Factorization of k-nonnegative Matrices

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Background

A matrix is totally nonnegative (TNN) if all of its minors are nonnegative. It is *k-nonnegative* (kNN) if all of its $i \times i$ submatrices with $i \le k$ have nonnegative determinant. The $n \times n$ invertible TNN matrices (resp. kNN matrices for $k \le n$) form a semigroup.

- **(1)** The subsemigroup of invertible TNN upper unitriangular matrices is generated by $e_i(a)$ for $i \in [n-1]$ and $a > 0$:

$$e_i(a) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & a & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

- **(2)** Relations of e_i generators are given by:

$$e_i(a)e_i(b) = e_i(\alpha)$$
$$e_i(a)e_{i+1}(b)e_i(c) = e_{i+1}(\alpha)e_i(\beta)e_{i+1}(\gamma)$$
$$e_i(a)e_j(b) = e_j(\alpha)e_i(\beta) \qquad |i-j| > 1$$

  The expressions for parameters are *subtraction-free* and, in the latter two, *bijective*. We can perform any valid relation on a factorization as long as it is *reduced*, i.e. it has minimal length.

- Consider a word w from the free monoid $\langle e_1, \dots, e_{n-1} \rangle$ subject to the above relations (disregarding parameters). Let $x_w : \mathbb{R}_{>0}^{\ell(w)} \to \mathrm{GL}_n(\mathbb{R})$ be defined by

$$x_w(a_1, \dots, a_{\ell(w)}) = w_1(a_1) \cdots w_{\ell(w)}(a_{\ell(w)})$$

  Let $U(w) := \mathrm{Im}(x_w)$ (called Bruhat cells). The set of $U(w)$ for distinct, reduced words w stratify the semigroup of TNN unitriangular $n \times n$ matrices.

- By noticing the identification $e_i \mapsto (i, i+1) \in S_n$ (cycle notation), we see that **(3)** cells are naturally indexed by elements in S_n, **(4)** the cells form a CW-complex, and **(5)** the corresponding closure poset is isomorphic to the Bruhat poset on S_n.

Poset Definitions

Subword order: $v \le w$ iff we can achieve v by replacing sections of w with a subword ($\emptyset$ is a subword of everything).

The *Bruhat order* on S_n is equivalent to the *subword order* on its generators $(i, i+1)$ (cycle notation).

Closure order: $A \le B$ iff $A \subset B$. When cells form a CW-complex, this means $B = \sqcup_A A$ for all $A \le B$.

Summary of Results

Goal: Generalize the Bruhat cell structure of totally nonnegative matrices to k-nonnegative matrices.

Results: Descriptions for cells for $(n-1)$NN and unitriangular $(n-2)$NN semigroups, and proof of desired topological properties.

	TNN unitriangular matrices (known)	$(n-2)$NN unitriangular matrices (new)	
Generating set	$\{e_i(a) \mid i \in [n-1],\ a \in \mathbb{R}_{>0}\}$	$\{e_i(a)\} \cup \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}_{>0}^{2n-5}\}$	**(1)**
Relations	complete, subtraction-free	complete, subtraction-free	**(2)**
Cell indexing scheme	S_n	$S_n \cup S_{n-2} \times \{0,1\}^2$	**(3)**
Cell topology	Forms a regular CW-complex	Forms a CW-complex	**(4)**
	Closure poset obeys subword order	Closure poset obeys subword order	**(5)**
	Closure poset graded and Eulerian	Closure poset graded, but *not* Eulerian	
	Homeomorphic to a ball	???	**(6)**

From Generators to Cells

(1) To generate the space of $(n-2)$NN unitriangular matrices, add the parametrized T-generator to the generating set:

$$T(\vec{a}, \vec{b}) = \begin{bmatrix} 1 & a_1 & a_1 b_1 & & & & & \\ & 1 & a_2 + b_1 & a_2 b_2 & & & & \\ & & 1 & \ddots & \ddots & & & \\ & & & \ddots & a_{n-3} + b_{n-4} & a_{n-3} b_{n-3} & & \\ & & & & 1 & b_{n-3} & b_{n-2} Y & \\ & & & & & 1 & b_{n-2} X & \\ & & & & & & 1 & \end{bmatrix}$$

$$Y = b_1 \cdots b_{n-3} \qquad X = \left| T_{[2,n-3],[3,n-2]} \right|$$

(2) This generating set is minimal and has the following relations (indices mod $n-1$):

- $e_i(x) T(\vec{a}, \vec{b}) = T(\vec{A}, \vec{B}) e_{i+2}(x')$
- $e_{n-1} e_{n-2} T = e_{n-2} e_{n-1} T \sqcup e_{n-2} \cdots e_1 e_{n-1} \cdots e_2 \sqcup e_{n-2} \cdots e_1 e_{n-1} \cdots e_1$

These relations are bijective and subtraction-free as desired. Thus, we can naturally extend the definitions of the map x_w and of $U(w)$ to define factorization cells for our space.

(3) Let $w_{0,[n-2]}$ be the long word of S_{n-2} embedded in S_n such that $w_{0,[n-2]} = (n-2, n-3, \dots, 1, n-1, n)$ (one-line notation). Our relations give the following list of reduced words; we prove that these are all the possible factorizations, and thus the full list of cells, for $(n-2)$NN unitriangular matrices.

$$\begin{cases} v\lambda & v \le w_{0,[n-2]} \\ & \lambda \in \{T, e_{n-1}T, e_{n-2}T,\ e_{n-2}e_{n-1}T\} \\ w & w \in S_n \end{cases}$$

Topology of Cells

(4) These cells form a CW-complex; namely,

- They partition the space of $(n-2)$NN matrices
- They are homeomorphic to open balls (namely, $U(w) \cong \mathbb{R}^{\ell(w)}$)
- The closure of a cell is the disjoint union of cells of lower dimension

(5) We can still describe the cell closure poset with a subword order. To do this, we extend the Bruhat order on S_n by defining the subwords of T.

- $m < \lambda \in \{T, e_{n-1}T, e_{n-2}T,\ e_{n-2}e_{n-1}T\}$ precisely when $m \le \alpha = e_{n-2} \cdots e_1 e_{n-1} \cdots e_1$ and satisfies the following:
- $m(1) \ne n$; if λ has no e_{n-1}, then $m(2) \ne n$ is relaxed; if λ has no e_{n-2}, then $m(1) \ne n-1$.

This description still defines a valid subword order.

Proof Method

Finding generators and relations requires a variant of the Loewner-Whitney theorem and careful examination of cell behavior. We prove topological properties by examining our cells in the context of the decomposition of $GL_n(\mathbb{R})$ into Bruhat cells. There can be as many as five of our cells in one Bruhat cell, but distinguishing the topology of these five cells is straightforward. We describe the subwords of T by comparing the Bruhat interval below $\alpha \in S_n$ to the algebraically determined subwords.

Results on $(n-1)$NN Matrices

Nearly all of the results shown here for the $(n-2)$NN unitriangular semigroup also apply to the $(n-1)$NN semigroup. The generating set includes an additional generator $K(\vec{a}, \vec{b})$. We show similar relations and give similar cells.

The only major difference is that the resulting cell decomposition is *not* a CW-complex, since the space has exactly two connected components: matrices with positive determinant, and matrices with negative determinant.

Discussion

TNN matrices are deep objects, appearing in stochastic processes, planar networks, Pólya frequency sequences, etc. kNN matrices, a classical and natural generalization, were introduced at the same time, and we have shown evidence for very similar structure in kNN matrices. Some further questions are:

- **(6)** What is the space of $(n-2)$NN invertible matrices topologically?
- Is the closure poset of cells of $(n-2)$NN matrices shellable or semi-Eulerian?
- What results can be generalized to k-nonnegative matrices?

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