 & h_1 & h_2 \\ 0 & 1 & h_1 \end{pmatrix}$$

which is a minor using columns $\{1, 3, 4\}$ of M and

$$s_{(4,3,1)} = \det \begin{pmatrix} h_1 & h_4 & h_6 \\ 1 & h_3 & h_5 \\ 0 & h_2 & h_4 \end{pmatrix}$$

which has column set $\{2, 5, 7\}$ (both minors use rows 2,3,4). The sets $\{1, 3, 4\}$ and $\{2, 5, 7\}$ are not weakly separated. However, the product $s_{(1,1)}s_{(4,3,1)}$ is extreme in C^3_{10}.

Acknowledgements

This research was part of the 2014 summer REU program at the University of Minnesota, Twin Cities, and was supported by RTG grant NSF/DMS-1148634. The authors would like to thank the math department of the university for its hospitality and the mentors for organizing the program. We would especially like to express our thanks to our mentor Victor Reiner for his guidance and advice throughout the project. We also want to thank Dennis White, Pavlo Pylyavskyy and Aaron Landesman for helpful comments and interesting conversations, and our TA Elise DelMas for her help in editing the paper.

References

[1] E. Schirmacher, *Log-concavity and the exponential formula*, Journal of Combinatorial Theory, Series A 85.2 (1999): 127-134.

[2] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, 1986.

[3] J. Scott, *Quasi-commuting families of quantum minors*, Journal of Algebra 290 (2005): 204220.

[4] R. P. Stanley, *Enumerative Combinatorics: Volume 1*, second edition, Cambridge Studies in Advanced Mathematics, 2011.

[5] R. P. Stanley, *Enumerative Combinatorics: Volume 2*, Cambridge Studies in Advanced Mathematics, 1999.

[6] D. E. White, *The Schur Cone and the Cone of Log Concavity*, arXiv:0903.2831v1.

Christian Gaetz, University of Minnesota - Twin Cities

E-mail address: gaetz022@umn.edu

Kyle Meyer, Northeastern University

E-mail address: meyer.ky@husky.neu.edu

Ka Yu Tam, Massachusetts Institute of Technology

E-mail address: garytam@mit.edu

Max Wimberley, Massachusetts Institute of Technology

E-mail address: maxilya4@mit.edu

Zijian Yao, Brown University

E-mail address: zijian_yao@brown.edu

Heyi Zhu, Haverford College

E-mail address: hzhu@haverford.edu