On Virtually Cohen–Macaulay Simplicial Complexes

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Introduction

Virtual resolutions of graded modules in the sense of [1] generalize free resolutions. The combinatorial properties of Cohen-Macaulay simplicial complexes, which are complexes whose Stanley-Reisner ideal have short free resolutions, are well-understood. **Virtually Cohen-Macaulay** (VCM) complexes are defined by virtual resolutions and therefore generalize Cohen-Macaulay complexes. In this project, we tried to understand the combinatorial properties of VCM simplicial complexes, with the following results:

- Balanced complexes are VCM;
- If the Stanley-Reisner ring of a complex is VCM by the Intersection Method, then the complex is CM up to irrelevant faces;
- VCM simplicial complexes are pure up to irrelevant facets;
- VCM simplicial complexes are not necessarily gallery-connected up to irrelevant facets.

Definitions

An **abstract simplicial complex** Δ on vertex set X is a collection of subsets of X such that $A \in \Delta$ whenever $A \subseteq B \in \Delta$.

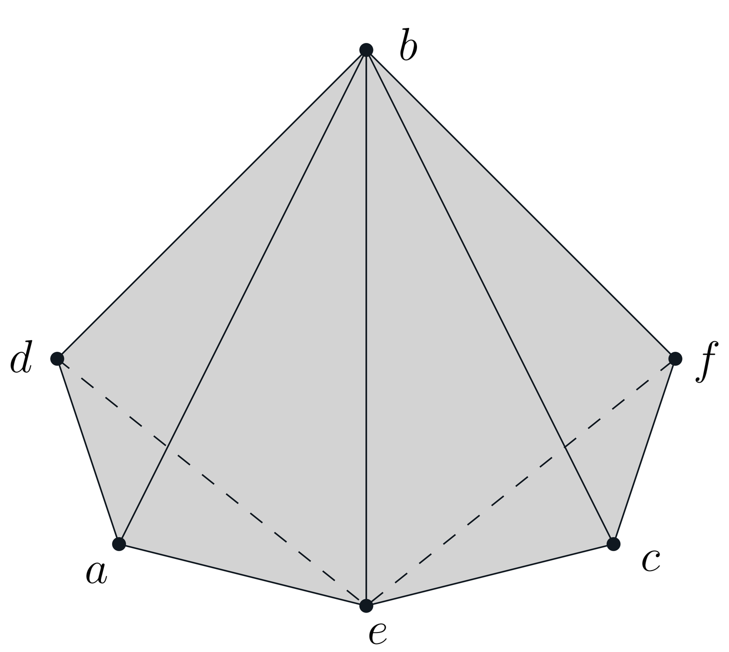

Fig. 1: $X = \{a, b, c, d, e, f\}$, $\Delta = 2^{\{a,b,d,e\}} \cup 2^{\{b,c,e,f\}}$, Facets: $\{a, b, d, e\}, \{b, c, e, f\}$, $I_\Delta = \langle c, f\rangle \cap \langle a, d\rangle$

Given a simplicial complex Δ on X, the **Stanley-Reisner ideal** of Δ is the following ideal in $\Bbbk[X]$:

$$I_\Delta = \bigcap_{A \in \Delta} (x_i : x_i \notin A) = (m_A : A \notin \Delta), m_A = \prod_{x_i \in A} x_i$$

We use the following notation:

- $\mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ where $\vec{n} = (n_1, \ldots, n_r)$.
- $S := \Bbbk[x_{i,j} : 1 \le i \le r, 0 \le j \le n_i]$ is the Cox ring of $\mathbb{P}^{\vec{n}}$.
- $B := \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i}\rangle$ is the **irrelevant ideal** of S. Note that $V(B) = \emptyset$.
- $X_{\vec{n}} = \bigcup_{i=1}^{r} \{x_{i,j} : 0 \le j \le n_i\}$ is the vertex set for simplicial complexes in $\mathbb{P}^{\vec{n}}$.
- The **Stanley-Reisner ring** of Δ is the quotient ring $\Bbbk[\Delta] := S/I_\Delta$.
- Given a vertex $x_{i,j} \in X_{\vec{n}}$, we say that i is the **component** of $x_{i,j}$.

A complex of free S-modules,

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_n} F_n,$$

is a **virtual resolution of** S/I if

1. $\operatorname{rad} \operatorname{ann} H_i\mathcal{F}_\bullet \supseteq B$ for all $i > 0$;
2. $\operatorname{ann} H_0\mathcal{F}_\bullet : B^\infty = I : B^\infty$.

A simplicial complex Δ on $X_{\vec{n}}$ is **virtually Cohen-Macaulay** if there exists a virtual resolution of $\Bbbk[\Delta]$ of length $\operatorname{codim} I_\Delta$.

Virtual Equivalence

Lemma 1. For two ideals $I, J \subset S$ with $V(I) = V(J)$, any virtual resolution r of S/J is a virtual resolution of S/I.

A face F of a simplicial complex Δ is relevant if it contains at least one vertex from every component; otherwise it is **irrelevant**. Since $I_\Delta = \bigcap_{A\in\Delta}(x_i : x_i \notin A)$, adding a face F to Δ is equivalent to intersecting I_Δ with the ideal $I = (x : x \notin F)$. Therefore, $V(I) = \varnothing$ if and only if F is irrelevant. If F is irrelevant, we say that $\Delta \cup \{F\}$ and $\Delta \setminus \{F\}$ (where $F \in \Delta$) are elementarily virtually equivalent to Δ. Two complexes Δ and Δ' are **virtually equivalent** if they are related by a sequence of elementary virtual equivalences.

Lemma 2. Let the complexes Δ, Δ' be virtually equivalent. Then Δ is VCM iff Δ' is VCM. In particular, if Δ' is CM, then Δ is VCM.

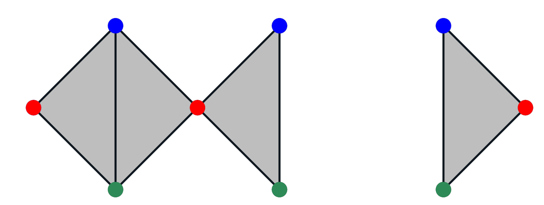

Fig. 2: Δ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$

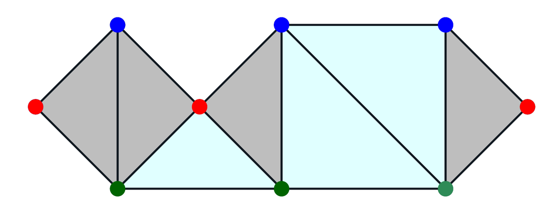

Fig. 3: $\Delta' = \Delta \cup \{\text{Irrelevant Facets}\}$

Balanced Complexes are VCM

Let Δ be a pure simplicial complex in $\mathbb{P}^{\vec{n}}$. We say that a facet $F \in \Delta$ is **balanced** if it contains exactly one vertex of every component and a simplicial complex is **balanced** if all of its facets are balanced. The main result of this project is the following theorem.

Theorem 1. If Δ is a balanced complex in the product projective space $\mathbb{P}^{\vec{n}}$, then Δ is virtually Cohen-Macaulay.

The proof uses the fact that the Stanley-Reisner ring of a pure shellable simplicial complex is Cohen-Macaulay. In particular, we show that:

- The irrelevant complex with only one balanced facet is shellable, where the *irrelevant complex* $\Delta_{irr}(\vec{n})$ supported on $X_{\vec{n}}$ is an $(r-1)$-dimensional complex defined by: for $\dim \sigma = r - 1$,

  $$\sigma \in \Delta_{irr}(\vec{n}) \Leftrightarrow \text{there is only one pair of vertices } \{v, w\} \subset \sigma \text{ of the same component;}$$

- Any other balanced facet has all its ridges contained in $\Delta_{irr}(\vec{n})$, so we can add any remaining balanced facets and get a shelling order of $\Delta_{irr}(\vec{n}) \cup \Delta$;
- So Δ is virtually equivalent to a shellable complex.

Constructing the Shelling

Let $\Delta_{\neg k} = \{\sigma \in \Delta_{irr}(\vec{n}) \mid x_{i,j} \in \sigma \Rightarrow i \neq k\}$. We illustrate the shelling with $\vec{n} = (3, 2, 2, 2)$. For convenience, we use abcd efg hij klm in place of the $x_{i,j}$ notation (for instance e represents $x_{2,0}$).

General Shelling

- First add the lexicographically first balanced facet.
- Then add the facets of $\Delta_{\neg r}, \ldots, \Delta_{\neg 1}$, in descending order, where for each k, add the facets of $\Delta_{\neg k}$ as follows:
  - We can write every facet $F \in \Delta_{\neg k}$ as $F = \text{Facet}(p_F, \vec{v_F})$, where p_F is its defining vertex pair and $\vec{v_F} \in \mathbb{Z}_{\ge 0}^{r-2}$ specifying the lexicographic order of the rest of its vertices.
  - Then given $F = \text{Facet}(p_F, \vec{v_F})$ and $G = \text{Facet}(p_G, \vec{v_G})$, $F < G$ iff $\vec{v_F} < \vec{v_G}$ or $\vec{v_F} = \vec{v_G}$ and $p_F < p_G$.
  - Add the facets of $\Delta_{\neg k}$ is ascending order.

Example

- The lexicographically first balanced facet is aehk.
- Add the facets of $\Delta_{\neg 4}, \ldots, \Delta_{\neg 1}$, in descending order:
  - E.g., in $\Delta_{\neg 4}$, Facet(ab, $(2, 3)$) = abfj.
  - E.g., fgah < efci, because

    fgah = Facet(fg, $(1, 1)$), efci = Facet(ef, $(3, 2)$).

  - The first facets of $\Delta_{\neg 4}$ in ascending order is:

    abeh < aceh < adeh < bceh < bdeh < cdeh
    < efah < egah < fgah < hiae < hjae < ijae
    < abei < acei < adei < bcei < bdei < cdei
    < efai < egai < fgai < hiaf < hjaf < ijaf

Other Results

A simplicial complex is *pure* if all of its facets have the same dimension. A pure simplicial complex is *gallery-connected* if for any two facets $F, F' \in \Delta$, there exists a path of facets $F = F_1, \ldots, F_{n-1}, F_n = F'$ such that for all $1 \le i \le n-1$, the intersection $F_i \cap F_{i+1}$ has codimension 1 in Δ. It is well-known that CM complexes are pure and gallery-connected. Naturally, we ask if the same is true for VCM complexes up to irrelevant faces.

Result I: VCM complexes are pure up to irrelevant facets.

Method of Proof: [1] provides a bound for the codimension of an associated prime of I_Δ, which is sharp for all virtually Cohen-Macaulay Δ. The codimension of an associated prime is linearly related to the dimension of the corresponding facet.

Result II: VCM complexes are not necessarily gallery-connected up to irrelevant facets. Consider the following counterexample:

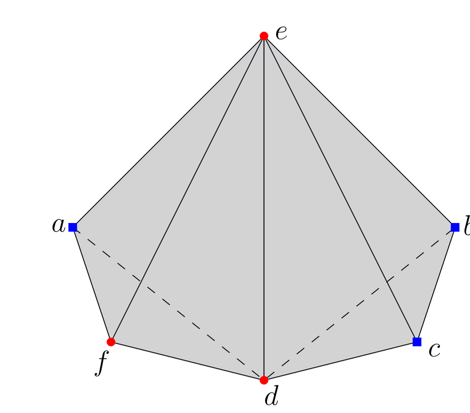

Fig. 4: VCM but not gallery-connected complex

One way to obtain virtual resolutions is using the **Intersection Method** as follows: If there exists J such that $V(J) = \emptyset$ and $I' = I \cap J$ is Cohen-Macaulay, then I is virtually Cohen-Macaulay. We showed that the Intersection Method for squarefree monomial ideals reduces to modifying the corresponding complex with irrelevant faces:

Result III: Let Δ be a simplicial complex on the product projective space $\mathbb{P}^{\vec{n}}$. If there exists J a monomial ideal with $V(J) = \varnothing$ such that $I_\Delta \cap J$ is Cohen-Macaulay, then there exists Δ' containing only irrelevant facets such that $\operatorname{rad}(J) = I_{\Delta'}$ and $I_\Delta \cap I_{\Delta'}$ is Cohen-Macaulay. In particular, this implies $\Delta \cup \Delta'$ is Cohen-Macaulay and Δ is virtually Cohen-Macaulay.

Method of proof: it is well-known (for example in [2]) that for a monomial ideal I, if I is Cohen-Macaulay, then $\operatorname{rad}(I)$ is also Cohen-Macaulay.

Conclusions and Future Work

- There are various different methods to obtain short virtual resolutions of corresponding Stanley-Reisner rings of simplicial complexes and we have no characterization of which method works best on which kind of simplicial complexes.
- We hope to work out a homological criterion for when complexes are VCM (analogous to Reisner's criterion).

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References

[1] Christine Berkesch, Daniel Erman, and Gregory G. Smith. "Virtual Resolutions for a Product of Projective Spaces". In: To appear in *Alg. Geom.* (2019). arXiv: 1703.07631 [math.AC].

[2] Jürgen Herzog, Yukihide Takayama, and Naoki Terai. "On the radical of a monomial ideal". In: *Archiv der Mathematik* 85 (Nov. 2005), pp. 397–408. DOI: 10.1007/s00013-005-1385-z.