ADVANCES IN MATHEMATICS 78, 42–75 (1989)

Generalized Rogers–Ramanujan Bijections

DAVID M. BRESSOUD*

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

AND

DORON ZEILBERGER

Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104

1. INTRODUCTION

In [5] we presented a short combinatorial proof of the first Rogers–Ramanujan identity which states that for any $n \geqslant 1$ the partitions of n into parts with difference at least two are equinumerous with the partitions of n in which all parts are congruent to $\pm 1 \pmod 5$. As an example, there are six partitions of 10 with difference at least two and six partitions of 10 with parts congruent to $\pm 1 \pmod 5$. Our proof established the following bijection between these two sets of partitions of 10:

10	6+1+1+1+1
9+1	4+4+1+1
8+2	9+1
7+3	6+4
6+4	1+1+1+1+1+1+1+1+1+1
6+3+1	4+1+1+1+1+1+1

The bijection given in [5] was presented in a telegraphic style. It is the purpose of this paper to explain the process by which this bijection was obtained: a direct combinatorialization of the "easy" analytic proof of the Rogers–Ramanujan identities given in [4]. More than this, we shall give a much more general bijection which will be shown to be equivalent to Eq. (14) of [4] with N taken to $+\infty$,

$$\prod_{i=1}^{\infty} (1+xq^{(2k+1)i-k})(1+x^{-1}q^{(2k+1)i-k-1})(1-q^{(2k+1)i})(1-q^i)^{-1}$$
$$= \sum_{s_1, \ldots, s_k} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-1} \\ s_k \end{bmatrix} \frac{(-xq)_{s_k}(-x^{-1})_{s_k}}{(q^{s_k+1})_{s_k}}, \tag{1.1}$$

* Partially supported by the NSF under Grant MCS-8101943.

0001-8708/89 $7.50

where

$$(a)_s \equiv \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{i+s})} \tag{1.2}$$

and thus $s \in Z^+ \Rightarrow (a)_s = (1-a)(1-aq)\cdots(1-aq^{s-1})$, while $s \in Z^- \Rightarrow (a)_s^{-1} = 0$; $[{s \atop t}]$ is the Gaussian polynomial defined by

$$\begin{bmatrix} s \\ t \end{bmatrix} \equiv \frac{(q)_s}{(q)_t (q)_{s-t}} \tag{1.3}$$

and is zero unless $s \geqslant t \geqslant 0$; and the summation is over all integral values of the indices, although the contribution to the summation is zero unless $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k \geqslant 0$. Here and throughout this paper, $|q| < 1$.

In Section 2 we shall state a combinatorial theorem, Theorem 2.7, which we shall show to be equivalent to (1.1). A direct, bijective proof of Theorem 2.7 which parallels the analytic argument of [4] will be given in Section 3.

If, in (1.1), we set $x = -1$, then the contribution to the summation is zero unless $s_k = 0$ and we have as a corollary

$$\left(\prod_{\substack{i=1 \\ i \not\equiv 0, \pm k \,(\mathrm{mod}\, 2k+1)}}^{\infty} (1-q^i) \right)^{-1} = \sum_{s_1, \ldots, s_{k-1}} \frac{q^{s_1^2 + \cdots + s_{k-1}^2}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-2} \\ s_{k-1} \end{bmatrix}. \tag{1.4}$$

In particular, for $k = 2$ we have the familiar analytic form of the first Rogers–Ramanujan identity

$$\left(\prod_{i=0}^{\infty} (1-q^{5i+1})(1-q^{5i+4}) \right)^{-1} = \sum_s \frac{q^{s^2}}{(q)_s}. \tag{1.5}$$

In Section 4 we shall show what setting $x = -1$ means combinatorially and how to derive a bijective proof of the combinatorial statement equivalent to (1.4) from the bijective proof of Theorem 2.7.

Section 5 contains the bijective proof of (1.4) in its simplest, most algorithmic form. While the bijection is a condensed form of what has been developed in the previous three sections, this section has been written to stand on its own and includes an independent proof of the validity of the bijection. For $k = 2$, the bijection given here reduces to what is essentially the bijection given in [5]. A detailed example will be worked out in Section 6.

It should be emphasized that this work was both inspired and greatly aided by the first Rogers–Ramanujan bijection ever found, that discovered by A. M. Garsia and S.C. Milne [6]. Their involution principle which was the crucial insight making their bijection possible is also the heart of our bijection.

2. The Combinatorial Theorem

The product side of (1.1) can be interpreted as a partition generating function using well-known arguments [1, (2.1.1), (2.1.2)].

Lemma 2.1. *For positive integral k, we have that*

$$\prod_{i=1}^{\infty} (1+xq^{(2k+1)i-k})(1+x^{-1}q^{(2k+1)i-k-1})(1-q^{(2k+1)i})(1-q^i)^{-1} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} A_k(m,n)\, x^m q^n, \tag{2.1}$$

where $A_k(m,n)$ counts the number of colored partitions of n into red parts not divisible by $2k+1$ and distinct blue parts which are congruent to $\pm k \pmod{2k+1}$ such that $m = (\#$ of blue parts $\equiv -k \pmod{2k+1}) - (\#$ of blue parts $\equiv k \pmod{2k+1})$.

To interpret the right side of (1.1), we shall need the notion of *successive Durfee squares* as defined by Andrews [2].

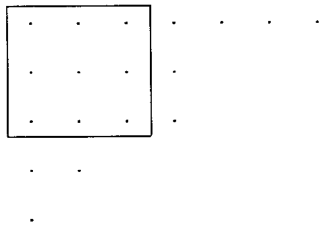
Definition 2.2. The *Ferrers graph* of a partition is a display of the partition in which each part is represented by a row of nodes. The rows are left justified and are placed in decreasing order from top to bottom.

Example. The Ferrers graph for $7+4+4+2+1$ is

$$\begin{array}{lllllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & & & & & \\
\cdot & & & & & &
\end{array}$$

Definition 2.3. The *Durfee square* of the Ferrers graph of a partition is the largest suqare of nodes in the upper left corner of the Ferrers graph.

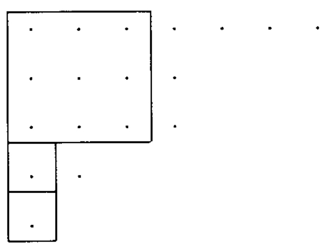
EXAMPLE. The Durfee square of the Ferrers graph given above is the 3×3 square in the upper left corner.



Remark. If the Durfee square is removed from a Ferrers graph, it leaves two smaller partitions: what is left of the rows strictly to the right of the Durfee square (in the example, this is the parts $4+1+1$), and what lies strictly below the Durfee square (in the example, $2+1$).

DEFINITION 2.4. The first Durfee square is the Durfee square. For $j > 1$, the jth Durfee suqare is the Durfee square of that portion of the Ferrers graph which lies strictly below the $(j-1)$st Durfee square.

EXAMPLE. The second and third Durfee squares of the Ferrers graph given above are both 1×1 squares. For $j > 3$, the jth Durfee square is a 0×0 square.



LEMMA 2.5. *Let $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k \geqslant 0$, then*

$$\frac{q^{s_1^2+s_2^2+\cdots+s_k^2}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-1} \\ s_k \end{bmatrix} = \sum_{n=0}^{\infty} d_k(n)\, q^n, \tag{2.2}$$

where $d_k(n)$ is the number of partitions of n such that for $1 \leqslant i \leqslant k$, the ith Durfee square of the Ferrers graph of the partition is an $s_i \times s_i$ square, and there are no parts below the kth Durfee square.

Proof. The term $q^{s_1^2+s_2^2+\cdots+s_k^2}$ generates the k Durfee squares of the required sizes. $(q)_{s_1}^{-1}$ generates partitions into at most s_1 parts, giving us the parts to the right of the first Durfee square. For $2 \leqslant i \leqslant k$, parts to the right of the ith Durfee square are bounded in magnitude by $s_{i-1} - s_i$ and in number by s_i. The generating function for such partitons is $\left[{s_{i-1} \atop s_i}\right]$, as will be proved in Section 3, or see [1, (3.2.1)].

LEMMA 2.6. *For positive integral k, we have that*

$$\sum_{s_1, \ldots, s_k} \frac{q^{s_1^2+s_2^2+\cdots+s_k^2}}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-1} \\ s_k \end{bmatrix} \frac{(-xq)_{s_k}(-x^{-1})_{s_k}}{(q^{s_k+1})_{s_k}}$$
$$= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} D_k(m, n)\, x^m q^n, \tag{2.3}$$

where $D_k(m, n)$ is the number of colored partitions of n into (1) red parts with nothing below the kth Durfee square whose size is, say, s_k, plus (2) yellow parts strictly larger than s_k and less than or equal to $2s_k$, plus (3) distinct green parts greater than or possibly including zero and strictly less than s_k, plus (4) distinct blue parts greater than or equal to 1 and less than or equal to s_k; such that $m = (\#$ of blue parts$) - (\#$ of green parts$)$.

Lemma 2.6 follows from Lemma 2.5 and the usual interpretations for $(-xq)_{s_k}$, $(-x^{-1})_{s_k}$ and $(q^{s_k+1})_{s_k}$ [1, 2.1.1), (2.1.2)].

Lemmas 2.1 and 2.6 establish that the following combinatorial theorem is equivalent to (1.1).

THEOREM 2.7. *Let m be an integer and let k and n be positive integers, then*

$$A_k(m, n) = D_k(m, n), \tag{2.4}$$

where $A_k(m, n)$ is defined in Lemma 2.1 and $D_k(m, n)$ is defined in Lemma 2.6.

3. PROOF OF THEOREM 2.7

The analytic proof of (1.1) can be broken into three steps:

$$\prod_{i=1}^{\infty} \frac{(1+x^i q^{(2k+1)i-k})(1+x^{-i} q^{(2k+1)i-k-1})(1-q^{(2k+1)i})}{(1-q^i)}$$
$$= \sum_m \frac{x^m q^{((2k+1)m^2+m)/2}}{(q)_\infty}$$

$$= \sum_{s_1, \ldots, s_k} q^{s_1^2 + s_2^2 + \cdots + s_k^2} \frac{1}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-1} \\ s_k \end{bmatrix} \frac{1}{(q^{s_k+1})_{s_k}}$$

$$\times \sum_m x^m q^{(m^2+m)/2} \begin{bmatrix} 2s_k \\ s_k - m \end{bmatrix}$$

$$= \sum_{s_1, \ldots, s_k} q^{s_1^2 + \cdots + s_k^2} \frac{1}{(q)_{s_1}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} s_{k-1} \\ s_k \end{bmatrix} \frac{(-xq)_{s_k}(-x^{-1})_{s_k}}{(q^{s_k+1})_{s_k}}, \quad (3.1)$$

all summations being over all integral values.

The first equality is Jacobi's Triple Product Identity [1, Theorem 2.8, p. 21, $q \to q^{(2k+1)/2}$, $z \to x$]. The third equality is Cauchy's Finite Form of the Triple Product Identity [1, Ex. 1, p. 49, $q \to q^{1/2}$, $x \to xq^{-1/2}$]. Both of these will be proved by a direct bijection discovered by Sylvester [9, Sects. 37–40] and also explained by MacMahon [7, Sect. 323, pp. 72–75].

The second equality follows from repeated applications of Lemma 2 in [4]:

$$\sum_j \frac{x^j q^{aj^2}}{(q)_{n-j}(q)_{n+j}} = \sum_s \frac{q^{s^2}}{(q)_{n-s}} \sum_j \frac{x^j q^{(a-1)j^2}}{(q)_{s-j}(q)_{s+j}}. \quad (3.2)$$

The first time (3.2) is applied, n is taken to be $+\infty$. Equation (3.2) itself is a consequence of two equalities

$$(q)_{n+j}^{-1} = \sum_i \frac{q^{i^2+2ij}}{(q)_{i+2j}} \begin{bmatrix} n-j \\ i \end{bmatrix}, \quad (3.3)$$

$$(q)_{n-j}^{-1} \begin{bmatrix} n-j \\ i \end{bmatrix} = (q)_i^{-1}(q)_{n-i-j}^{-1}. \quad (3.4)$$

Sylvester [9, Sect. 34] and MacMahon [7, Sect. 265] have given direct bijective proofs of (3.3). While (3.4) is trivial analytically, it is far from trivial combinatorially if $\left[\begin{smallmatrix} A+B \\ A \end{smallmatrix}\right]$ is to be interpreted as the generating function for partitions bounded in numbers by B and in magnitude by A. The proof we give here, found by D.Z., is entirely new.

PROPOSITION 3.1. *There exists a one-to-one correspondence between colored partitions of n into at most $A+B$ red parts plus blue parts bounded in number by B and in magnitude by A and colored partitions of n into at most A red parts and at most B blue parts. Furthermore, if we let $A \geqslant \lambda(1) \geqslant \lambda(2) \geqslant \cdots \geqslant \lambda(B) \geqslant 0$ be the blue parts bounded in number by B and in magnitude by A (if there are fewer than B parts we admit zeroes to get exactly B parts), if we let $\mu(1) \geqslant \cdots \geqslant \mu(A+B) \geqslant 0$ be the red parts* *bounded in number by* $A+B$, $\rho(1)\geqslant\rho(2)\geqslant\cdots\geqslant\rho(B)\geqslant 0$ *be the blue parts bounded in number by* B, *and* $\sigma(1)\geqslant\cdots\geqslant\sigma(A)\geqslant 0$ *be the red parts bounded in number by* A, *then the correspondence described above is given explicitly by*

$$\rho(i)=\lambda(i)+\mu(A-\lambda(i)+i),\qquad 1\leqslant i\leqslant B, \tag{3.5}$$

$$\sigma(j)=\mu(k_j),\qquad 1\leqslant j\leqslant A, \tag{3.6}$$

where k_j *is the* jth *smallest element of* $\{1,2,...,A+B\}-\{A-\lambda(i)+i\mid 1\leqslant i\leqslant B\}$. *In the other direction we have that*

$$\lambda(i)=\min\{0\leqslant x\leqslant A\mid \rho(i)-x\leqslant\sigma(A-x)\},\qquad 1\leqslant i\leqslant B,\ \sigma(0)\equiv+\infty, \tag{3.7}$$

$$\begin{aligned}\mu(j)=j\text{th } \textit{largest part in the multiset } &\{\sigma(j)\mid 1\leqslant j\leqslant A\}\\ &\cup\{\rho(i)-\lambda(i)\mid 1\leqslant i\leqslant B\},\qquad 1\leqslant j\leqslant A+B,\end{aligned} \tag{3.8}$$

the union taken to be a multiset union, meaning that the number of times an element appears in the union equals the sum of the number of times it appears in each set.

Since the generating function for paritions into at most M parts is $(q)_M^{-1}$, it will follow from Proposition 3.1 that

COROLLARY 3.2. *The generating function for partitions bounded in number by B and in magnitude by A is given by*

$$\frac{(q)_{A+B}}{(q)_A(q)_B}=\begin{bmatrix}A+B\\A\end{bmatrix}=\begin{bmatrix}A+B\\B\end{bmatrix}.$$

Proof of Proposition 3.1. We begin with a colored partition of n into parts $\lambda(i)$ and $\mu(j)$ as described in the proposition. We note that $i>j$ implies that $\lambda(i)\leqslant\lambda(j)$ and thus $(A-\lambda(i)+i)>(A-\lambda(j)+j)$. If we define $\rho(i)$, $1\leqslant i\leqslant B$, and $\sigma(j)$, $1\leqslant j\leqslant A$, by (3.5) and (3.6), then $i>j$ implies that $\rho(i)\leqslant\rho(j)$ and that there are precisely A values for the $\sigma(j)$ (although some of them might be zero). It is easily seen from (3.5) and (3.6) that

$$\sum_{i=1}^{A}\rho(i)+\sum_{j=1}^{B}\sigma(j)=\sum_{i=1}^{A}\lambda(i)+\sum_{j=1}^{A+B}\mu(j)=n. \tag{3.9}$$

As an example, if $A=5$ and $B=3$ and

$$\begin{array}{ll} \lambda(1)=4 & \mu(1)=10 \\ \lambda(2)=2 & \mu(2)=10 \\ \lambda(3)=0 & \mu(3)=5 \\ & \mu(4)=3 \\ & \mu(5)=2 \\ & \mu(6)=1 \\ & \mu(7)=1 \\ & \mu(8)=0, \end{array}$$

then the correspondence yields

$$\begin{array}{ll} \rho(1)=4+10=14 & \sigma(1)=10 \\ \rho(2)=2+2=4 & \sigma(2)=5 \\ \rho(3)=0+0=0 & \sigma(3)=3 \\ & \sigma(4)=1 \\ & \sigma(5)=1. \end{array}$$

We shall next show that this algorithm is uniquely reversed by (3.7) and (3.8). If we have parts $\rho(i)$, $1 \leqslant i \leqslant B$, and $\sigma(j)$, $1 \leqslant j \leqslant A$, which have been defined by (3.5) and (3.6) from parts $\lambda(i) \leqslant A$, $1 \leqslant i \leqslant B$, and $\mu(j)$, $1 \leqslant j \leqslant A+B$, we shall show that these parts satisfy (3.7) and (3.8).

By the definition of k_j given in (3.6), we see that

$$k_{A-\lambda(i)} \leqslant A-\lambda(i)+i \leqslant k_{A-\lambda(i)+1}. \tag{3.10}$$

Using these inequalities in (3.6) gives us

$$\sigma(A-\lambda(i)+1) \leqslant \mu(A-\lambda(i)+i) \leqslant \sigma(A-\lambda(i)). \tag{3.11}$$

We now apply these inequalities to (3.5) to get

$$\sigma(A-\lambda(i)+1) \leqslant \rho(i)-\lambda(i) \leqslant \sigma(A-\lambda(i)). \tag{3.12}$$

This last pair of inequalities is equivalent to (3.7):

$$\lambda(i)=\min\{0 \leqslant x \leqslant A \mid \rho(i)-x \leqslant \sigma(A-x)\}.$$

Thus, if the $\rho(i)$ and $\sigma(j)$ can be defined in terms of $\lambda(i)$ and $\mu(j)$ by (3.5) and (3.6) then the $\lambda(i)$ are uniquely determined by (3.7).

Once the $\lambda(i)$ have been determined, the $\mu(j)$ are uniquely determined, the multiset $\{\mu(j) \mid 1 \leqslant j \leqslant A+B\}$ being the multiset union

$$\{\sigma(j) \mid 1 \leqslant j \leqslant A\} \cup \{\rho(i)-\lambda(i) \mid 1 \leqslant i \leqslant B\},$$

and this is how the $\mu(j)$ are defined in (3.8).

Finally, if we are given the $\rho(i)$ and $\sigma(j)$ and if we define $\lambda(i)$ and $\mu(j)$ by (3.7) and (3.8), respectively, and set

$$\rho'(i) = \lambda(i) + \mu(A - \lambda(i) + i),$$
$$\sigma'(j) = \mu(k_j),$$

then, by (3.7),

$$\sigma(A - \lambda(i)) \geqslant \rho(i) - \lambda(i) \geqslant \sigma(A - \lambda(i) + 1). \tag{3.13}$$

Since $i < j$ implies that $\rho(i) - \lambda(i) \geqslant \rho(j) - \lambda(j)$, (3.13) guarantees that the $(A - \lambda(i) + i)$th largest element of the multiset union

$$\{\sigma(j) \mid 1 \leqslant j \leqslant A\} \cup \{\rho(i) - \lambda(i) \mid 1 \leqslant i \leqslant B\}$$

is equal to $\rho(i) - \lambda(i)$. That is to say, by (3.8),

$$\mu(A - \lambda(i) + i) = \rho(i) - \lambda(i).$$

This implies that $\rho'(i) = \rho(i)$ and $\sigma'(j) = \sigma(j)$, concluding the proof of Proposition 3.1.

We shall now prove Theorem 2.7, beginning with a combinatorial statement and proof of the first equality in (3.1). After each step of the bijection we shall illustrate what it does to the particular colored partition of $A_3(2, 373)$ ($k = 3$, $m = 2$, $n = 373$) whose red parts are given by $19 + 17 + 17 + 16 + 15 + 13 + 11 + 11 + 10 + 6 + 5 + 5 + 4 + 3 + 3 + 2 + 2 + 1 + 1 + 1$ and whose blue parts are given by $53 + 39 + 32 + 31 + 25 + 17 + 11 + 3$.

LEMMA 3.3. *Given integral m and positive integral k and n, there exists a one-to-one correspondence between colored partitions counted by $A_k(m, n)$ (see (2.1)) and colored partitions of n into arbitrary red parts plus blue parts lying in an arithmetic progression with difference $2k + 1$ and smallest part either k or $k + 1$. If $m \geqslant 0$, then there are m blue parts and the smallest is $k + 1$. If $m < 0$, then there are $-m$ blue parts, and the smallest is k.*

Proof of Lemma 3.3. Following Sylvester [9, Sects. 37–40], we begin with the blue parts in the colored partition counted by $A_k(m, n)$. Let us first assume that (# of blue parts $\equiv -k$)$-$(# of blue parts $\equiv +k$)$=$ $m \geqslant 0$. Let $a(1) > a(2) > \cdots > a(m) > a(m+1) > \cdots > a(m+t)$ be the blue parts $\equiv -k \pmod{2k+1}$, and let $b(1) > b(2) > \cdots > b(t)$ be the blue parts $\equiv k \pmod{2k+1}$, $t \geqslant 0$.

For $1 \leqslant i \leqslant t$, we define

$$\alpha(m+i) = a(m+i) + k, \tag{3.14}$$

$$\beta(i) = b(i) - k. \tag{3.15}$$

For $1 \leqslant j \leqslant m$, we set

$$\alpha(j) = a(j) - ((m-j)(2k+1) + k + 1). \tag{3.16}$$

We now have that $2k+1$ divides all of the $\alpha(i)$, $1 \leqslant i \leqslant m+t$, and all of the $\beta(i)$, $1 \leqslant i \leqslant t$. Also, we see that

$$\alpha(1) \geqslant \alpha(2) \geqslant \cdots \geqslant \alpha(m) \geqslant \alpha(m+1) > \alpha(m+2) > \cdots > \alpha(m+t) > 0$$

(if $t=0$, $\alpha(m) \geqslant 0$), and

$$\beta(1) > \beta(2) > \cdots > \beta(t) \geqslant 0.$$

Most importantly, we have that

$$\sum_{i=1}^{m+t} a(i) + \sum_{i=1}^{t} b(i)$$
$$= \sum_{i=1}^{m+t} \alpha(i) + \sum_{i=1}^{t} \beta(i) + \sum_{j=1}^{m} ((j-1)(2k+1) + k + 1). \tag{3.17}$$

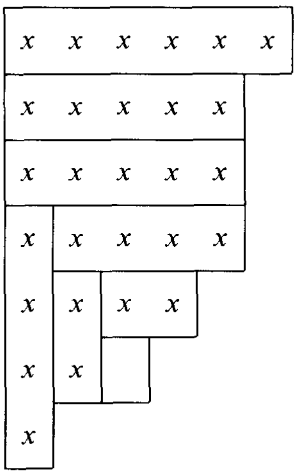
We have thus pulled out our arithmetic series of m blue parts with difference $2k+1$ and smallest part $k+1$.

Example. We have that $k=3$ and so $2k+1=7$. Among the blue parts, five are congruent to $-3 \pmod 7$: $a(1)=53$, $a(2)=39$, $a(3)=32$, $a(4)=25$, $a(5)=11$; and three are congruent to $3 \pmod 7$: $b(1)=31$, $b(2)=17$, $b(3)=3$. This yields $\alpha(1)=42$, $\alpha(2)=35$, $\alpha(3)=35$, $\alpha(4)=28$, $\alpha(5)=14$; $\beta(1)=28$, $\beta(2)=14$, $\beta(3)=0$; plus the arithmetic series of length 2: $4+11$.

It now only remains to demonstrate a bijection between our $\alpha(i)$ and $\beta(j)$ on the one hand and red parts divisible by $2k+1$ on the other.

Each $\alpha(i)$ and $\beta(j)$ can be replaced by a row of x's, each x being counted as $2k+1$ (e.g., $7(2k+1)$ is represented as $xxxxxxx$). The rows of x's are placed in a diagram as follows: We first place the row representing $\alpha(1)$, below that the row representing $\alpha(2)$, ..., below that the row representing $\alpha(m+1)$. The x's representing $\beta(1)$ are now placed in a column directly below the first x in the representation of $\alpha(m+1)$. The x's representing $\alpha(m+2)$ are placed in a row directly to the right of the first x in the representation of $\beta(1)$. We continue this alternating placement. In general, the x's representing $\beta(i)$ are placed in a column directly below the first x in the representation of $\alpha(m+1)$, the x's representing $\alpha(m+i+1)$ are placed in a row directly to the right of the first x in the representation of $\beta(i)$.

EXAMPLE. We have that $\alpha(1)=6\cdot 7$, $\alpha(2)=5\cdot 7$, $\alpha(3)=5\cdot 7$, $\alpha(4)=4\cdot 7$, $\alpha(5)=2\cdot 7$; $\beta(1)=4\cdot 7$, $\beta(2)=2\cdot 7$, $\beta(3)=0\cdot 7$. The resulting diagram is



The diagram is read by rows, each x representing $2k+1$, to yield red parts whose only restriction is that they are divisible by $2k+1$. The original blue parts are uniquely recoverable from the arithmetic progression and the red parts divisible by $2k+1$ since the length of the series tells us how far down to read the rows of parts idivisble by $2k+1$ before we start decomposing those parts by both rows and columns.

EXAMPLE. Our colored partition now has red parts: 42 + 35 + 35 + 35 + 28 + 19 + 17 + 17 + 16 + 15 + 14 + 13 + 11 + 11 + 10 + 7 + 6 + 5 + 5 + 4 + 3 + 3 + 2 + 2 + 1 + 1 + 1; and blue parts: $4+11$.

If $m<0$, we do exactly the same procedure except that the roles of the $a(i)$ and $b(j)$ are reversed and we have that

$$\alpha(i)=a(i)-k-1, \qquad 1\leqslant i\leqslant t, \tag{3.18}$$

$$\beta(-m+i)=b(-m+i)+k+1, \qquad 1\leqslant i\leqslant t, \tag{3.19}$$

$$\beta(j)=b(j)-((m-j)(2k+1)+k), \qquad 1\leqslant j\leqslant -m. \tag{3.20}$$

This concludes the proof of Lemma 3.3 since this procedure is uniquely reversible.

The next lemma, which gives us the combinatorial interpretation and proof of the second equality in (3.1), relies very heavily on the bijection described in Proposition 3.1.

LEMMA 3.4. *Given integral m and positive integral k and n, there exists a one-to-one correspondence between colored partions of n into arbitrary red parts plus blue parts lying in an arithmetic progression of length $|m|$ with difference $2k+1$ and smallest part $k+1$ if $m\geqslant 0$, or k if $m<0$ and colored partitions of n into red parts with nothing below the* kth *Durfee square whose size is, say, s_k, plus yellow parts strictly larger than s_k and less than or equal to $2s_k$, plus blue parts lying in an arithmetic progression of length $|m|$ with difference 1 and smallest part 1 if $m\geqslant 0$, or 0 if $m<0$, plus green parts bounded in number by s_k+m and in magnitude by s_k-m.*

Proof of Lemma 3.4. We begin with the colored partition into only red and blue parts, and consider the Ferrers graph for the red parts. From the upper left corner of this graph, we remove the largest *rectangle* of nodes whose horizontal length exceeds its vertical width by $2m$ (equivalently, whose vertical width exceeds its horizontal length by $-2m$). Let n_1 be the vertical width of this rectangle. The length and width of this rectangle are always taken to be nonnegative, but may be zero. In particular, if the largest such rectangle is empty, then $n_1=0$ if $m\geqslant 0$ and $n_1=-2m$ if $m<0$. What remains of the rows to the right of this rectangle are recolored a shade of orange we denote by orange 1. These parts are bounded in number by n_1. Below the rectangle we still have red parts, now bounded in magnitude by n_1+2m.

From the graph of the remaining red parts, we again remove the largest rectangle whose length exceeds its width by $2m$. We let n_2 be the width of this rectangle. The parts to the right of this rectangle are recolored orange 2. The orange 2 parts are bounded in number by n_2 and in magnitude by n_1-n_2. The remaining red parts are bounded in magnitude by n_2+2m.

We continue in this manner until we have removed k rectangles whose length exceeds their width by $2m$. Some or all of these might be empty. We have parts in k shades of orange, orange 1 through orange k, and for $2\leqslant i\leqslant k$ the parts colored orange i are bounded in number by n_i and in magnitude by $n_{i-1} - n_i$, where n_i, $1 \leqslant i \leqslant k$, is the width of the ith rectangle. The remaining red parts are bounded in magnitude by $n_k + 2m$.

EXAMPLE. In the Ferrers graph of our red parts, which we do not explicitly include here because of limitations on space, the largest rectangle for which the length exceeds the width by $2m = 4$ is a 10×14 rectangle, and the orange 1 parts are $28 + 21 + 21 + 21 + 14 + 5 + 3 + 3 + 2 + 1$. Below this, the largest such rectangle is a 5×9 rectangle, and the orange 2 parts are $5 + 4 + 2 + 2 + 1$. Below this, the largest such rectangle is a 2×6 rectangle, and the orange 3 parts are $1 + 0$. The remaining red parts are $5 + 5 + 4 + 3 + 3 + 2 + 2 + 1 + 1 + 1$. We have that $n_1 = 10$, $n_2 = 5$, and $n_3 = 2$.

We now transform the orange and red parts. This is the key step of our bijection, and we have named it "Ringing the Changes."

Ringing the Changes, Part I: Going Down. The orange 1 parts are bounded in number by n_1, while the orange 2 parts are bounded in number by n_2 and in magnitude by $n_1 - n_2$. Using the algorithm described in Proposition 3.1, we have a bijection between these and orange 1 parts bounded in number by $n_1 - n_2$ plus orange 2 parts bounded only in number by n_2. We now take the orange 2 parts bounded in number by n_2 and the orange 3 parts bounded in number by n_3 and in magnitude by $n_2 - n_3$, and we use the same bijection to obtain orange 2 parts bounded in number by $n_2 - n_3$ plus orange 3 parts bounded in number by n_3. This procedure is continued until for all i, $1 \leqslant i \leqslant k-1$, the orange i parts are bounded in number by $n_i - n_{i+1}$ and unbounded in magnitude. The orange k parts are now bounded in number by n_k and unbounded in magnitude.

Since the red parts are bounded in magnitude by $n_k + 2m$, we can conjugate the Ferrers graph of the red parts to get red parts bounded in number by $n_k + 2m$. We consider these together with the orange k parts bounded in number by n_k. By the inverse to the bijection used above, we can transform these into orange k parts which are bounded in number by $2n_k + 2m$, plus red parts bounded in number by $n_k + 2m$ and in magnitude by n_k. These last red parts we recolor green.

Thus, at the end of the first part of Ringing the Changes, we have orange i parts, $1 \leqslant i \leqslant k-1$, which are bounded in number by $n_i - n_{i+1}$, plus orange k parts bounded in number by $2n_k + 2m$, plus green parts bounded in number by $n_k + 2m$ and in magnitude by n_k.

EXAMPLE. Each of the columns below consists of all of the orange and red and, in the last column, green parts at each step of Ringing the Changes, Going Down. Parts of different colors or shades are separated by a horizontal line.

$$
\begin{array}{ccccccccc}
28 & & 21 & & 21 & & 21 & & 21 \\
21 & & 21 & & 21 & & 21 & & 21 \\
21 & & 14 & & 14 & & 14 & & 14 \\
21 & & 3 & & 3 & & 3 & & 3 \\
14 & & \underline{1} & & \underline{1} & & \underline{1} & & \underline{1} \\
5 & & 33 & & 33 & & 33 & & 33 \\
3 & & 25 & & 25 & & 25 & & 25 \\
3 & \longrightarrow & 7 & \longrightarrow & \underline{5} & \longrightarrow & \underline{5} & \longrightarrow & \underline{5} \\
2 & & 5 & & 8 & & 8 & & 8 \\
\underline{1} & & \underline{3} & & \underline{3} & & \underline{3} & & 8 \\
5 & & 1 & & 5 & & 10 & & 6 \\
4 & & \underline{0} & & 5 & & 7 & & 4 \\
2 & & 5 & & 4 & & 5 & & 3 \\
2 & & 5 & & 3 & & 3 & & 3 \\
\underline{1} & & 4 & & 3 & & 2 & & 2 \\
1 & & 3 & & 2 & & 0 & & \underline{0} \\
\underline{0} & & 3 & & 2 & & & & 2 \\
5 & & 2 & & 1 & & & & 1 \\
5 & & 2 & & 1 & & & & 1 \\
4 & & 1 & & 1 & & & & 0 \\
3 & & 1 & & & & & & 0 \\
3 & & 1 & & & & & & 0 \\
2 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
1 & & & & & & & & \\
1 & & & & & & & &
\end{array}
$$

At the end of Ringing the Changes Going Down we have $10-5=5$ orange 1 parts: $21+21+14+3+1$; $5-2=3$ orange 2 parts: $33+25+5$; $2\cdot 2+2\cdot 2=8$ orange 3 parts: $8+8+6+4+3+3+2+0$; plus green parts bounded in number by $2+2\cdot 2=6$ and in magnitude by 2: $2+1+1+0+0+0$.

Ringing the Changes, Part II: Coming Up. We conjugate the Ferrers graph of the orange k parts to get orange k parts bounded in magnitude by $2n_k+2m$. Those parts which are strictly larger than n_k+m are recolored yellow. We conjugate the remaining orange k parts so that we now have yellow parts strictly larger than n_k+m and less than or equal to $2n_k+2m$, plus orange k parts bounded in number by n_k+m.

EXAMPLE. The orange 3 parts are $8+8+6+4+3+3+2+0$ which, when the partition is conjugated, become $7+7+6+4+3+3+2+2$. The parts $7+7+6$ are recolored yellow. The conjugate partition to $4+3+3+2+2$ is $5+5+3+1$.

We now consider the orange k parts together with the orange $(k-1)$ parts which are bounded in number by $n_{k-1}-n_k$. Using the algorithm in the proof of Proposition 3.1, we have a bijection between these and orange $(k-1)$ parts bounded in number by $n_{k-1}+m$ plus orange k parts bounded in number by n_k+m and in magnitude by $n_{k-1}-n_k$. We next take the orange $(k-1)$ parts with the orange $(k-2)$ parts which are bounded in number by $n_{k-2}-n_{k-1}$. By the same bijection as above, we get orange $(k-2)$ parts bounded in number by $n_{k-2}+m$ plus orange $(k-1)$ parts bounded in number by $n_{k-1}+m$ and in magnitude by $n_{k-2}-n_{k-1}$. This procedure is continued until for all i, $2\leqslant i\leqslant k$, the orange i parts are bounded in number by n_i+m and in magnitude by $n_{i-1}-n_i$, and the orange 1 parts are bounded in number by n_1+m.

Thus, at the end of the second and last part of Ringing the Changes, we have orange 1 parts bounded in number by n_1+m plus orange i parts, $2\leqslant i\leqslant k$, bounded in number by n_i+m and in magnitude by $n_{i-1}-n_i$, plus yellow parts strictly greater than n_k+m and less than or equal to $2n_k+2m$, plus green parts bounded in number by n_k+2m and in magnitude by n_k.

EXAMPLE. Each of the columns below consists of all of the orange parts at each step of Ringing the Changes Going Up after the yellow parts have been removed. Since the yellow and green parts are unchanged, they are not included here. Parts of different shades are separated by a horizontal line.

21	21	28
21	21	21
14	14	21
3	3	21
$\underline{1}$	$\underline{1}$	14
33	33	3
25	25	3
$\underline{5}$	5	3
5	5	3
5	5	2
3	3	1
1	$\underline{1}$	$\underline{1}$
	0	5
	0	4
	0	2
	0	2
		2
		1
		$\underline{0}$
		0
		0
		0
		0

Once we have rung the changes, the only thing left to do is to piece things back together: the blue parts lying in an arithmetic progression with difference $2k+1$, the k rectangles of width n_i and length n_i+2m, $1 \leqslant i \leqslant k$, the orange i parts, $1 \leqslant i \leqslant k$, the yellow parts and the green parts.

We take the blue parts and subtract k from the smallest, $3k$ from the second smallest, ..., $(2i-1)k$ from the ith smallest, ..., $(2m-1)k$ from the largest. This leaves us with blue parts lying in an arithmetic progression of length $|m|$ with difference 1 and smallest part 1 if $m \geqslant 0$, 0 if $m < 0$. The amount we have subtracted, $k + 3k + 5k + \cdots + (2m-1)k$, equals km^2. We add m^2 nodes to each of the n_i by $n_i + 2m$ rectangles to get k squares, each $n_i + m$ by $n_i + m$, $1 \leqslant i \leqslant k$.

For $1 \leqslant i \leqslant k$, the orange i parts fit precisely to the right of the $n_i + m$ by $n_i + m$ square, and we obtain a partition, now recolored red, whose ith Durfee square is an $n_i + m$ by $n_i + m$ square and which has no parts below the kth Durfee square.

Setting $s_k = n_k + m =$ size of kth Durfee square, we see that we have the red, yellow, blue, and green parts needed for our bijection. Since each step of this procedure was uniquely reversible, the bijection is established and the lemma is proved.

EXAMPLE. We have blue parts: $4+11$; three rectangles: 10×14, 5×9, and 2×6; $10+2=12$ orange 1 parts: $28+21+21+21+14+3+3+3+3+2+1+1$; $5+2=7$ orange 2 parts: $5+4+2+2+2+1+0$ bounded in magnitude by $10-5=5$; $2+2=4$ orange 3 parts: $0+0+0+0$ bounded in magnitude by $5-2=3$; yellow parts strictly greater than 4 and less than or equal to 8: $7+7+6$; and green parts bounded in number by 6 and in magnitude by 2: $2+1+1+0+0+0$. The blue parts become $1+2$. The reconstructed red parts are $40+33+33+33+26+15+15+15+15+14+13+13+12+11+9+9+9+8+7+4+4+4+4$. The yellow and green parts are unchanged. We have that $s_3 = 4$.

This third lemma, giving a combinatorial interpretation and proof of the third equality of (3.1) and so completing the bijective proof of Theorem 2.7, is proved with only minor modifications on the proof of Lemma 3.3.

LEMMA 3.5. *Given integral m and positive integral k and n, there exists a one-to-one correspondence between colored partitions of n into red parts with nothing below the kth Durfee square whose size is, say, s_k, plus yellow parts strictly larger than s_k and less than or equal to $2s_k$, plus blue parts lying in an aritmetic progression of length $|m|$ with difference 1 and smallest part 1 if $m \geqslant 0$, or 0 if $m < 0$, plus green parts bounded in number by $s_k + m$ and in magnitude by $s_k - m$ and colored partitions of n which are counted by $D_k(m, n)$ (see (2.3)).*

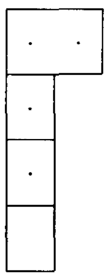
Proof of Lemma 3.5. It is clear that we do not want to touch the red or yellow parts. Let us begin with the blue parts in arithmetic progression and the green parts bounded in both number and magnitude. For the present, we shall assume that $m \geqslant 0$.

From the Ferrers graph for the green parts, we inductively define new blue parts, $\alpha(i)$, and new green parts, $\beta(j)$, as follows: For $1 \leqslant i \leqslant m+1$, $\alpha(i) = i$th row of the Ferrers graph. For $j \geqslant 1$, $\beta(j) =$ length of the column of nodes directly below the leftmost node in the representation of $\alpha(m+j)$; $\alpha(m+j+1) =$ length of row of nodes directly to the right of the top most node in the representation of $\beta(j)$. We continue until we reach the smallest t such that $\alpha(m+t+1) = 0$. Our blue parts are the

$$\begin{aligned} s_k - m \geqslant \alpha(1) \geqslant \alpha(2) \geqslant \cdots \geqslant \alpha(m) \\ \geqslant \alpha(m+1) > \alpha(m+2) > \cdots > \alpha(m+t) > 0 \end{aligned}$$

and our green parts are $s_k > \beta(1) > \beta(2) > \cdots > \beta(t) \geqslant 0$.

EXAMPLE. Since $m = 2$, there Ferrers graph for the green parts



yields $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 1$, $\beta(1) = 0$.

We now set

$$a(i) = \begin{cases} \alpha(i) + m - i + 1, & 1 \leqslant i \leqslant m, \\ \alpha(i), & m+1 \leqslant i \leqslant m+t, \end{cases} \tag{3.21}$$

$$b(j) = \beta(j), \qquad 1 \leqslant j \leqslant t. \tag{3.22}$$

Thus $s_k \geqslant a(1) > a(2) > \cdots > a(m+t) > 0$ and $s_k > b(1) > b(2) > \cdots > b(t) \geqslant 0$, which are the desired blue and green parts.

EXAMPLE. The blue parts will be $1 + 2 + 4$ and the green parts will be the single part 0.

If $m < 0$, we first conjugate the Ferrers graph of the original green parts to get parts bounded in number by $s_k - m$ and in magnitude by $s_k + m$. We read off the parts as before except that parts read from rows are colored green while parts from columns are colored blue and m is replaced by $-m$. This gives us green parts $s_k+m \geqslant \beta(1) \geqslant \cdots \geqslant \beta(-m) \geqslant \beta(-m+1) > \beta(-m+2) > \cdots > \beta(-m+t) > 0$, and blue parts $s_k > \alpha(1) > \cdots > \alpha(t) \geqslant 0$. We shall have the desired blue and green parts if we now set

$$b(i) = \begin{cases} \beta(i) - m - i, & 1 \leqslant i \leqslant -m, \\ \beta(i) - 1, & -m+1 \leqslant i \leqslant -m+t, \end{cases} \tag{3.23}$$

$$a(j) = \alpha(j) + 1, \qquad 1 \leqslant j \leqslant n. \tag{3.24}$$

This procedure is a bijection and so we have proved Lemma 3.5. This concludes our bijective proof of Theorem 2.7.

EXAMPLE. The colored partition counted by $D_3(2, 373)$ has red parts: $40 + 33 + 33 + 33 + 26 + 5 + 15 + 15 + 5 + 14 + 3 + 13 + 12 + 1 + 9 + 9 + 9 + 8 + 7 + 4 + 4 + 4 + 4$; $s_3 = 4$; yellow parts: $7+7+6$; green parts: 0; and blue parts: $1+2+4$.

4. THE COMBINATORICS OF $x = -1$

As described in the Introduction, when x is set equal to -1 in (1.1) we get the corollary (1.4). This corollary has the following very appealing interpretation:

PROPOSITION 4.1. *For positive integers k and n, the partitions of n in which* no *parts are congruent to* 0 *or* $\pm k \pmod{2k+1}$ *are equinumerous with the partitions of n which have no parts below the $(k-1)$st Durfee square.*

Remark 4.2. When $k=2$, this proposition is equivalent to the first Rogers–Ramanujan identity since there is a very natural one-to-one correspondence between partitions with difference at least two and partitions with no parts below the Durfee square, i.e., partitions for which the smallest part is at least as large as the number of parts. If $\lambda(1) > \lambda(2) > \cdots > \lambda(t) > 0$ are the parts with difference at least two, $i < j \Rightarrow \lambda(i) \geqslant \lambda(j) + 2$, then the correspondence transforms these into

$$\lambda(1) - t + 1 \geqslant \lambda(2) - t + 3 \geqslant \cdots$$
$$\geqslant \lambda(i) - t + 2i - 1 \geqslant \cdots \geqslant \lambda(t) + t - 1 \geqslant t.$$

This correspondence is easily seen to be one-to-one.

Remark 4.3. For $k > 2$, Proposition 4.1 is equivalent to Gordon's theorem [1, Theorem 7.5] with $i = k$. That is to say, there exists a known, constructive one-to-one correspondence between partitions with no parts below the $(k-1)$st Durfee square and partitions for which, with f_i denoting the number of times i appears as a part in the partition, $f_i+f_{i+1}\leqslant k-1$ for all i. This bijection is much more difficult than the $k=2$ case. It is essentially the bijection given in [3], $\delta=1$, $r=k$.

Proposition 4.1 is, of course, a corollary of (1.4). As we shall show in this section, it can also be obtained directly as a corollary of Theorem 2.7, and thus has a purely combinatorial proof. More than this, we shall be able to turn our combinatorial proof of Proposition 4.1 into an actual bijection between the two types of partitions being counted.

As an outline for constructing our combinatorial proof, we take (1.4) and break it into three steps:

$$\prod_{\substack{i=1 \\ i\not\equiv k \pmod{2k+1}}}^{\infty} \frac{1}{(1-q^i)}$$
$$=\frac{\prod_{i=1}^{\infty}(1-q^{(2k+1)i-k})(1-q^{(2k+1)i-k-1})(1-q^{(2k+1)i})}{\prod_{i=1}^{\infty}(1-q^i)}$$
$$=\sum_{s_1,\ldots,s_k}\frac{q^{s_1^2+\cdots+s_k^2}}{(q)_{s_1}}\begin{bmatrix}s_1\\s_2\end{bmatrix}\cdots\begin{bmatrix}s_{k-1}\\s_k\end{bmatrix}\frac{(q)_{s_k}(1)_{s_k}}{(q^{s_k+1})_{s_k}}$$
$$=\sum_{s_1,\ldots,s_{k-1}}\frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q)_{s_1}}\begin{bmatrix}s_1\\s_2\end{bmatrix}\cdots\begin{bmatrix}s_{k-2}\\s_{k-1}\end{bmatrix}. \tag{4.1}$$

These four equal analytic statements are respectively the generating functions for the following four combinatorial functions, n and k positive integers.

$\alpha_k(n)\equiv$ The number of partitions of n into parts not congruent to 0 or $\pm k$ (mod $2k+1$).

$\beta_k(n)\equiv$ The number of partitions counted by $A_k(m,n)$ with m even, less the number of partitions counted by $A_k(m,n)$ with m odd.

$\gamma_k(n)\equiv$ The number of partitions counted by $D_k(m,n)$ with m even, less the numbers of partitions counted by $D_k(m,n)$ with m odd.

$\delta_k(n)\equiv$ The number of partitions of n with no parts below the $(k-1)$st Durfee square.

We shall prove Proposition 4.1 by showing that $\alpha_k(n)=\beta_k(n)=\gamma_k(n)=\delta_k(n)$ for all n and k. The equality of $\beta_k(n)$ and $\gamma_k(n)$ is an immediate corollary of Theorem 2.7: $A_k(m,n)=D_k(m,n)$.

LEMMA 4.4. $\alpha_k(n)=\beta_k(n)$.

Proof. $\alpha_k(n)$ counts those colored partitions counted by $A_k(m, n)$ for which there are no blue parts and no red parts $\equiv \pm k \pmod{2k+1}$. If a colored partition counted by $A_k(m, n)$ has blue parts and/or red parts $\equiv \pm k \pmod{2k+1}$ and if the largest blue part is greater than or equal to the largest of the red parts which are $\equiv \pm k \pmod{2k+1}$, then we pair this partition with the colored partition obtained by changing the color of this largest blue part from blue to red. It is clear that this procedure uniquely pairs all colored partitions counted by $A_k(m, n)$ with blue parts and/or red parts $\equiv \pm k \pmod{2k+1}$, and the two partitions in each pair have values of m of opposite parity.

LEMMA 4.5. $\gamma_k(n) = \delta_k(n)$.

Proof. $\delta_n(n)$ counts those colored partitions counted by $D_k(m, n)$ for which the kth Durfee square is empty ($s_k = 0$). If a colored partition counted by $D_k(m, n)$ has a non-empty kth Durfee square ($s_k \geqslant 1$) and if 0 is one of the green parts, then we pair this partition with the colored partition obtained by deleting the green zero. It is clear that this procedure uniquely pairs all colored partitions counted by $D_k(m, n)$ with non-empty kth Durfee square, and the two partitions in each pair have values of m of opposite parity.

This concludes the proof of Proposition 4.1. We are now in a position to set up a bijection between partitions counted by $\alpha_k(n)$ and partitions counted by $\delta_k(n)$.

DEFINITION 4.6. Let $\mathscr{B}_k(m, n)$ denote the set of colored partitions of n into arbitrary red parts plus blue parts lying in an arithmetic progression of length m with difference $2k+1$ and smallest part $k+1$ if $m \geqslant 0$, k if $m < 0$. Let $B_k(m, n) = |\mathscr{B}_k(m, n)|$. Let $\mathscr{B}_k(n) = \bigcup_{m \in Z} \mathscr{B}_k(m, n)$.

DEFINITION 4.7. Let $\mathscr{A}_k(n)$ be those partitions in $\mathscr{B}_k(0, n)$ for which no red parts are congruent to 0 or $\pm k \pmod{2k+1}$. Let $\mathscr{D}_k(n)$ be those partitions in $\mathscr{B}_k(0, n)$ for which the red parts have an empty kth Durfee square.

DEFINITION 4.8. Let $\varphi: \mathscr{B}_k(n) - \mathscr{D}_k(n) \to \mathscr{B}_k(n) - \mathscr{D}_k(n)$ be the mapping defined by taking a colored partition in $\mathscr{B}_k(n) - \mathscr{D}_k(n)$ and applying the following three transformations in order:

(1) We perform the bijective transformations described in the proof of Lemmas 3.4 and 3.5 to obtain a colored partition counted by $D_k(m, n)$ for some m. The red parts will have non-empty kth Durfee square since

$$m < 0 \Rightarrow n_k \geqslant -2m \Rightarrow n_k + m > 0,$$

$$m = 0 \Rightarrow n_k > 0 \Rightarrow n_k + m > 0 \Rightarrow n_k + m > 0.$$

(2) By the proof of Lemma 4.5, this partition is paired with another with opposite parity on m, for which we exchange it.

(3) We reverse the bijective transformations described in Lemmas 3.5 and 3.4 to return to a colored partition counted by $\mathscr{B}_k(n)-\mathscr{D}_k(n)$.

DEFINITION 4.9. Let $\psi: \mathscr{B}_k(n)-\mathscr{A}_k(n) \to \mathscr{B}_k(n)-\mathscr{A}_k(n)$ be the mapping defined by taking a colored partition in $\mathscr{B}_k(n)-\mathscr{A}_k(n)$ and applying the following three transformations in order:

(1) We perform the reverse of the bijective transformation described in the proof of Lemma 3.3 to obtain a colored partition counted by $A_k(m, n)$ for some m. Since the original partition was in $\mathscr{B}_k(n)-\mathscr{A}_k(n)$, the new partition will have some blue parts and/or some red parts $\equiv \pm k \pmod{2k+1}$.

(2) By the proof of Lemma 4.4, this partition is paired with another with opposite parity on m, for which we exchange it.

(3) We apply the bijective transformation described in Lemma 3.3 to return to a colored partition counted by $\mathscr{B}_k(n)-\mathscr{A}_k(n)$.

Remark 4.10. The mapping φ is an involution on $\mathscr{B}_k(n)-\mathscr{D}_k(n)$, while ψ is an involution on $\mathscr{B}_k(n)-\mathscr{A}_k(n)$. Both φ and ψ change the parity of m.

PROPOSITION 4.11. *The mapping* $\pi: \mathscr{A}_k(n) \to \mathscr{D}_k(n)$ *is a bijection, where* $\pi(\lambda)$, $\lambda \in \mathscr{A}_k(n)$, *is defined to be the last well-defined element in* $((\psi\varphi)^r(\lambda))_{r=0}^{\infty}$.

Proof. π is an iteration of the mapping $\psi\varphi$ which is repeated until arrival in $\mathscr{D}_k(n)$. π is well-defined since φ and ψ are involutions and therefore if $(\psi\varphi)^r(\lambda)=(\psi\varphi)^s(\lambda)$, $r>s\geqslant 0$, then $(\psi\varphi)^{r-s}(\lambda)=\lambda$, which implies that λ is in the range of ψ which is $\mathscr{B}_k(n)-\mathscr{A}_k(n)$, a contradiction. This implies that no two elements of $((\psi\varphi)^r(\lambda))_{r=0}^{\infty}$ are equal. Since $\mathscr{B}_k(n)$ is a finite set, this sequence must have a last well-defined element. π is a mapping into $\mathscr{D}_k(n)$. This follows from the fact that φ and ψ both change the parity of m and $\lambda \in \mathscr{A}_k(n)$ has $m=0$, which is even. Thus $\varphi(\psi\varphi)^r(\lambda)$, if it is defined, has m odd and therefore $\neq 0$ and so $(\psi\varphi)^{r+1}(\lambda)$ will also be defined. This implies that $(\psi\varphi)^{r+1}(\lambda)$ is not defined if and only if $(\psi\varphi)^r(\lambda)$ is not in the range of φ. In other words, $(\psi\varphi)^R(\lambda)$ is the last well-defined element in $((\psi\varphi)^r(\lambda))_{r=0}^{\infty}$ if and only if $(\psi\varphi)^R(\lambda) \in \mathscr{D}_k(n)$. We see that π is one-to-one for if $(\psi\varphi)^r(\lambda)=(\psi\varphi)^s(\mu)$, $r\geqslant s$, then $(\psi\varphi)^{r-s}(\lambda)=\mu$. Since μ is not in the range of ψ, $r=s$ and so $\lambda=\mu$. By similar arguments, $\pi^{-1}(\eta)$, $\eta \in \mathscr{D}_k(n)$, is the last well-defined element in $((\varphi\psi)^r(\eta))_{r=0}^{\infty}$, has its range contained in $\mathscr{A}_k(n)$, and is one-to-one. Thus π is a bijection between $\mathscr{A}_k(n)$ and $\mathscr{D}_k(n)$.

Remark 4.12. This construction of a bijection from a pair of involutions such as φ and ψ was first observed by Garsia and Milne [6]. The specific involutions we have described here are new.

While our bijection for Proposition 4.1 is now very well motivated and explained, it is hopeless from the practical point of view of actually constructing these bijections for given partitions. The next step is to take the involutions φ and ψ and see if they can be streamlined. Fortunately, they can be. The streamlined involutions, Φ and Ψ, are given in algorithmic form in the next section. They are not precisely the same mappings as φ and ψ, although Φ is still an involution on $\mathscr{B}_k(n)-\mathscr{D}_k(n)$ and Ψ is an involution on $\mathscr{B}_k(n)-\mathscr{A}_k(n)$, both changing the parity of m, and so they define a bijection Π: $\mathscr{A}_k(n)\to\mathscr{D}_k(n)$, where $\Pi(\lambda)$ is the last well-defined element in $((\Psi\Phi(\lambda))^r)_{r=0}^{\infty}$. The remainder of this section will be devoted to outlining how Φ and Ψ are derived from φ and ψ. Since Φ and Ψ will be proven in Section 5 to be involutions on the correct domains, details of the following derivation are left to the reader.

It is convenient to consider the involution φ in three stages: (1) the bijection of Lemma 3.4 up to the point where we have orange k parts bounded in number by, say, n_k and unbounded in size plus red parts unbounded in number but bounded in size by n_k+2m, (2) the remainder of the bijection of Lemma 3.4, the bijection of Lemma 3.5, the creation or elimination of a green zero, the inverse of the bijection of Lemma 3.5, and, finally, the inverse of the bijection of Lemma 3.4 up to the point where we have orange k parts bounded in number by $n_k+\delta$, $\delta=\pm 1$, and unbounded in size plus red parts unbounded in number but bounded in size by $n_k+2m-\delta$, and (3) the remainder of the inverse of the bijection of Lemma 3.4.

The second stage is an involution which has the effect of leaving the orange 1 through orange $k-1$ parts untouched, of changing the rectangles $n_1\times(n_1+2m)$ through $n_k\times(n_k+2m)$ into rectangles $(n_1+\delta)\times(n_1+2m-\delta)$ through $(n_k+\delta)\times(n_k+2m-\delta)$, of changing the number of orange k parts from n_k $+\delta$, of changing the upper bound on the size of the red parts from n_k+2m to $n_k+2m-\delta$, of changing the sum of the blue parts from $((2k+1)m^2+m)/2$ to $((2k+1)(m-\delta)^2+(m-\delta))/2$, and of leaving unchanged the sum of all of the parts and rectangles. Any other involution with the same domain and with these same effects may be substituted for this second stage. Making such a substitution creates a new mapping Φ which will still be an involution on $\mathscr{B}_k(n)-\mathscr{D}_k(n)$ and which changes the parity of m. In particular, if we let τ be the total number of red parts and L be the largest orange k part, we may substitute for the second stage the following algorithm:

A. If $L-m>\tau$, then $\delta=-1$. We remove the largest orange k part, add 1 to each of the red parts, create $L-m-\tau-1$ new red parts of 1; if

$m \geqslant 0$ then we create a new blue part $(2k+1)m+k+1$, if $m<0$ then we eliminate the largest blue part $-(2k+1)m-k-1$; and we remove a row from and then add a column to each of the rectangles.

B. If $L-m \leqslant \tau$, then $\delta=1$. We subtract 1 from each of the red parts, create a new orange k part of size $\tau+m$; if $m \geqslant 0$ then we eliminate the largest blue part $(2k+1)m-k$, if $m \leqslant 0$ then we create a new blue part $-(2k+1)m+k$; and we remove a column from and then add a row to each of the rectangles.

It is a straightforward exercise to verify that this is an involution of the desired type. When it is substituted for the second stage of φ, it yields Φ as described in Section 5. We note that the involution described above depends only on the red parts and the largest of the orange k parts, and so we need not explicitly work through all of stages one and three, but only those aspects of these stages which affect or are affected by the largest of the orange k parts.

The involution ψ has the effect of changing m to $m+\varepsilon$, $\varepsilon=\pm 1$. If $\varepsilon=+1$, then the blue part $(2k+1)m+k+1$ is created when $m \geqslant 0$ and the blue part $-(2k+1)m-k-1$ is eliminated when $m<0$. If $\varepsilon=-1$, then the blue part $(2k+1)m-k$ is eliminated when $m>0$ and the blue part $-(2k+1)m+k$ is created when $m \leqslant 0$. In either case, the red parts are changed so that the sum of the red and blue parts remains constant. Again, any other involution defined on $\mathscr{B}_k(n)-\mathscr{A}_k(n)$ with this same effect may be substituted for ψ. In particular, we may substitute for ψ the involution Ψ described in Section 5.

It is worth noting that in many instances ψ and Ψ will have the same effect. They are both, in essence, bijections which establish the Jacobi triple product identity. Such bijections have a long history of rediscoveries, beginning with Sylvester [9, Sects. 37–40] and Hathaway [9, Sect. 62] and including in recent years Wright [10], Sudler [8], and Zolnowski [11].

5. An Algorithm for the Generalized Rogers–Ramanujan Bijection

THEOREM 5.1. *For positive integral k and n, and for* $\mathscr{A}_k(n)$, $\mathscr{B}_k(n)$, *and* $\mathscr{D}_k(n)$ *as defined in Section* 4 *and Definitions* 4.6 *and* 4.7, *we have a bijection*

$$\Pi\colon \mathscr{A}_k(n) \to \mathscr{D}_k(n),$$

where $\Pi(\lambda)$, $\lambda \in \mathscr{A}_k(n)$, *is defined to be the last well-defined element in* $((\Psi\Phi)^r(\lambda))_{r=0}^{\infty}$, Φ *and* Ψ, *defined below, are involutions on* $\mathscr{B}_k(n)-\mathscr{D}_k(n)$ *and* $\mathscr{B}_k(n)-\mathscr{A}_k(n)$, *respectively, and both* Φ *and* Ψ *change the parity of m. The effect of* Π *is to iterate the mapping* $\Psi\Phi$ *until we have an element of* $\mathscr{D}_k(n)$.

DEFINITION 5.2. We shall denote an element of $\mathscr{B}_k(n)$ by $(m; \lambda(1), ..., \lambda(t))$, where $\lambda(1) \geqslant \cdots \geqslant \lambda(t) > 0$ are the red parts; $|m|$ is the number of blue parts and $m > 0$ if the smallest blue part is $k+1$, $m < 0$ if the smallest blue part is k; and $((2k+1)m^2 + m)/2 + \lambda(1) + \cdots + \lambda(t) =$ sum of all red and blue parts $= n$.

DEFINITION 5.3. Given an element $(m; \lambda(1), ..., \lambda(t))$ of $\mathscr{B}_k(n)$, we define the following constants and functions. In all cases, $\lambda(i) \equiv +\infty$ for $i \leqslant 0$, $\lambda(i) \equiv 0$ for $i > t$.

$$\begin{aligned}
N_1 &= \max\{n \geqslant 0 \mid \lambda(n) \geqslant n + 2m\}, \\
N_i &= \max\{n \geqslant 0 \mid \lambda(n) \geqslant n - N_{i-1} + 2m\}, \qquad 1 < i \leqslant k, \\
n_1 &= N_1, \qquad n_i = N_i - N_{i-1}, \qquad 1 < i \leqslant k, \\
G(z) &= z - \lambda(z) + 2m, \qquad z \in \mathbb{Z}, \\
A_1 &= N_{k-1} + 1, \\
A_i &= G^{i-1}(A_1), \qquad 1 < i \leqslant k. \\
L &= \sum_{i=1}^{k} \lambda(A_i), \\
L_1 &= t + (2k+1)m - k, \\
L_i &= L_1 - \sum_{r=1}^{i-1} M_r, \qquad 1 < i \leqslant k, \\
P_1 &= A_1, \qquad P_i = P_{i-1} + 2m - 1, \qquad 1 < i \leqslant k, \\
M_i &= \min\left\{0 \leqslant z \,\middle|\, L_i + k - i \leqslant z + \sum_{r=0}^{k-i-1} \lambda(G^r(P_i + 2m - 2 - z))\right\}, \\
&\qquad\qquad 1 \leqslant i \leqslant k, \\
t_0 &= \text{number of parts } \lambda(i),\ 1 \leqslant i \leqslant t, \text{ which are divisible by } 2k+1, \\
m_0 &= \max\{0\} \cup \{\lambda(i) \equiv 0 \ (\mathrm{mod}\ 2k+1) \mid 1 \leqslant i \leqslant t\}, \\
m_k &= \max\{0\} \cup \{\lambda(i) \equiv k \ (\mathrm{mod}\ 2k+1) \mid 1 \leqslant i \leqslant t\}, \\
m_{k+1} &= \max\{0\} \cup \{\lambda(i) \equiv k+1 \ (\mathrm{mod}\ 2k+1) \mid 1 \leqslant i \leqslant t\}, \\
t^* &= t + \sum_{i=1}^{k} \chi(\lambda(A_i) = 0),
\end{aligned}$$

where $\chi(X) = 1$ if X is true, $= 0$ if X is false.

DEFINITION 5.4. The mapping Φ is defined according to which of the following two cases holds.

Case I. $L-(2k+1)m>t^*$. Delete the parts $\lambda(A_1), \lambda(A_2), ..., \lambda(A_k)$; add 1 to each of the remaining red parts; create $L-t^*-(2k+1)m-1$ new red parts of 1 each; change m to $m+1$.

Case II. $L-(2k+1)m\leqslant t^*$. Subtract 1 from each of the red parts; create k new red parts: $M_1, M_2, ..., M_k$; change m to $m-1$.

DEFINITION 5.5. The mapping Ψ is defined according to which of the following four cases holds.

Case Ia. $t_0+m>0$ or $m_{k+1}>0$; $(2k+1)(t_0+m)\geqslant m_{k+1}+k$. Subtract $2k+1$ from each red part divisible by $2k+1$; create a new red part: $(2k+1)(t_0+m)-k$; change m to $m-1$.

Case Ib. $t_0+m>0$ or $m_{k+1}>0$; $(2k+1)(t_0+m)<m_{k+1}+k$. Eliminate the red part m_{k+1}; add $2k+1$ to each red part divisible by $2k+1$; create $(m_{k+1}-(2k+1)(t_0+m)-k-1)/(2k+1)$ new red parts of $2k+1$ each; change m to $m+1$.

Case IIa. $t_0+m\leqslant 0$; $m_{k+1}=0$; $m_k+(2k+1)m\geqslant m_0+k$. Add $(2k+1)m-k$ to m_k; change m to $m-1$.

Case IIb. $t_0+m\leqslant 0$; $m_{k+1}=0$; $m_k+(2k+1)m<m_0+k$. Subtract $(2k+1)m+k+1$ from m_0; change m to $m+1$.

Proof of Theorem 5.1. From the arguments of Section 4 (or see Garsia and Milne [6]) it follows that Π will be a bijection if we can establish that Φ and Ψ are involutions on $\mathscr{B}_k(n)-\mathscr{D}_k(n)$ and $\mathscr{B}_k(n)-\mathscr{A}_k(n)$, respectively. By definition, we know that both Φ and Ψ change the parity of m. The verification that Ψ is an involution is straightforward, and is left as an exercise for the reader. It should be kept in mind that Ψ is *not* defined on $\mathscr{A}_k(n)$. Thus, if $m=0$, then $\max\{m_0, m_k, m_{k+1}\}>0$, and so Cases IIa and IIb will not present any problems. The verification that Φ is an involution is somewhat more troublesome.

Let $(m; \lambda(1), ..., \lambda(t))\in\mathscr{B}_k(n)$ satisfy $L-(2k+1)m>t^*$, so that Φ of this element is determined by Case I, and let the image under Φ be $(\bar{m}; \bar{\lambda}(1), ..., \bar{\lambda}(\bar{t}))$, the new values of N_i, n_i, A_i, etc., denoted by $\bar{N}_i$, $\bar{n}_i$, $\bar{A}_i$, etc. Then we see that

$$\bar{m}=m+1, \tag{5.1}$$

$$\bar{\lambda}(i)=1+\lambda(i+\max\{j\geqslant 1 \mid A_{k-j+1}<i+j\}), \tag{5.2}$$

$$\bar{n}_i=n_i-1, \tag{5.3}$$

$$\bar{N}_i = N_i - i, \tag{5.4}$$

$$\bar{A}_1 = A_1 - k + 1, \tag{5.5}$$

$$\begin{aligned}\bar{A}_2 &= \bar{A}_1 - \bar{\lambda}(\bar{A}_1) + 2\bar{m} = A_1 - k + 1 - 1 - \lambda(A_1 + 1) + 2m + 2\\ &= A_1 - k + 2 - \lambda(A_1 + 1) + 2m \geqslant A_2 - k + 2,\end{aligned} \tag{5.6}$$

$$\bar{t} = L - (2k+1)m - k - 1, \tag{5.7}$$

$$\bar{L}_1 = \bar{t} + (2k+1)\bar{m} - k = L. \tag{5.8}$$

Claim 5.6. For $1 \leqslant i \leqslant k$, $\bar{A}_i \geqslant A_i - k + i$.

Proof. As we see above, this is true for $i = 1$ or 2. We shall prove it by induction.

$$\begin{aligned}\bar{A}_i &= \bar{A}_{i-1} - \bar{\lambda}(\bar{A}_{i-1}) + 2\bar{m}\\ &\geqslant A_{i-1} - k + i - 1 - \bar{\lambda}(A_{i-1} - k + i - 1) + 2m + 2\\ &= A_{i-1} - k + i - \lambda(A_{i-1} + 1) + 2m \geqslant A_i - k + i.\end{aligned} \tag{5.9}$$

Claim 5.7. $\bar{L} - (2k+1)\bar{m} \leqslant \bar{t}^*$.

Proof. We evaluate the left side of the claimed inequality.

$$\begin{aligned}\bar{L} - (2k+1)\bar{m} &= \bar{\lambda}(\bar{A}_1) + \cdots + \bar{\lambda}(\bar{A}_k) - (2k+1)m - (2k+1)\\ &\leqslant \bar{\lambda}(A_1 - k + 1) + \cdots + \bar{\lambda}(A_k) - (2k+1)m - (2k+1)\\ &= \lambda(A_1 + 1) + \cdots + \lambda(A_k + 1) - (2k+1)m - k - 1\\ &\leqslant L - (2k+1)m - k - 1 = \bar{t} \leqslant \bar{t}^*.\end{aligned} \tag{5.10}$$

We thus see that if we apply Φ to $(\bar{m}; \bar{\lambda}(1), ..., \bar{\lambda}(\bar{t}))$, Φ is determined by Case II. We shall now verify that if $L - (2k+1)m > t^*$, then $\Phi^2((m; \lambda(1), ..., \lambda(t))) = (m; \lambda(1), ..., \lambda(t))$. It is sufficient to verify that $\bar{M}_i = \lambda(A_i)$, $1 \leqslant i \leqslant k$. We begin with the following pair of implications.

Claim 5.8.

$$z \geqslant A_i - k + i \Rightarrow \sum_{r=0}^{k-i} \bar{\lambda}(\bar{G}^r(z)) \leqslant \sum_{r=0}^{k-i} \lambda(A_{i+r}) + k - i + 1, \tag{5.11}$$

$$z \leqslant A_i - k + 1 \Rightarrow \sum_{r=0}^{k-i} \bar{\lambda}(\bar{G}^r(z)) \geqslant \sum_{r=0}^{k-i} \lambda(A_{i+r}) + k - i + 1. \tag{5.12}$$

Proof. We first establish by induction that

$$z \geqslant A_i - k + i \Rightarrow \bar{G}^r(z) \geqslant A_{i+r} - k + i + r, \tag{5.13}$$

$$z \leqslant A_i - k + i - 1 \Rightarrow \bar{G}^r(z) \leqslant A_{i+r} - k + i + r - 1. \tag{5.14}$$

This is clearly true for $r=0$. We assume that (5.13) is valid up to and including $r-1$, then

$$\begin{aligned}
\bar{G}^r(z) &= \bar{G}(\bar{G}^{r-1}(z)) \\
&= \bar{G}^{r-1}(z) - \bar{\lambda}(\bar{G}^{r-1}(z)) + 2\bar{m} \\
&\geqslant A_{i+r-1} - k + i + r - 1 - \bar{\lambda}(A_{i+r-1} - k + i + r - 1) + 2m + 2 \\
&= A_{i+r-1} - k + i + r - 1 - 1 - \lambda(A_{i+r-1} + 1) + 2m + 2 \\
&\geqslant A_{i+r-1} - \lambda(A_{i+r-1}) + 2m - k + i + r \\
&= A_{i+r} - k + i + r.
\end{aligned}$$

The inequality (5.14) is proved similarly. It now follows from (5.13) that if $z \geqslant A_i - k + i$, then

$$\begin{aligned}
\sum_{r=0}^{k-i} \bar{\lambda}(\bar{G}^r(z)) &\leqslant \sum_{r=0}^{k-i} \bar{\lambda}(A_{i+r} - k + i + r) \\
&= \sum_{r=0}^{k-i} (1 + \lambda(A_{i+r} + 1)) \\
&\leqslant k - i + 1 + \sum_{r=0}^{k-i} \lambda(A_{i+r}).
\end{aligned}$$

By a similar argument, (5.12) follows from (5.14).

We shall prove that $\bar{M}_i = \lambda(A_i)$, $1 \leqslant i \leqslant k$, by first establishing it for $i=1$, and then proceeding by induction. By definition, we have that

$$\bar{P}_1 = \bar{A}_1 = A_1 - k + 1, \tag{5.15}$$

and therefore

$$\begin{aligned}
\bar{M}_1 &= \min\left\{ z \,\middle|\, \bar{L}_1 + k - 1 \leqslant z + \sum_{r=0}^{k-2} \bar{\lambda}(\bar{G}^r(\bar{P}_1 + 2\bar{m} - 2 - z)) \right\} \\
&= \min\left\{ z \,\middle|\, L + k - 1 \leqslant z + \sum_{r=0}^{k-2} \bar{\lambda}(\bar{G}^r(A_1 - k + 1 + 2m - z)) \right\}.
\end{aligned} \tag{5.16}$$

If $z = \lambda(A_1)$, then we have that

$$\begin{aligned}
&\lambda(A_1) + \sum_{r=0}^{k-2} \bar{\lambda}(\bar{G}^r(A_1 - k + 1 + 2m - \lambda(A_1))) \\
&\quad = \lambda(A_1) + \sum_{r=0}^{k-2} \bar{\lambda}(\bar{G}^r(A_2 - k + 1)),
\end{aligned}$$

which, by (5.12), is greater than or equal to

$$\lambda(A_1)+\sum_{r=0}^{k-2}\lambda(A_{2+r})+k-1=L+k-1.$$

Thus, $\bar{M}_1\leqslant\lambda(A_1)$. On the other hand, if $z=\lambda(A_1)-1$, then

$$\begin{aligned}\lambda(A_1)-1+\sum_{r=0}^{k-2}\bar{\lambda}(\bar{G}^r(A_1-k+1+2m-\lambda(A_1)+1))\\=\lambda(A_1)-1+\sum_{r=0}^{k-2}\bar{\lambda}(\bar{G}^r(A_2-k+2)),\end{aligned}$$

which, by (5.11), is less than or equal to

$$\lambda(A_1)-1+\sum_{r=0}^{k-2}\lambda(A_{2+r})+k-1=L+k-2.$$

Thus, $\bar{M}_1>\lambda(A_1)-1$, from which it follows that $\bar{M}_1=\lambda(A_1)$, as desired.

We now assume that $\bar{M}_j=\lambda(A_j)$ for $1\leqslant j<i$. This implies that

$$\bar{L}_i=\bar{L}_1-\sum_{r=1}^{i-1}\bar{M}_r=L-\sum_{r=1}^{i-1}\lambda(A_r)=\lambda(A_i)+\lambda(A_{i+1})+\cdots+\lambda(A_k). \tag{5.17}$$

Our first task is to compute $\bar{P}_i$.

Claim 5.9. $\bar{P}_i=A_i-k+i$, $1\leqslant i\leqslant k$.

Proof. This is true for $i=1$. We proceed by induction

$$\begin{aligned}\bar{P}_i&=\bar{P}_{i-1}-\bar{M}_{i-1}+2\bar{m}-1\\&=A_{i-1}-k+i-1-\lambda(A_{i-1})+2m+2-1\\&=A_i-k+i.\end{aligned} \tag{5.18}$$

We now compute $\bar{M}_i$ from its definition.

$$\begin{aligned}\bar{M}_i&=\min\left\{z\,\middle|\,\bar{L}_i+k-i\leqslant z+\sum_{r=0}^{k-i-1}\bar{\lambda}(\bar{G}^r(\bar{P}_i+2\bar{m}-2-z))\right\}\\&=\min\left\{z\,\middle|\,\lambda(A_i+\cdots+\lambda(A_k)+k-i\right.\\&\qquad\left.\leqslant z+\sum_{r=0}^{k-i-1}\bar{\lambda}(\bar{G}^r(A_i-k+i+2m-z))\right\}.\end{aligned} \tag{5.19}$$

If we set $z=\lambda(A_i)$, then, by (5.12),

$$\begin{aligned}
\lambda(A_i) + \sum_{r=0}^{k-i-1} \bar{\lambda}(\bar{G}^r(A_i - k + i + 2m - \lambda(A_i)))& \\
= \lambda(A_i) + \sum_{r=0}^{k-i-1} \bar{\lambda}(\bar{G}^r(A_{i+1} - k + i))& \\
\geqslant \lambda(A_i) + \lambda(A_{i+1}) + \cdots + \lambda(A_k) + k - i.& \qquad (5.20)
\end{aligned}$$

On the other hand, if we set $z = \lambda(A_i) - 1$, then, by (5.11),

$$\begin{aligned}
\lambda(A_i) - 1 + \sum_{r=0}^{k-i-1} \bar{\lambda}(\bar{G}^r(A_i - k + i + 2m - \lambda(A_i) + 1))& \\
= \lambda(A_i) - 1 + \sum_{r=0}^{k-i-1} \bar{\lambda}(\bar{G}^r(A_{i+1} - k + i + 1))& \\
< \lambda(A_i) + \lambda(A_{i+1}) + \cdots + \lambda(A_k) + k - i.& \qquad (5.21)
\end{aligned}$$

Thus, we have, as desired,

$$\bar{M}_i = \lambda(A_i). \qquad (5.22)$$

It remains to be shown that Φ acts as an involution on elements $(m; \lambda(1), ..., \lambda(t)) \in \mathscr{B}_k(n)$ for which $L - (2k+1)m \leqslant t^*$. Let us consider such an element, and let its image under Φ be $(\tilde{m}; \tilde{\lambda}(1), ..., \tilde{\lambda}(\tilde{t}))$ with the new values of N_i, n_i, A_i, etc., denoted by $\tilde{N}_i, \tilde{n}_i, \tilde{A}_i$, etc. We shall first demonstrate that $\tilde{\lambda}(\tilde{A}_i) = M_i$.

Claim 5.10. $L_1 = M_1 + M_2 + \cdots + M_k$.

Proof. By definition, we have that

$$M_k \equiv \min\{z \mid L_k \leqslant z\} = L_k \equiv L_1 - (M_1 + \cdots + M_{k-1}), \qquad (5.23)$$

which is the claim.

As a corollary to Claim 5.10, we see that

$$L_i = M_i + M_{i+1} + \cdots + M_k. \qquad (5.24)$$

Claim 5.11. $\lambda(P_i) - 1 \leqslant M_i \leqslant \lambda(P_i - 1) - 1$, $1 \leqslant i \leqslant k$.

Proof. By the definition of M_i, we have that

$$\begin{aligned}
M_i + \cdots + M_k + k - i = L_i + k - i& \\
\leqslant M_i + \sum_{r=0}^{k-i-1} \lambda(G^r(P_i + 2m - 2 - M_i))& \\
= M_i + \sum_{r=0}^{k-i-1} \lambda(G^r(P_{i+1} - 1)),& \qquad (5.25)
\end{aligned}$$

and therefore

$$M_{i+1} + \cdots + M_k + k - i \leqslant \sum_{r=0}^{k-i-1} \lambda(G^r(P_{i+1} - 1)). \tag{5.26}$$

Also, by the definition of M_i, we have strict inequality going the other way if we set $z = M_i - 1$, that is to say

$$\begin{aligned} M_i + \cdots + M_k + k - i &= L_i + k - i \\ &> M_i - 1 + \sum_{r=0}^{k-i-1} \lambda(G^r(P_i + 2m - 1 - M_i)) \\ &= M_i - 1 + \sum_{r=0}^{k-i-1} \lambda(G^r(P_{i+1})), \end{aligned} \tag{5.27}$$

and therefore

$$M_{i+1} + \cdots + M_k + k - i \geqslant \sum_{r=0}^{k-i-1} \lambda(G^r(P_{i+1})). \tag{5.28}$$

By setting $i = k - 1$ in (5.26) and (5.28), we see that our claim is valid for $i = k$. Let us assume that for some $i < k$, we have $\lambda(P_i - 1) \leqslant M_i$. This implies that

$$\begin{aligned} G(P_i - 1) &= P_i - 1 - \lambda(P_i - 1) + 2m \\ &\geqslant P_i - M_i + 2m - 1 = P_{i+1}, \end{aligned} \tag{5.29}$$

from which it follows by (5.26) that

$$\begin{aligned} M_i + M_{i+1} + \cdots + M_k + k - i + 1 &\leqslant \sum_{r=0}^{k-i} \lambda(G^r(P_i - 1)) \\ &= \lambda(P_i - 1) + \sum_{r=1}^{k-i} \lambda(G^r(P_i - 1)) \\ &\leqslant \lambda(P_i - 1) + \sum_{r=0}^{k-i-1} \lambda(G^r(P_{i+1})). \end{aligned} \tag{5.30}$$

If we now use (5.28), we see that

$$M_i + 1 \leqslant \lambda(P_i - 1),$$

contradicting our assumption. A similar argument establishes the other inequality in Claim 5.11.

Since our mapping Φ acts on an element $(m; \lambda(1), ..., \lambda(t))$ with

$L-(2k+1)m \leqslant t^*$ by subtracting 1 from each $\lambda(i)$ and then inserting $M_1 \geqslant M_2 \geqslant \cdots \geqslant M_k$ where they fit, it is easily seen that

$$\tilde{\lambda}(P_i+k-i)=M_i \tag{5.31}$$

is a consequence of Claim 5.11. Therefore, in order to prove that $\tilde{\lambda}(\tilde{A}_i)=M_i$, it is sufficient to prove that $\tilde{A}_i=P_i+k-i$. For $i=1$, we have that

$$\tilde{A}_1=\tilde{N}_1+1=N_1+k=A_1+k-1=P_1+k-1. \tag{5.32}$$

We assume that $\tilde{A}_j=P_j+k-j$ for $1 \leqslant j<i$. We then have that

$$\begin{aligned}\tilde{A}_i&=\tilde{A}_{i-1}-\tilde{\lambda}(\tilde{A}_{i-1})+2\tilde{m}\\&=P_{i-1}-k-i+1-\tilde{\lambda}(P_{i-1}+k-i+1)+2m-2\\&=P_{i-1}+k-i-1-M_{i-1}+2m\\&=P_i+k-i.\end{aligned} \tag{5.33}$$

This concludes the proof that $\tilde{\lambda}(\tilde{A}_i)=M_i$. It only remains to be shown that $(\tilde{m}; \tilde{\lambda}(1), ..., \tilde{\lambda}(t))$ satisfies $\tilde{L}-(2k+1)\tilde{m}>\tilde{t}^*$, and thus is acted on by Φ using Case I. We first observe that

$$\begin{aligned}\tilde{t}&\leqslant t+k-\sum_{i=1}^{k}\chi(M_i=0)\\&=t+k-\sum_{i=1}^{k}\chi(\tilde{\lambda}(\tilde{A}_i)=0),\end{aligned} \tag{5.34}$$

and thus

$$\tilde{t}^*=\tilde{t}+\sum_{i=1}^{k}\chi(\tilde{\lambda}(\tilde{A}_i)=0)\leqslant t+k. \tag{5.35}$$

On the other side, we have that

$$\begin{aligned}\tilde{L}-(2k+1)\tilde{m}&=\sum_{i=1}^{k}\tilde{\lambda}(\tilde{A}_i)-(2k+1)m+2k+1\\&=\sum_{i=1}^{k}M_i-(2k+1)m+2k+1\\&=L_1-(2k+1)m+2k+1\\&=t+k+1.\end{aligned} \tag{5.36}$$

The desired inequality follows from a comparison of (5.35) and (5.36).

We have thus demonstrated that in all cases, Φ is an involution, and this completes our proof of Theorem 5.1.

6. Example and Conclusion

It should be noted that while Π is a bijection between the sets $\mathscr{A}_k(n)$ and $\mathscr{D}_k(n)$, we have not given a bijective proof that $|\mathscr{A}_k(n)| = |\mathscr{D}_k(n)|$ in the sense that Section 3 provides a bijective proof of Theorem 2.7. The problem is that Π is known to be a bijection solely by virtue of the fact that Φ and Ψ are known to be involutions which change the parity of m. As Garsia and Milne have observed [6], once Φ and Ψ are known to be involutions on $\mathscr{B}_k(n) - \mathscr{D}_k(n)$ and $\mathscr{B}_k(n) - \mathscr{A}_k(n)$, respectively, both changing the parity of m, it is an immediate consequence that

$$\begin{aligned}\left|\bigcup_{m \text{ even}} \mathscr{B}_k(m, n) - \mathscr{D}_k(n)\right| &= \left|\bigcup_{m \text{ odd}} \mathscr{B}_k(m, n)\right| \\ &= \left|\bigcup_{m \text{ even}} \mathscr{B}_k(m, n) - \mathscr{A}_k(n)\right|, \end{aligned} \tag{6.1}$$

and thus

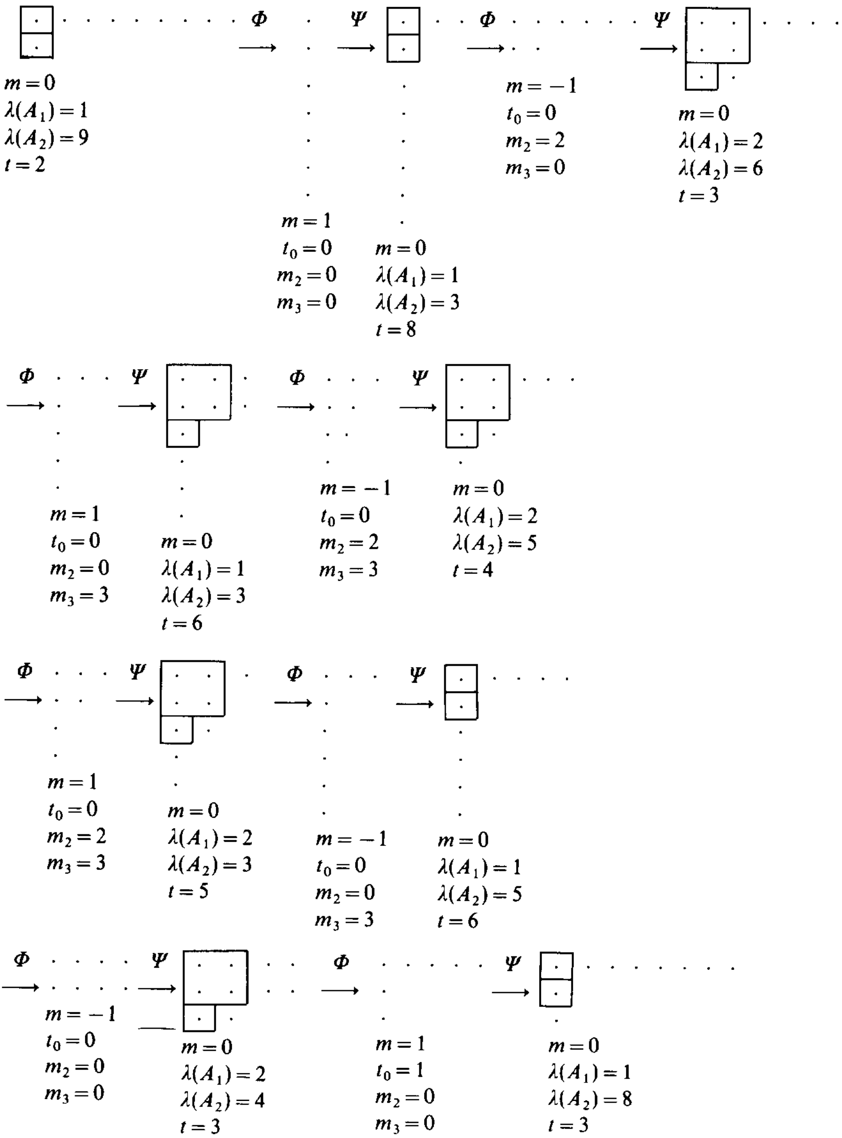
$$|\mathscr{D}_k(n)| = |\mathscr{A}_k(n)|. \tag{6.2}$$

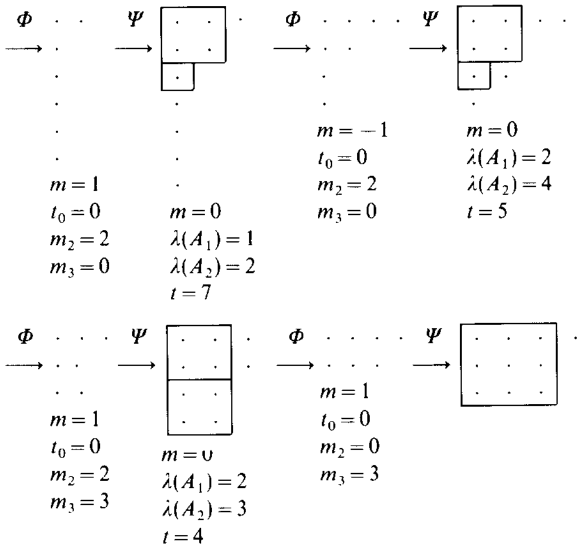
The bijection Π is in some sense superfluous.

The point in considering Π is the hope that, since it does arise naturally, it may be possible to understand and motivate it independent of the involutions Φ and Ψ. If this could be done, then we would have a truly bijective proof of the first Rogers–Ramanujan identity in the sense usually meant by the phrase "bijective proof": a constructable bijection between the sets $\mathscr{D}_2(n)$ and $\mathscr{A}_2(n)$ which is known to be a bijection by an argument which at no point relies on the fact that these two sets have the same cardinality.

With the hope of inspiring someone to such an understanding of Π, we conclude this paper with a detailed illustration of how Π maps the partition $9+1 \in \mathscr{A}_2(10)$ (partitions of 10 with parts $\equiv \pm 1 \pmod 5$)) to $4+3+3 \in \mathscr{D}_2(10)$ (partitions of 10 with no parts below the first Durfee square). As shown in Section 4, there is a natural bijection between partitions with no parts under the Durfee square and partitions with difference at least two. Under this bijection, $4+3+3$ corresponds to $6+3+1$. It should be noted that this is *not* the same correspondence as would be produced by the bijection described in [5] and illustrated in the Introduction to this paper. One obvious truth to be kept in mind is that partition bijections are highly non-unique.

In the following example, we denote the red parts of an element in $\mathscr{B}_2(10)$ by their Ferrers graph. For elements acted on by Φ, we have outlined the first two Durfee rectangles. Some of the important parameters for the element are listed below it.





References

1. G. E. Andrews, "Theory of Partitions," Addison–Wesley, Reading, MA, 1976.
2. G. E. Andrews, Partitions and Durfee dissection, *Amer. J. Math.* **101** (1979), 735–742.
3. D. M. Bressoud, An analytic generalization of the Rogers–Ramanujan identities with interpretation, *Quart. J. Math. Oxford (2)* **31** (1980), 385–399.
4. D. M. Bressoud, An easy proof of the Rogers–Ramanujan identities, *J. Number Theory* **16** (1983), 235–241.
5. D. M. Bressoud and D. Zeilberger, A short Rogers–Ramanujan bijection, *Discrete Math.* **38** (1982), 313–315.
6. A. M. Garsia and S. C. Milne, Method for constructing bijections for classical partition identities, *Proc. Nat. Acad. Sci. U.S.A.* **78** (1981), 2026–2028.
7. P. A. MacMahon, "Combinatory Analysis," Vol. II, Cambridge Univ. Press, London/New York, 1916; reprinted by Chelsea, New York, 1960.
8. C. Sudler, Two enumerative proofs of an identity of Jacobi, *Proc. Edinburgh Math. Soc.* **15** (1966), 67–71.
9. J. J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact and an exodion, *Amer. J. Math.* **5** (1982); reprinted in "The Collected Mathematical Papers of J. J. Sylvester," Vol. 4, pp. 1–83, Cambridge Univ. Press, London/New York, 1912.
10. E. M. Wright, An enumerative proof of an identity of Jacobi, *J. London Math. Soc.* **40** (1965), 55–57.
11. J. Zolnowski, A direct combinatorial proof of the Jacobi identity, *Discrete. Math.* **9** (1974), 293–298.