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An Easy Proof of the Rogers–Ramanujan Identities

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A proof of the Rogers–Ramanujan identities is presented which is brief, elementary, and well motivated; the "easy" proof of whose existence Hardy and Wright had despaired. A multisum generalization of the Rogers–Ramanujan identities is shown to be a simple consequence of this proof.

The Rogers–Ramanujan identities are a pair of analytic identities first discovered by Rogers [9] and then rediscovered by Ramanujan (see [5, p. 91]), Schur [10], and, in 1979, by the physicist Baxter [2]. They are

$$\sum_0^\infty \frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)} = \prod_0^\infty \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}, \tag{1}$$

$$\sum_0^\infty \frac{q^{m^2+m}}{(1-q)(1-q^2)\cdots(1-q^m)} = \prod_0^\infty \frac{1}{(1-q^{5m+2})(1-q^{5m+3})}. \tag{2}$$

Here and throughout this paper $|q|$ is strictly less than one.

Two new proofs of these identities have recently been announced. The first, by Lepowsky and Wilson [7], uses a Lie algebraic interpretation of the identities. The second, by Garsia and Milne [4], relies on the combinatorial interpretation and establishes the correspondence between the partitions which are counted by each side. Both of these proofs are enlightening but difficult. What is being presented here is the one type of proof Hardy and Wright despaired of ever finding when they wrote [6, p. 292]: "No proof is really easy (and it would perhaps be unreasonable to expect an easy proof)."

The Rogers–Ramanujan identities bear at least superficial resemblance to the triple product identity of Jacobi:

$$\sum_{-\infty}^\infty x^m q^{(m^2+m)/2} = \prod_1^\infty (1+xq^m)(1+x^{-1}q^{m-1})(1-q^m). \tag{3}$$

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In fact, the first step in almost every proof of the Rogers–Ramanujan identities is to use the triple product identity with q replaced by q^5 and x set equal to $-q^{-2}$ or $-q^{-1}$ to rewrite (1) and (2) as

$$\sum_0^\infty \frac{q^{m^2}}{(1-q)\cdots(1-q^m)} = \left(\prod_1^\infty (1-q^m)\right)^{-1} \sum_{-\infty}^{\infty} (-1)^m q^{(5m^2+m)/2}, \tag{4}$$

$$\sum_0^\infty \frac{q^{m^2+m}}{(1-q)\cdots(1-q^m)} = \left(\prod_1^\infty (1-q^m)\right)^{-1} \sum_{-\infty}^{\infty} (-1)^m q^{(5m^2+3m)/2}. \tag{5}$$

The triple product identity has an easy proof found by Cauchy [3]. It relies on the observation that (3) is the limit as N tends to ∞ of

$$\sum_{-N}^{N} x^m q^{(m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix} = \prod_1^N (1+xq^m)(1+x^{-1}q^{m-1}), \tag{6}$$

where $\left[\begin{smallmatrix} N \\ m \end{smallmatrix}\right]$ is the Gaussian polynomial defined to be 1 if $m=0$ or N, 0 if $m<0$ or $m>N$, and

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)}, \qquad 0<m<N.$$

The Gaussian polynomial becomes the binomial coefficient $\binom{N}{m}$ in the limit as q approaches 1, and it satisfies a similar recursion,

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{bmatrix} N-1 \\ m \end{bmatrix} + q^{N-m} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}, \tag{7}$$

While (6) may look more difficult to prove than (3), it is in fact much easier, being simply the case $n=2N$, $y=xq^{-N}$ of the q-binomial theorem

$$\sum_0^\infty y^m q^{(m^2+m)/2} \begin{bmatrix} n \\ m \end{bmatrix} = (1+yq)(1+yq^2)\cdots(1+yq^n). \tag{8}$$

The q-binomial theorem can be proved either by observing that both sides are equal for $n=1$ and satisfy the same recursion

$$f_n(y;q) = f_{n-1}(y;q) + yq^n f_{n-1}(y;q) \tag{9}$$

or by noting that if each side of (8) is expanded as a power series in both y and q, then for both sides the coefficient of $y^s q^t$ counts the number of partitions of t into s distinct parts, each less than or equal to n [8, Sect. 246, pp. 9–10].

It is the same sort of "simplification" that we shall use to prove (1) and (2). What we shall actually prove is that

$$\sum_{N\geqslant s\geqslant t\geqslant 0} \frac{q^{s^2+t^2}(1+xq)(1+xq^2)\cdots(1+xq^t)}{(1-q)\cdots(1-q^{N-s})(1-q)\cdots(1-q^{s-t})} \times \frac{(1+x^{-1})(1+x^{-1}q)\cdots(1+x^{-1}q^{t-1})}{(1-q)\cdots(1-q^{2t})}$$

$$= \left(\prod_{1}^{2N}(1-q^m)\right)^{-1} \sum_{-N}^{N} x^m q^{(5m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix}. \tag{10}$$

When $x=-1$, the left-hand summation in zero except when $t=0$. If we let N approach ∞ and multiply both sides by $\prod_1^\infty(1-q^m)$, we get (4) which is equivalent to (1). If $x=-q$, then the left-hand summation is zero except when $t=0$ or 1. After letting N approach ∞ and multiplying both sides by $\prod_1^\infty(1-q^m)$, the left-hand summation is

$$\sum_0^\infty \frac{q^{s^2}}{(1-q)\cdots(1-q^s)} + \sum_1^\infty \frac{q^{s^2+1}(1-q^2)(1-q^{-1})}{(1-q)\cdots(1-q^{s-1})(1-q)(1-q^2)}$$

which equals

$$\sum_0^\infty \frac{q^{s^2}}{(1-q)\cdots(1-q^s)} - \sum_0^\infty \frac{q^{s^2}(1-q^s)}{(1-q)\cdots(1-q^s)}$$

which is easily seen to equal the left-hand summation of (5). Thus (2) is equivalent to (5) which is (10) with $x=-q$.

Remark. The equation we shall prove, (10), is entirely new and differs quite markedly from the usual way of introducing an extra parameter x into the Rogers–Ramanujan identities,

$$\sum_0^\infty \frac{x^s q^{s^2}}{(1-q)\cdots(1-q^s)} = \left(\prod_1^\infty(1-xq^n)\right)^{-1} \times \sum_0^\infty \frac{(-x^2)^m q^{(5m^2-m)/2}(1-xq^{2m})\cdot(1-xq)\cdots(1-xq^{m-1})}{(1-q)\cdots(1-q^m)}. \tag{11}$$

See, for example, [6, Sect. 19.14].

One of the advantages of our proof is that it generalizes quite easily. For convenience, we shall first introduce some notation.

DEFINITION. We let $(q)_s$ denote the "q-factorial"

$$(q)_s \equiv \prod_{1}^{\infty} \frac{(1-q^m)}{(1-q^{s+m})} \tag{12}$$

defined formally for all complex s. In practice, s will be an integer. If s is a positive integer, then $(q)_s = (1-q)(1-q^2)\cdots(1-q^s)$, while if s is a negative integer, then $(q)_s^{-1} = 0$.

Remark. The Gaussian polynomial may be defined in terms of the q-factorial

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{(q)_N}{(q)_m\,(q)_{N-m}}. \tag{13}$$

From here on, all summations will be over all integral values of the index (positive and negative), although in general all but a finite number of terms in the summation will be zero. For example, $\sum_s (q^{s^2}/(q)_s\,(q)_{n-s})$ is, in effect, only a summation from 0 to n since $(q)_s^{-1} = 0$ for $s < 0$ and $(q)_{n-s}^{-1} = 0$ for $s > n$.

THEOREM. *Given positive integers k and N, we have that*

$$\begin{aligned} \sum_{s_1,\ldots,s_k} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2}}{(q)_{N-s_1}\,(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}\,(q)_{2s_k}} \prod_{1}^{s_k} (1+xq^m)(1+x^{-1}q^{m-1}) \\ = (q)_{2N}^{-1} \sum_m x^m q^{((2k+1)m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix}. \end{aligned} \tag{14}$$

Remark. When $k = 2$, the theorem becomes (10). For $x = -1$, we have

$$\begin{aligned} \sum_{s_1,\ldots,s_{k-1}} \frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q)_{n-s_1}\,(q)_{s_1-s_2}\cdots(q)_{s_{k-2}-s_{k-1}}\,(q)_{s_{k-1}}} \\ = (q)_{2N}^{-1} \sum_m (-1)^m\, q^{((2k+1)m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix}, \end{aligned} \tag{15}$$

a result which can also be obtained as a corollary of [1, Theorem 4, p. 199].

We shall prove (10) by taking the obvious approach. The summation on the right-hand side of (10) is identical to the summation side of (6), except that in the exponent of q the coefficient of m^2 is $\frac{5}{2}$ instead of $\frac{1}{2}$. All we need to do is to find a simple means of decreasing this coefficient until we can apply (6), which has already been established. Our reduction procedure rests on another well-known q-analog of the binomial theorem.

LEMMA 1. *For positive integral k, we have that*

$$\left(\prod_1^k (1-xq^j)\right)^{-1} = \sum_j \frac{x^j q^{j^2}}{(1-xq)\cdots(1-xq^j)} \begin{bmatrix} k \\ j \end{bmatrix}. \tag{16}$$

Remark. For $q = 1$, this becomes

$$(1-x)^{-k} = \left(1 + \frac{x}{1-x}\right)^k = \sum_j \left(\frac{x}{1-x}\right)^j \binom{k}{j}.$$

Proof. As with the q-binomial theorem (8), this can be proved either by observing that both sides are equal for $k = 1$ and satisfy the same recursion

$$g_k(x;q) = g_{k-1}(x;q) + \frac{xq^k}{1-xq} g_{k-1}(xq;q), \tag{17}$$

or by noting that if each side of (16) is expanded as a power series in both x and q, then for both sides the coefficient of $x^s q^t$ counts the number of partitions of t into s parts, each less than or equal to k, [11, Sect. 34; 8, Sect. 265, p. 26].

LEMMA 2. *For positive integral n and completely arbitrary a, we have the following formula for reducing the coefficient of m^2 in the exponent of q*:

$$\sum_m \frac{x^m q^{am^2}}{(q)_{n-m}(q)_{n+m}} = \sum_s \frac{q^{s^2}}{(q)_{n-s}} \sum_m \frac{x^m q^{(a-1)m^2}}{(q)_{s-m}(q)_{s+m}}. \tag{18}$$

Remark. Lemma 2 is new and should have a wide range of applicability.

Proof. If, in Lemma 1, we set $k = n - m$ and $x = q^{2m}$ and then multiply both sides by $(q)_{2m}^{-1}$, we get

$$(q)_{n+m}^{-1} = \sum_j \frac{q^{j^2+2mj}}{(q)_{j+2m}} \frac{(q)_{n-m}}{(q)_j (q)_{n-m-j}}. \tag{19}$$

If we make this substitution for $(q)_{n+m}^{-1}$ in the left-hand side of (18), we have that

$$\begin{aligned} \sum_m \frac{x^m q^{am^2}}{(q)_{n-m}(q)_{n+m}} &= \sum_m \frac{x^m q^{am^2}}{(q)_{n-m}} \sum_j \frac{q^{j^2+2mj}(q)_{n-m}}{(q)_{j+2m}(q)_j(q)_{n-m-j}} \\ &= \sum_{m,j} \frac{q^{(m+j)^2}}{(q)_{n-m-j}} \frac{x^m q^{(a-1)m^2}}{(q)_j (q)_{2m+j}}. \end{aligned} \tag{20}$$

If, instead of summing over m and j, we sum over m and $s = m + j$, we have precisely the right-hand side of (18). This concludes the proof.

We now prove (10) by starting with the right-hand side and repeatedly applying Lemma 2 until the coefficient of m^2 is $\frac{1}{2}$:

$$
\begin{aligned}
(q)_{2N}^{-1} \sum_m x^m q^{(5m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix}
&= \sum_m \frac{(xq^{1/2})^m q^{5m^2/2}}{(q)_{N-m}(q)_{N+m}} \\
&= \sum_s \frac{q^{s^2}}{(s)_{N-s}} \sum_m \frac{(xq^{1/2})^m q^{3m^2/2}}{(q)_{s-m}(q)_{s+m}} \\
&= \sum_s \frac{q^{s^2}}{(q)_{N-s}} \sum_t \frac{q^{t^2}}{(q)_{s-t}} \sum_m \frac{(xq^{1/2})^m q^{m^2/2}}{(q)_{t-m}(q)_{t+m}}. \qquad (21)
\end{aligned}
$$

We can now sum the innermost sum using (6) with $N=t$ and both sides divided by $(q)_{2t}$ to get

$$
\begin{aligned}
(q)_{2N}^{-1} \sum_m x^m q^{(5m^2+m)/2} \begin{bmatrix} 2N \\ N-m \end{bmatrix}
&= \sum_{s,t} \frac{q^{s^2+t^2}}{(q)_{N-s}(q)_{s-t}} \frac{\prod_1^t (1+xq^m)(1+x^{-1}q^{m-1})}{(q)_{2t}}
\end{aligned}
$$

which is (10).

It is easily seen that the Theorem is proved in exactly the same manner, except that Lemma 2 is applied k times to the right-hand side instead of merely twice.

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