A FEW APPROACHES TO SOLVING THE BORWEIN CONJECTURE

GAKU LIU

1. Introduction

Here we list a few attempted approaches to solving Borwein's conjecture, namely that the polynomials defined by

$$A_N(q) = \sum_l (-1)^l q^{l(9l-1)/2} \begin{bmatrix} 2N \\ N-3l \end{bmatrix}_q$$
$$B_N(q) = \sum_l (-1)^l q^{l(9l-5)/2} \begin{bmatrix} 2N \\ N+1-3l \end{bmatrix}_q$$
$$C_N(q) = \sum_l (-1)^l q^{l(9l-7)/2} \begin{bmatrix} 2N \\ N+1-3l \end{bmatrix}_q$$

have nonnegative integer coefficients. This is a special case of the following conjecture by Bressoud [2]:

Conjecture 1.1. *Let M, N and k be positive integers with $k \geq 2$, and α and β be positive rational numbers such that $k\alpha$ and $k\beta$ are integers. If $1 \leq \alpha+\beta \leq 2k-1$, with strict inequalities when $k=2$, and $-k+\beta \leq N-M \leq k-\alpha$, then the polynomial*

$$D_k(M,N,\alpha,\beta;q) = \sum_l (-1)^l q^{l(k(\alpha+\beta)l-k(\alpha-\beta)/2)} \begin{bmatrix} M+n \\ M-kl \end{bmatrix}_q$$

has nonnegative integer coefficients.

In particular, $A_n(q) = D_3(N,N,5/3,4/3;q)$, $B_n(q) = D_3(N+1,N-1,7/3,2/3;q)$, and $C_n(q) = D_3(N+1,N-1,8/3,1/3;q)$.

2. Related polynomials

Conjecture 1 has been proven for integer values of α, β; in fact, in this case, Andrews et al. [1] proved that $D_k(M,N,\alpha,\beta;q)$ is the generating function for partitions with certain prescribed "hook differences." Given a partition λ, label the box in the i-th row and j-th column of λ as (i,j). Define the *hook difference* at (i,j) to be the size of row i minus the size of column j; that is, $\lambda_i - \lambda'_j$. Let the c-th diagonal of λ be the set of boxes (i,j) such that $i-j=c$. The theorem is as follows

Theorem 2.1. *For α, β positive integers, $D_k(M,N,\alpha,\beta;q)$ is the generating function for partitions with at most M parts each of size at most N, and such that the hook differences on diagonal $\alpha-1$ are at most $k-\alpha-1$ and the hook differences on diagonal $1-\beta$ are at most $-k+\beta+1$. If $\alpha=0$ and $\beta \neq 0$, $D_k(M,N,\alpha,\beta;q)$ is the generating function for partitions satisfying the above conditions with the additional condition that the number of parts is at least $N-k+1$. If $\beta=0$ and $\alpha \neq 0$, the additional condition is that the largest part is at least $M-k+1$.*

The hope is that by better understanding what occurs in the integer case, we might be able to extend this understanding to cases with non-integer values of α and β. In particular, we look at the following two polynomials, which seem related to A_N:

$$X_N(q) = D_3(N,N,2,1;q) = \sum_l (-1)^l q^{\frac{l(9l-3)}{2}} \begin{bmatrix} 2N \\ N-3l \end{bmatrix}_q$$

$$Y_N(q) = D_3(N,N,3,0;q) = \sum_l (-1)^l q^{\frac{l(9l-9)}{2}} \begin{bmatrix} 2N \\ N-3l \end{bmatrix}_q.$$

Proposition 2.1. *We have the factorizations*

$$X_N(q) = (1+q^N)\prod_{i=1}^{N-1}(1+q^i+q^{2i})$$

$$Y_N(q) = q^{N-2}(1+q^N)(1+q+q^2)\prod_{i=1}^{N-2}(1+q^i+q^{2i}).$$

Proof. We will provide a bijective proof of these two identities using Theorem 2.1. First consider $X_N(q) = D_3(N,N,2,1;q)$. Every partition λ has an associated Frobenius representation

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ b_1 & b_2 & \cdots & b_t \end{pmatrix}$$

where $a_i = \lambda_i - i$, and $b_i = \lambda'_i - i$, t is the largest integer such that $\lambda_t \geq t$, and the number being partitioned is $t + \sum_{i=1}^{t}(a_i + b_i)$. The hook difference condition for $k = 3$, $\alpha = 2$, $\beta = 1$ can be written as

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq b_2 \geq \cdots \geq a_t + 1 \geq b_t.$$

We also have the original Frobenius representation conditions $N \geq a_1+1 > a_2+1 > \cdots > a_t+1 \geq 1$ and $N-1 \geq b_1 > b_2 > \cdots > b_t \geq 0$. Thus, there is a bijection between the partitions counted by $X_N(q)$ and weakly decreasing positive integer sequences $a_1+1 \geq b_1 \geq a_2+1 \geq \ldots \geq a_t+1 \geq b_t$ (where we leave out b_t if $b_t = 0$), such that the first term is at most N, the number N appears at most once, and all other numbers appear at most twice. In addition, the weight of the partition is simply the sum of the terms of its associated sequence. The generating function of such sequences (with the weight being the sum of its terms) is clearly

$$(1+q^N)\prod_{i=1}^{N-1}(1+q^i+q^{2i}).$$

which proves the identity for X_N.

The proof for $Y_N(q) = D_3(N,N,3,0;q)$ is similar. For $k = 3$, $\alpha = 3$, $\beta = 0$, the hook difference condition can be restated as requiring the largest part to be at least $N-2$, and that when the largest part is removed, the partition remaining satisfies the hook difference conditions for $D_3(N-1,N,2,1;q)$. If we let p be the largest part and

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ b_1 & b_2 & \cdots & b_t \end{pmatrix}$$

be the Frobenius representation for the partition formed by removing the largest part, then we have a bijection between the desired partitions and weakly decreasing positive integer sequences $p \geq a_1+1 \geq b_1 \geq \ldots \geq a_t+1 \geq b_t$ (where we leave out b_t if $b_t = 0$) with $N-2 \leq p \leq N$, $b_1 \leq N-2$, no number appearing more than twice in the subsequence $a_1+1 \geq b_1 \geq \ldots \geq a_t+1 \geq b_t$, and the sum of the terms being the weight of the associated partition. The generating function for such sequences is

$$(q^{N-2}+q^{N-1}+q^{N}+q^{N-1}q^{N-1}+q^{N-1}q^{N}+q^{N}q^{N})\prod_{i=1}^{N-2}(1+q^{i}+q^{2i})$$

which equals the product given in the Proposition. □

Guo and Zhang give another combinatorial proof of this result in [3].

3. A possible combinatorial approach

By the Jacobi triple product formula, we can write

$$\sum_{l}(-1)^{l}q^{l(9l-1)/2}=\prod_{\substack{n\equiv 0,4,5 \bmod 9\\ n>0}}(1-q^{n}).$$

But we can re-expand the product on the right side, giving a sum with many more terms than the original sum

$$\sum_{l}(-1)^{l}q^{l(9l-1)/2}=\sum_{r\geq 0}\sum_{\substack{0<n_1<n_2<\cdots<n_r\\ n_1,\ldots,n_r\equiv 0,4,5 \bmod 9}}(-1)^{r}q^{n_1+\cdots+n_r}.$$

While at first it does not seem helpful to add in so many new terms, this can be helpful in interpreting things combinatorially; for example, if $P(q)$, is the generating function for partitions (without restriction), it is not immediately clear how to interpret

$$\sum_{l}(-1)^{l}q^{l(9l-1)/2}P(q).$$

However, if we rewrite this as

$$\sum_{r\geq 0}\sum_{\substack{0<n_1<n_2<\cdots<n_r\\ n_1,\ldots,n_r\equiv 0,4,5 \bmod 9}}(-1)^{r}q^{n_1+\cdots+n_r}P(q)$$

then we see that $q^{n_1+\cdots+n_r}P(q)$ is the generating function for partitions which contain at least one part each of sizes $n_1,\ldots,n_r$, and thus the sum is an inclusion-exclusion that counts the number of partitions with no parts congruent to 0, 4, or 5 modulo 9. Following this example, if we were to construct a family of generating functions $P_n(q)$ such that

$$P_{l(9l-1)/2}=q^{l(9l-1)/2}\begin{bmatrix}2N\\ N-3l\end{bmatrix}$$

for integer l, then we would have

$$A_N(q)=\sum_{r\geq 0}\sum_{\substack{0<n_1<n_2<\cdots<n_r\\ n_1,\ldots,n_r\equiv 0,4,5 \bmod 9}}(-1)^{r}P_{n_1+\cdots+n_r}(q).$$

For an appropriate choice of the $P_n(q)$, we might obtain a sum more easily recognizable as an inclusion-exclusion or involution.

4. Recurrences and "finite differences"

We wish to prove that the coefficients of $A_N(q)$ are nonnegative, but more seems to be true: the coefficient of each q^r appears to weakly increase as N increases. We thus might try to prove that appropriately defined "finite differences" of the $A_N(q)$ have nonnegative coefficients, which is the approach in this section.

For simplicity, we abbreviate $A_N(q)$ as A_N, and similarly for other polynomials. The polynomials A_N, B_N, and C_N satisfy the recurrences

$$\begin{aligned}
A_N &= (1+q^{2N-1})A_{N-1} + q^N B_{N-1} + q^N C_{N-1} \\
B_N &= (1+q^{2N-1})B_{N-1} + q^{N-1} A_{N-1} - q^N C_{N-1} \\
C_N &= (1+q^{2N-1})C_{N-1} + q^{N-1} A_{N-1} - q^{N-1} B_{N-1}.
\end{aligned}$$

Solving these relations for A_N gives

$$\begin{aligned}
A_N = (1+q+q^2)(1+q^{2N-3})A_{N-1} &- q(1+q+q^2)(1+q^{2N-4}+q^{4N-8})A_{N-2} \\
&+ q^3(1-q^{3N-8}-q^{3N-7}+q^{6N-15})A_{N-3}.
\end{aligned}$$

This seems complicated, but we can simplify the relation by considering successive "finite differences" of A_N, which we define as follows:

$$\begin{aligned}
K_N &= A_N - (1+q^{2N-1})A_{N-1} \\
L_N &= K_N - (q+q^{2N-2})A_{N-1} \\
R_N &= L_N - (q^2+q^{2N-3})L_{N-1}.
\end{aligned}$$

Using the recurrence for A_N, we have the following relations:

$$\begin{aligned}
K_N &= q(1+q)(1+q^{2N-4})K_{N-1} - q^3(1-q^{2N-5}+q^{4N-10})K_{N-2} - (q^{3N-5}+q^{3N-4})A_{N-3} \\
L_N &= (q^2+q^{2N-3})L_{N-1} + 3q^{2N-2}K_{N-2} - (q^{3N-5}+q^{3N-4})A_{N-3} \\
R_N &= 3q^{2N-2}K_{N-2} - (q^{3N-5}+q^{3N-4})A_{N-3}.
\end{aligned}$$

In the end, we have been able to remove all negative terms except $(q^{3N-5}+q^{3n-4})A_{N-3}$. Because of the way K_N, L_N, and R_N are defined, the fact that any one of them has nonnegative coefficients (along with some easily verifiable base cases) would imply that A_N has nonnegative coefficients. This in fact seems to be true for K_N and L_N, and it seems true for R_N with the excepton of its first and last coefficients, which appear to be -1. We state this as a conjecture:

Conjecture 4.1. *K_N and L_N have nonnegative coefficients, while R_N has nonnegative coefficients with the exception of the coefficients of q^{3N-5} and q^{N^2-3N+5}, which are -1.*

Because of the simplicity of the expression for R_N, one might be able to tackle the problem by inductively proving appropriate bounds on the coefficients R_N, which would translate into bounds on L_N, K_N, and A_N.

It might also be of interest to note that the polynomial X_N, which we defined in Section 2 and whose factorization and combinatorial interpretation are known, satisfies a similar recurrence as A_N does, with identical initial conditions:

$$\begin{aligned}
X_N = (1+q+q^2)(1+q^{2N-3})X_{N-1} &- q(1+q+q^2)(1+q^{2N-4}+q^{4N-8})X_{N-2} \\
&+ q^3(1-q^{3N-9}-q^{3N-6}+q^{6N-15})X_{N-3}.
\end{aligned}$$

The only difference is that we have $-(q^{3N-6}+q^{3N-3})X_{N-3}$ instead of $-(q^{3N-5}+q^{3n-4})A_{N-3}$. Recall that when we took finite differences, $-(q^{3N-5}+q^{3n-4})A_{N-3}$ was the only negative term that remained in the end; and indeed, if we were to employ the same strategy of finite differences on X_N, we would get the same relations except with the term $-(q^{3N-6} + q^{3N-3})X_{N-3}$ instead of $-(q^{3N-5}+q^{3n-4})A_{N-3}$. Understanding why the X_N that arise from this recurrence have nonnegative coefficients might help us understand why the A_N do.

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E-mail address: `xueliu@princeton.edu`