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Path description of conserved quantities of generalized periodic box-ball systems

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We investigate conserved quantities of periodic box-ball systems (PBBS) with arbitrary kinds of balls and box capacity greater than or equal to 1. We introduce the notion of nonintersecting paths on the two dimensional array of boxes, and give a combinatorial formula for the conserved quantities of the generalized PBBS using these paths. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1842354]

I. INTRODUCTION

The box-ball system (BBS) is a reinterpretation of a soliton cellular automaton proposed by Takahashi–Satsuma[1] as a dynamical system of balls in a one dimensional array of boxes.[2] Hence, the BBS shows both a feature of cellular automata (CA) and that of solitons.

CAs are mathematical idealizations of physical systems in which space and time are discrete, and physical quantities take on a finite set of discrete values. The CAs were originally introduced by von Neumann and Ulam as a possible idealization of biological systems, with the particular purpose of modeling biological self-reproduction. Physical systems containing many discrete elements with local interactions are often conveniently modeled as the CAs. Many biological systems have been modeled by the CAs. The CAs have also been used to study problems in number theory and their applications to tapestry design. The CAs play an important role in various fields like these.

On the other hand, the notion of a soliton arose from a peculiar solution of partial differential equations.[3,4] Actually, the system in which solitons exist has continuous and smooth mathematical structures, such as an inverse scattering method, a pseudodifferential operator, an algebraic manifold, an infinite-dimensional Lie group and so on. Because of these rich structures, the soliton systems play an important role in various fields of mathematics and physics.

The reason why the BBS has these two completely different features is well explained by the notion of ultradiscretization.[5] Ultradiscretization is a limiting procedure through which we can construct piecewise linear equations or CAs from continuous equations. By taking the ultradiscrete limit, the rich mathematical structures of soliton systems are introduced to the CAs. On the other hand, the useful properties of the CAs for computer simulation are introduced to the continuous systems by inverse ultradiscretization. Using this limiting procedure, the BBSs are obtained from

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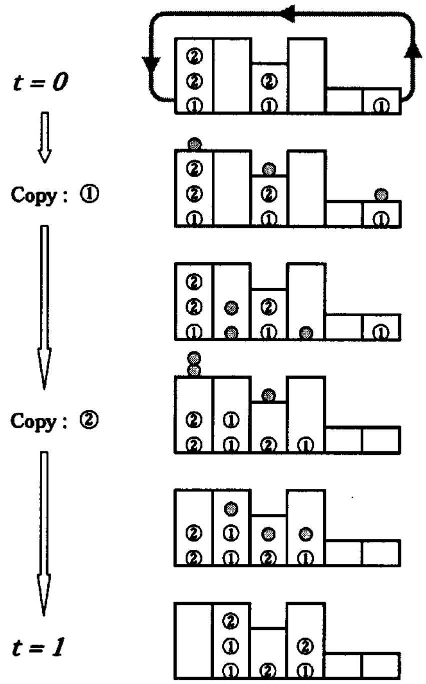

$$N = 6,\ M = 2,\ \theta_1 = \theta_2 = \theta_4 = 3,\ \theta_3 = 2,\ \theta_5 = \theta_6 = 1.$$

FIG. 1. Time evolution rule for the PBBS.

the soliton equations (the KdV equation and the Toda equation).[6,7] Thus the BBS has N soliton solutions and an infinite number of conserved quantities and the BBS is called an "integrable" CA.

The periodic box-ball system (PBBS) is the BBS in which the updating rule is extended to be compatible with a periodic boundary condition.[8] Let us consider a one-dimensional array of N boxes. A periodic boundary condition is imposed by assuming that the Nth box is adjacent to the first one. (We may imagine that the boxes are arranged in a circle.) In the generalized PBBS (gPBBS), the capacity of the nth $(1 \leqslant n \leqslant N)$ box is denoted by a positive integer θ_n, and we suppose that there are M kinds of balls distinguished by an integer index j $(1 \leqslant j \leqslant M)$. When $^{\forall} n \theta_n = 1$ and $M = 1$, the gPBBS coincides with the PBBS. Then, the rule for the time evolution of the gPBBS from time step t to $t+1$ is given as follows:

(1) At each box, create the same number of copies of the balls with index 1.
(2) Choose one of the copies arbitrarily and move it to the nearest box with an available space to the right of it.
(3) Choose one of the remaining copies and move it to the nearest available box on the right of it.
(4) Repeat the above procedure until all the copies have been moved.
(5) Delete all the original balls with index 1.
(6) Perform the same procedure for the balls with index 2.
(7) Repeat this procedure successively until all of the balls are moved.

An example of the time evolution of the gPBBS according to this rule is shown in Fig. 1.

In Ref. 9, we have established an algorithm to construct the conserved quantities of the gPBBS by means of the ultradiscretization of the nonautonomous discrete KP (ndKP) equation.[10] Using this algorithm, we obtain an expression for the conserved quantities of the gPBBS in the case of one kind of balls ($M=1$). We have also proved that, when box capacities are all one, our conserved quantities for $M=1$ coincide with those described by the Young diagram.[11]

In this paper, using a path description and the results obtained in Ref. 9, we investigate the conserved quantities of the gPBBS for arbitrary M. In Sec. II, we derive the path description of the characteristic polynomial of a particular matrix. In Sec. III, we briefly summarize the results of Ref. 9, which we will use in the subsequent sections. In Sec. IV, we treat the ndKP equation which corresponds to the gPBBS. We shall obtain an explicit expression for the conserved quantities of the ndKP equation. Using the results in Sec. IV, we construct the conserved quantities of the gPBBS in Sec. V. In Sec. VI, we discuss algebraic aspects of the gPBBS with respect to the affine Weyl group and the crystals of quantum affine algebra. Section VII is devoted to concluding remarks.

II. PATH DESCRIPTION OF CHARACTERISTIC POLYNOMIAL FOR A PARTICULAR MATRIX

For a particular matrix A which contains a parameter μ in the upper half elements, we give a combinatorial description for coefficients of the characteristic polynomial $\det(\lambda I-A)$ in λ and μ in terms of nonintersecting paths (Theorem II.1). The result will be used in the subsequent sections to obtain a combinatorial formula for conserved quantities of the gPBBS.

We denote by S_X the set of all permutations of elements in $X\subset\{1,2,\ldots,N\}$. Let A be an arbitrary $N\times N$ matrix, and $A_{n,m}$ denote the (n,m) element of A. The characteristic polynomial of A is

$$\begin{aligned}\det(\lambda I-A) &= \sum_{\sigma\in S_{\{1,2,\ldots,N\}}} \operatorname{sgn}(\sigma)\prod_{i=1}^{N}(\lambda\delta_{i,\sigma(i)}-A_{i,\sigma(i)})\\ &= \sum_{k=0}^{N}(-1)^{N-k}\lambda^k \sum_{\substack{X\subset\{1,2,\ldots,N\}\\ \sharp X=N-k}}\sum_{\sigma\in S_X}\operatorname{sgn}(\sigma)\prod_{i\in X}A_{i,\sigma(i)},\end{aligned}$$

where $\delta_{n,m}$ is Kronecker's delta. For $J\subset X$, we set

$$S_X^J := \left\{\sigma\in S_X \middle| \begin{array}{ll} i<\sigma(i) & (i\in J),\\ i\geqslant\sigma(i) & (i\in X-J). \end{array}\right\}.$$

Since

$$S_X = \bigcup_{j=0}^{N}\bigcup_{\substack{J\subset X\\ \sharp J=j}} S_X^J \quad (\text{disjoint}),$$

we have

$$\begin{aligned}\det(\lambda I-A) &= \sum_{k=0}^{N}(-1)^{N-k}\lambda^k \sum_{\substack{X\subset\{1,2,\ldots,N\}\\ \sharp X=N-k}}\sum_{j=0}^{N-k}\sum_{\substack{J\subset X\\ \sharp J=j}}\sum_{\sigma\in S_X^J}\operatorname{sgn}(\sigma)\prod_{i\in X}A_{i,\sigma(i)}\\ &= \sum_{k=0}^{N}(-1)^{N-k}\lambda^k\sum_{j=0}^{N-k}\sum_{\substack{X\subset\{1,2,\ldots,N\}\\ \sharp X=N-k}}\sum_{\substack{J\subset X\\ \sharp J=j}}\sum_{\sigma\in S_X^J}\operatorname{sgn}(\sigma)\left(\prod_{i\in J}A_{i,\sigma(i)}\right)\left(\prod_{i\in X-J}A_{i,\sigma(i)}\right). \qquad (1)\end{aligned}$$

Now we assume

$$A = (D_0 - \Upsilon)(D_1 - \Upsilon)\cdots(D_M - \Upsilon), \tag{2}$$

where $D_i = \mathrm{diag}(x_{1,i}, x_{2,i}, \ldots, x_{N,i})$ $(i=0,1,\ldots,M)$ and

$$\Upsilon := \begin{bmatrix} & & & & \mu \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}.$$

If we set $D_i^{(0)} := D_i$ and $D_i^{(r)} := \mathrm{diag}(x_{N-r+1,i}, x_{N-r+2,i}, \ldots, x_{N,i}, x_{1,i}, \ldots, x_{N-r,i})$ for $0 < r < N$, we have $D_i^{(r+1)}\Upsilon = \Upsilon D_i^{(r)}$. Hereafter, for $i=0$, we define $\sum_{c_1<c_2<\cdots<c_i}\cdots := 1$. Then

$$A = \sum_{\ell=0}^{M+1} (-1)^\ell \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-\ell+1}\le M}} D_{h_1}^{(h_1)} D_{h_2}^{(h_2-1)} \cdots D_{h_{M-\ell+1}}^{(h_{M-\ell+1}-M+\ell)} \right) \Upsilon^\ell.$$

We assume $M+1 < N$. The (n,m) element of A is the following.

(i) if $m=n$, $x_{n,0}x_{n,1}\cdots x_{n,M}$;
(ii) if $m=N+n-\ell$ $(\ell=1,2,\ldots,M)$,

$$(-1)^\ell \mu \sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-\ell+1}\le M}} \prod_{i=1}^{M-\ell+1} x_{n-h_i+i-1,h_i};$$

(ii′) if $m=n-\ell$ $(\ell=1,2,\ldots,M)$,

$$(-1)^\ell \sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-\ell+1}\le M}} \prod_{i=1}^{M-\ell+1} x_{n-h_i+i-1,h_i};$$

(iii) if $m=N+n-M-1$, $(-1)^{M+1}\mu$;
(iii′) if $m=n-M-1$, $(-1)^{M+1}$;
(iv) otherwise, 0.

Hence, from (1), we have

$$\begin{aligned} \det(\lambda I - A) = \sum_{k=0}^{N} (-1)^{N-k} \lambda^k \sum_{j=0}^{N-k} \mu^j \sum_{\substack{X\subset\{1,2,\ldots,N\} \\ \sharp X=N-k}} \sum_{\substack{J\subset X \\ \sharp J=j}} \sum_{\sigma\in S_X^J} \mathrm{sgn}(\sigma) \\ \times\left(\prod_{n\in J} \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-N-n+\sigma(n)+1}\le M}} \prod_{i=1}^{M-N-n+\sigma(n)+1} x_{n-h_i+i-1,h_i} \right)\right) \\ \times\left(\prod_{n\in X-J} \left(\sum_{\substack{0\le h_1<h_2<\cdots \\ \cdots<h_{M-n+\sigma(n)+1}\le M}} \prod_{i=1}^{M-n+\sigma(n)+1} x_{n-h_i+i-1,h_i} \right)\right). \end{aligned} \tag{3}$$

A combinatorial description of the coefficients is possible.

By $C_{N,M+1}$ we denote the $N\times(M+1)$ boxes in Fig. 2 and by (n,m)-box the box at the nth column in the $(m+1)$th row. We assume that the Nth column is adjacent to the first one.

Let a and b be column indices $(a,b=1,2,\ldots,N)$. A *path* connecting the *initial point* a and the

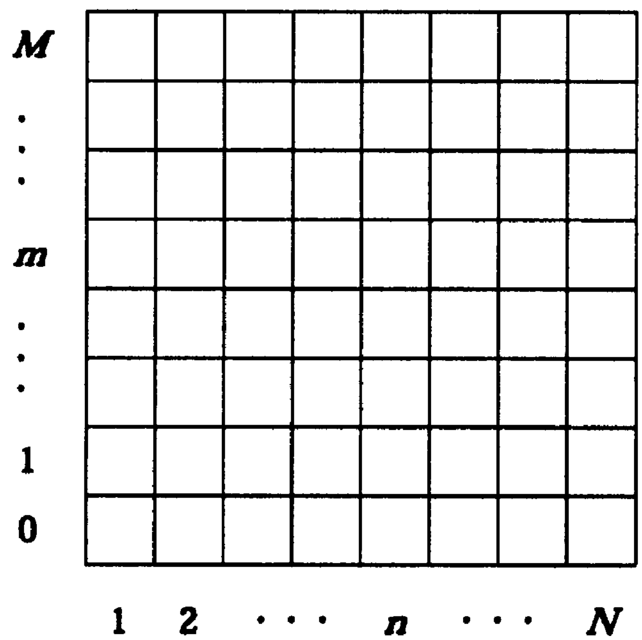

FIG. 2. $N\times(M+1)$ boxes.

end point b is a (continuous) polygonal line from the initial point a to the end point b which consists of (i), (ii) or (iii) in Fig. 3 locally; here by the initial point a we mean the middle point of the south edge of $(a,0)$-box and by the end point b the middle point of the north edge of (b,M)-box. For example, the left-hand part in Fig. 4 shows a path connecting the initial point 1 and the end point 1, and the right-hand part shows a path from 5 to 2.

There is a natural correspondence between $\Pi_{i=1}^{M-\ell+1}x_{n-h_i+i-1,h_i}$ $(0\leqslant h_1<h_2<\cdots<h_{M-\ell+1}\leqslant M)$ and a path on $C_{N,M+1}$. To put it concretely, we draw the line (i) on $(n-h_i+i-1,h_i)$-box $(i=1,2,\ldots,M-\ell+1)$; for each r, $h_i<r<h_{i+1}$, we draw the line (ii) on $(n+i-r,r)$-box and the line (iii) on $(n+i-r-1,r)$-box where $h_0=-1$ and $h_{M-\ell+2}=M+1$; then we obtain a path. For example, for $N=8$ and $M=5$, $x_{1,0}x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{1,5}$ and $x_{5,0}x_{4,2}x_{2,5}$ correspond to paths in Fig. 4, respectively.

Let $X=\{d_1,d_2,\ldots,d_{N-k}\}$ $(1\leqslant d_1<d_2<\cdots<d_{N-k}\leqslant N)$; we denote by $\mathcal{P}(d;\sigma)$ the set of all paths which connect the initial point d and the end point $\sigma(d)$ $(d\in X,\ \sigma\in S_X$; cf. Fig. 5). Define $\xi_{n,m}:\mathcal{P}(d;\sigma)\to\{x_{n,m},1\}$ as

$$\xi_{n,m}(P) := \begin{cases} x_{n,m} & (P \text{ has the vertical line on the}(n,m)\text{-box of } C_{N,M+1}),\\ 1 & (\text{otherwise}),\end{cases} \tag{4}$$

where $P\in\mathcal{P}(d;\sigma)$. Then, we obtain

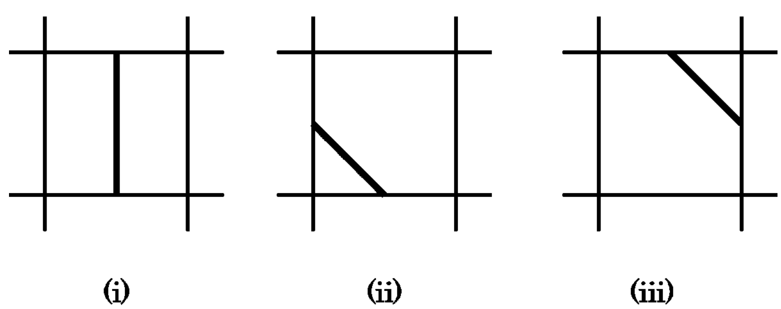

FIG. 3. A line can pass through a box in three possible ways.

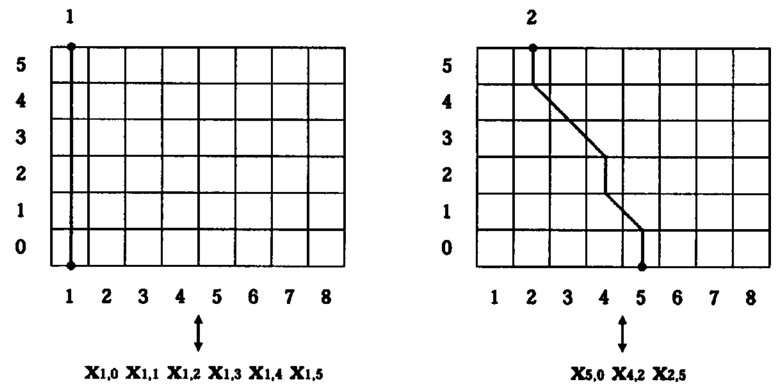

FIG. 4. Paths corresponding to $x_{1,0}x_{1,1}x_{1,2}x_{1,3}x_{1,4}x_{1,5}$ and $x_{5,0}x_{4,2}x_{2,5}$.

$$\sum_{\substack{X\subset\{1,2,\ldots,N\}\\ \sharp X=N-k}}\ \sum_{\substack{J\subset X\\ \sharp J=j}}\ \sum_{\sigma\in S_X^J}\operatorname{sgn}(\sigma)\left(\prod_{n\in J}\left(\sum_{\substack{0\leqslant h_1<h_2<\cdots\\ \cdots<h_{M-N-n+\sigma(n)+1}\leqslant M}}\ \prod_{i=1}^{M-N-n+\sigma(n)+1} x_{n-h_i+i-1,h_i}\right)\right)$$

$$\times\left(\prod_{n\in X-J}\left(\sum_{\substack{0\leqslant h_1<h_2<\cdots\\ \cdots<h_{M-n+\sigma(n)+1}\leqslant M}}\ \prod_{i=1}^{M-n+\sigma(n)+1} x_{n-h_i+i-1,h_i}\right)\right)$$

$$=\sum_{\substack{1\leqslant d_1<d_2<\cdots\\ \cdots<d_{N-k}\leqslant N}}\ \sum_{\substack{J\subset X\\ \sharp J=j}}\ \sum_{\sigma\in S_X^J}\ \sum_{P_1\in\mathcal{P}(d_1;\sigma)}\cdots\sum_{P_{N-k}\in\mathcal{P}(d_{N-k};\sigma)}\operatorname{sgn}(\sigma)\prod_{i=1}^{N-k}\prod_{n=1}^{N}\prod_{m=0}^{M}\xi_{n,m}(P_i). \tag{5}$$

If we draw $N-k$ paths $P_1\in\mathcal{P}(d_1;\sigma),\ldots,P_{N-k}\in\mathcal{P}(d_{N-k};\sigma)$ on $C_{N,M+1}$, some paths may go

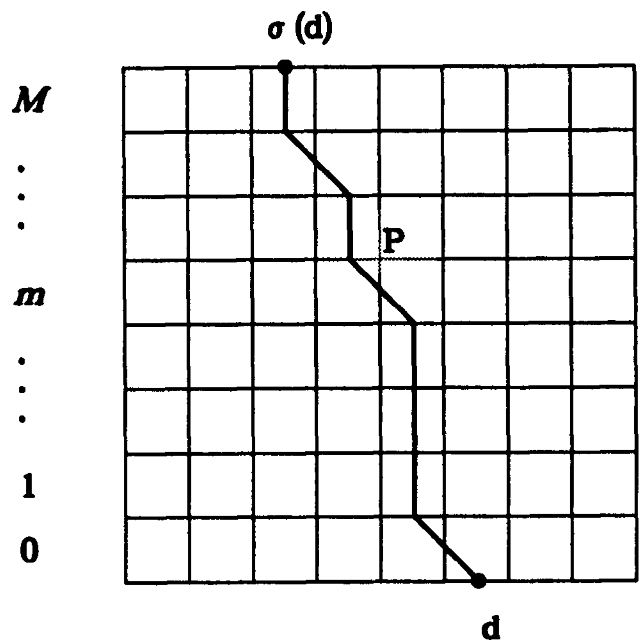

FIG. 5. A path $P\in\mathcal{P}(d;\sigma)$.

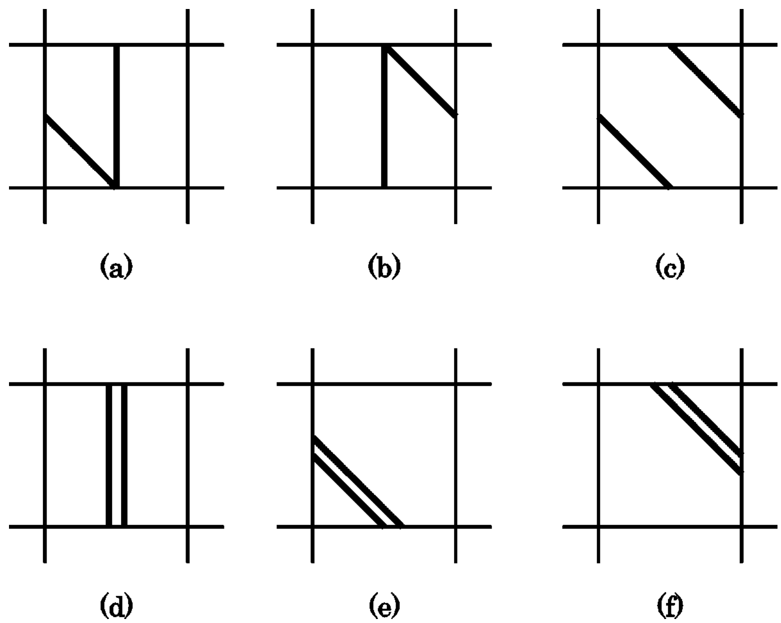

FIG. 6. Two lines can pass through a box in six possible ways.

through the same box (cf. Fig. 6). When two paths pass through a single box, there are six possible states as shown in Fig. 6. Except for the state (c) in Fig. 6, the lines touch each other. When the lines touch each other, we say that the lines *intersect*.

Now we show that, in (5), terms corresponding to intersecting paths cancel out. Let $P_{i_1} \in \mathcal{P}(d_{i_1};\sigma)$ and $P_{i_2} \in \mathcal{P}(d_{i_2};\sigma)$ be paths which intersect $(i_1 < i_2)$. Then at some box, the state (a) occurs as in the left-hand part of Fig. 7. Let P'_{i_1} and P'_{i_2} denote new paths constructed from P_{i_1} and P_{i_2} by exchanging lines in the box as shown in Fig. 7, where $P'_{i_1} \in \mathcal{P}(d_{i_1};\sigma')$, $P'_{i_2} \in \mathcal{P}(d_{i_2};\sigma')$ and

$$\sigma' = \begin{pmatrix} d_1 & \cdots & d_{i_1} & \cdots & d_{i_2} & \cdots & d_{N-k} \\ \sigma(d_1) & \cdots & \sigma(d_{i_2}) & \cdots & \sigma(d_{i_1}) & \cdots & \sigma(d_{N-k}) \end{pmatrix}.$$

Since $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$ and $\xi_{n,m}(P'_{i_1})\xi_{n,m}(P'_{i_2}) = \xi_{n,m}(P_{i_1})\xi_{n,m}(P_{i_2})$ for any $P_i \in \mathcal{P}(d_i;\sigma)$ $(i \neq i_1, i_2)$, the terms corresponding to $(P_1,\ldots,P_{i_1},\ldots,P_{i_2},\ldots,P_{N-k})$ and $(P_1,\ldots,P'_{i_1},\ldots,P'_{i_2},\ldots,P_{N-k})$ cancel each other out in (5).

If any two of $P_1,\ldots,P_{N-k}$ do not intersect, we say that the paths are nonintersecting. Hence only terms corresponding to nonintersecting paths contribute to (5).

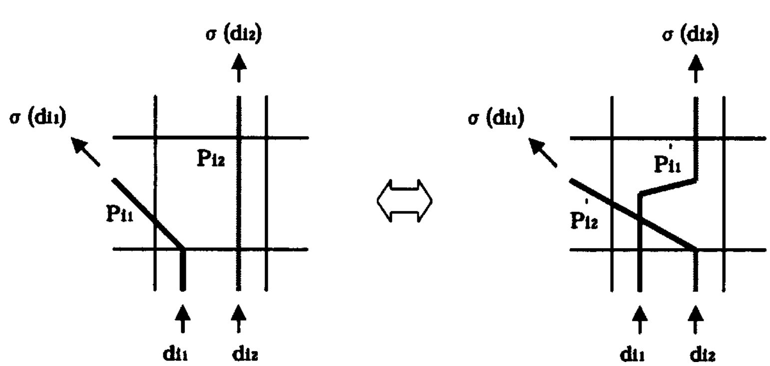

FIG. 7. Definition of P'_{i_1} and P'_{i_2}.

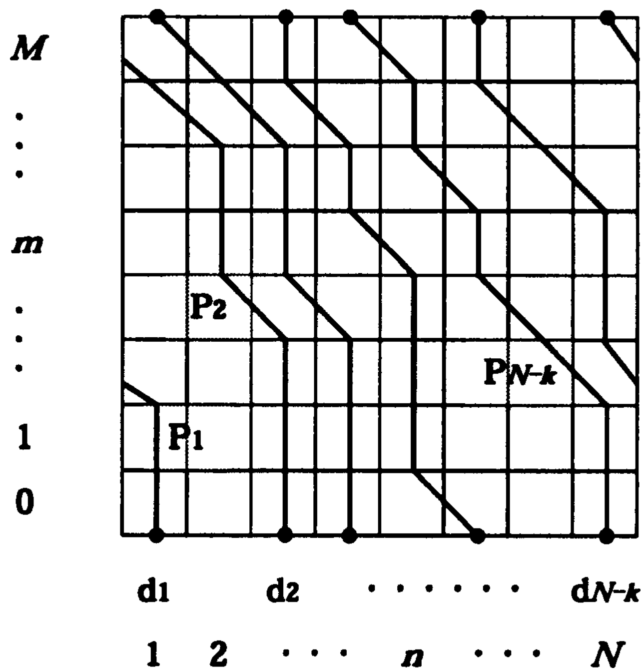

FIG. 8. Nonintersecting paths $P_1, \ldots, P_{N-k}$.

When $P_1, \ldots, P_{N-k}$ are nonintersecting $[P_1 \in \mathcal{P}(d_1;\sigma), \ldots, P_{N-k} \in \mathcal{P}(d_{N-k};\sigma)$ and $\sigma \in S_X^J, \sharp J = j]$,

$$\sigma(d_i) = \begin{cases} d_{i-j} & (i-j \geqslant 1), \\ d_{N+i-j} & (i-j \leqslant 0) \end{cases}$$

(cf. Fig. 8 and Fig. 9), and therefore $\mathrm{sgn}(\sigma) = (-1)^{j(N-k-1)}$.

In (3), the upper bound of the summation over j is $\{[(N-k)/N](M+1)\}$ where $[\ell]$ denotes the largest integer which does not exceed ℓ.

From (3) and (5), we obtain the following theorem.

Theorem II.1: *For A defined in (2), it holds that*

$$\det(\lambda I - A) = \sum_{k=0}^{N} (-1)^{N-k} \lambda^k \sum_{j=0}^{[(N-k)(M+1)/N]} (-1)^{j(N-k-1)} \mu^j \sum_{\substack{1 \leqslant d_1 < d_2 < \cdots \\ \cdots < d_{N-k} \leqslant N}} \sum_{\substack{(P_1,\ldots,P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1,\ldots,d_{N-k})}} \prod_{i=1}^{N-k} \prod_{n=1}^{N} \prod_{m=0}^{M} \xi_{n,m}(P_i),$$

where $\xi_{n,m}$ are defined in (4) and

$$\mathcal{P}^{(j)}(d_1, \ldots, d_{N-k}) := \left\{ (P_1, \ldots, P_{N-k}) \,\middle|\, \begin{array}{l} P_i \text{ connects the initial point } d_i \text{ and the} \\ \text{end point } d_{N+i-j} (1 \leqslant i \leqslant j); P_i \\ \text{connects } d_i \text{ and } d_{i-j} (j < i \leqslant N-k). \\ \text{Any two of them are nonintersecting} \end{array} \right\}. \qquad (6)$$

■

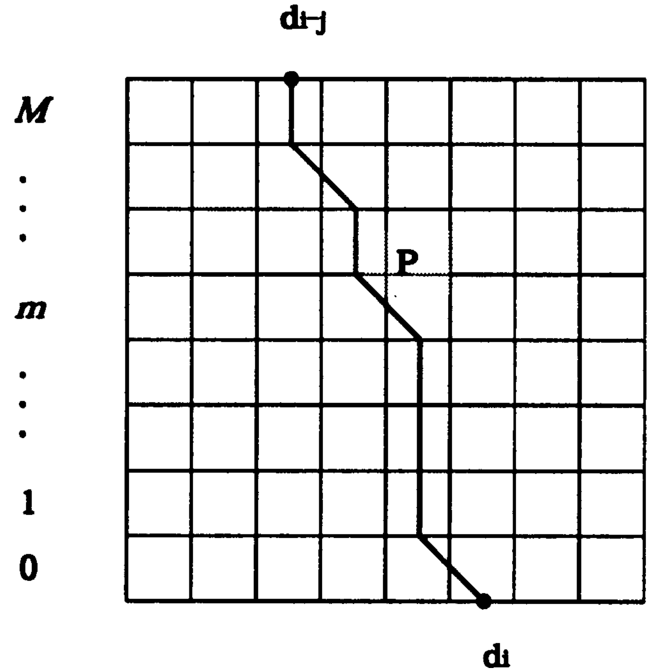

FIG. 9. A path $P \in \mathcal{P}(d_i;\sigma)$, $\sigma \in S_X^J$, $\sharp J=j$, which connects d_i and $\sigma(d_i)=d_{i-j}$.

III. gPBBS AND ndKP EQUATION

We briefly summarize the results obtained in Ref. 9 to fix the notations used in the subsequent sections.

A. gPBBS and its equation of motion

In order to describe the dynamics of the gPBBS, we introduce a new independent variable s ($s \in \mathbb{Z}$). As any integer s can be uniquely expressed as $s=Mt+j$ ($t \in \mathbb{Z}, 1 \leqslant j \leqslant M$), we denote by u_n^s the number of balls with index $j \equiv s \bmod M$ in the nth box at time step $t=[(s-1)/M]$, where $[x]$ denotes the largest integer which does not exceed x. In other words, the new *time* variable s is a refinement of the original time, indicating explicitly when balls with index j move.

We assume that θ_n and u_n^s satisfy the relation

$$\sum_{n=1}^{N} \theta_n - \sum_{j=1}^{M} \sum_{n=1}^{N} u_n^j \geqslant \sum_{n=1}^{N} u_n^k \quad (k=1,2,\ldots,M). \tag{7}$$

The first and second terms of the left-hand side of (7) represent the number of spaces and the number of balls in the gPBBS, respectively, hence the left-hand side is the total number of free spaces of the gPBBS. The right-hand side of (7) is the number of balls with index k. Thus (7) requires the total number of free spaces of the gPBBS to be larger than the number of copies of any type of ball in the time evolution process.

Let us consider the process at time s, i.e., the movement of the balls with index j at time step t where $s=Mt+j$; we often use s instead of j, i.e., we treat the indices modulo M. If we define κ_n^s, which denotes the number of spaces of the nth box at s, by

$$\kappa_n^s := \theta_n - (u_n^s + u_n^{s-1} + \cdots + u_n^{s-M+1}),$$

condition (7) is rewritten as

$$\sum_{n=1}^{N}\kappa_n^s \geqslant \sum_{n=1}^{N} u_n^{s-M+k} \quad (k=1,2,\ldots,M).$$

Theorem III.1 (Ref. 9): *The time evolution of the gPBBS is described by an ultradiscrete equation,*

$$u_n^{s+1}-\kappa_n^s = \max_{k=1,\ldots,N}\left[\sum_{j=1}^{k} u_{n-j}^{s-M+1}-\kappa_{n-j+1}^s\right]-\max\left[0,\max_{k=1,\ldots,N-1}\left[\sum_{j=1}^{k} u_{n-j}^{s-M+1}-\kappa_{n-j+1}^s\right]\right]. \tag{8}$$

■

B. From ndKP equation to gPBBS

The ndKP equation is obtained from the generating formula of the KP hierarchy.[12,13] It is given as

$$(b(m)-c(n))\tau(l+1,m,n)\tau(l,m+1,n+1)+(c(n)-a(l))\tau(l,m+1,n)\tau(l+1,m,n+1)+(a(l)-b(m))\tau(l,m,n+1)\tau(l+1,m+1,n)=0, \tag{9}$$

where l, m, $n\in\mathbb{Z}$ are independent variables, the tau function $\tau:\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ (or $\mathbb{C}$) is dependent variable and the coefficients $a(l)$, $b(m)$, $c(n)$ are arbitrary functions which depend on the independent variables l, m, n, respectively.

In order to relate the ndKP equation to the gPBBS, we take $a(l)=0$, $b(m)=1$, $c(n)=1+\delta_n$ and impose the following constraint on $\tau(l,m,n)$:

$$\tau(l,m,n)=\tau(l-M,m-1,n).$$

If we define $\sigma_n^s := \tau(s-1,0,n)$, (9) turns into

$$\frac{\sigma_{n+1}^{s+M-1}\sigma_{n+1}^{s}}{\sigma_{n+1}^{s+M}\sigma_{n+1}^{s-1}}-(1+\delta_{n+1})\frac{\sigma_n^{s-1}\sigma_{n+1}^{s}}{\sigma_{n+1}^{s-1}\sigma_n^{s}}=-\delta_{n+1}\frac{\sigma_n^{s+M}\sigma_{n+1}^{s}}{\sigma_n^{s}\sigma_{n+1}^{s+M}}. \tag{10}$$

Furthermore, we define U_n^s and K_n^s as

$$U_n^s := \frac{\sigma_{n+1}^{s}\sigma_n^{s+1}}{(1+\delta_{n+1})\sigma_n^{s}\sigma_{n+1}^{s+1}}, \quad \frac{1}{K_n^s}=\delta_{n+1}\cdot\prod_{j=1}^{M}U_n^{s-j+1}$$

and impose the following periodic condition on U_n^s:

$$U_n^s = U_{n+N}^s. \tag{11}$$

Then, from (10), we have

$$\frac{U_n^{s+1}}{K_n^s}=\frac{\displaystyle\sum_{k=1}^{N}\prod_{j=1}^{k}\frac{U_{n-j}^{s-M+1}}{K_{n-j+1}^s}}{1+\displaystyle\sum_{k=1}^{N-1}\prod_{j=1}^{k}\frac{U_{n-j}^{s-M+1}}{K_{n-j+1}^s}}. \tag{12}$$

To take the ultradiscrete limit, we set $U_n^s=e^{u_n^s/\epsilon}$, $K_n^s=e^{\kappa_n^s/\epsilon}$, $1/\delta_{n+1}=e^{\theta_n/\epsilon}$. Then, we found the following.

Theorem III.2 (Ref. 9): *The ultradiscrete limit of the constrained ndKP equation with the periodic boundary condition [i.e., (11) and (12)] coincides with the time evolution equation of the gPBBS (8).* ■

IV. CONSERVED QUANTITIES OF ndKP EQUATION

In Ref. 9 we derived the Lax representation for the ndKP equation when it has period N in the spatial variable n. In short, the equation (12) is equivalent to the matrix equation

$$\tilde{M}(s)L(M;s) = L(M;s-1)\tilde{M}(s),$$

where $\tilde{M}(s)=G_{U;s}-\tilde{\Upsilon}$,

$$L(M;s) = (-G_{K;s}+\tilde{\Upsilon})(G_{U;s-M+1}-\tilde{\Upsilon})(G_{U;s-M+2}-\tilde{\Upsilon})\cdots(G_{U;s}-\tilde{\Upsilon}), \tag{13}$$

$G_{K;s}=\mathrm{diag}(1/K_1^s,1/K_2^s,\ldots,1/K_N^s)$, $G_{U;s}=\mathrm{diag}(1/U_1^s,1/U_2^s,\ldots,1/U_N^s)$, and

$$\tilde{\Upsilon} := \begin{bmatrix} & & & & (1+\delta_N)\cdot\eta \\ 1+\delta_1 & & & & \\ & 1+\delta_2 & & & \\ & & \ddots & & \\ & & & 1+\delta_{N-1} & \end{bmatrix};$$

here, η is an arbitrary parameter.

This means

$$\det(\lambda I + L(M;s)) = \det(\lambda I + L(M;s-1));$$

therefore, the coefficients e_k of the characteristic polynomial

$$\det(\lambda I + L(M;s)) = \lambda^N + e_{N-1}\lambda^{N-1} + e_{N-2}\lambda^{N-2} + \cdots + e_1\lambda + e_0$$

are conserved in time s. Furthermore, since η is arbitrary and e_k contain η, if we define $e_k^{[j]}$ by

$$e_k = \sum_j e_k^{[j]}\eta^j, \tag{14}$$

then $e_k^{[j]}$ are also conserved.

Let $\Delta := \Pi_{i=1}^N(1+\delta_i)$,

$$\Upsilon := \begin{bmatrix} & & & & \eta\Delta \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix},$$

and $D_\delta := \mathrm{diag}(1,1+\delta_1,(1+\delta_1)(1+\delta_2),\ldots,\Pi_{i=1}^{N-1}(1+\delta_i))$. Since $\Upsilon=(D_\delta)^{-1}\tilde{\Upsilon}D_\delta$ we have (Ref. 9)

$$\det(\lambda I + L(M;s)) = \det(\lambda I + L_0(M;s)), \tag{15}$$

where

$$L_0(M;s) := (-G_{K;s}+\Upsilon)(G_{U;s-M+1}-\Upsilon)(G_{U;s-M+2}-\Upsilon)\cdots(G_{U;s}-\Upsilon).$$

From Theorem II.1, we obtain a combinatorial formula for $e_k^{[j]}$ immediately.

Theorem IV.1: *Set*

$$x_{n,0} = \frac{1}{K_n^s}, \quad x_{n,m} = \frac{1}{U_n^{s-M+m}} \quad (m \neq 0)$$

in (4) and set $\mu=\eta\Delta$. *Then, for* $k=0,1,\ldots,N$ *and* $j=0,1,\ldots,[(N-k)(M+1)/N]$, *it holds that*

$$e_k^{[j]} = (-1)^{\ell(k,j)}\Delta^j \sum_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}} \sum_{\substack{(P_1,\ldots,P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1,\ldots,d_{N-k})}} \prod_{i=1}^{N-k}\prod_{n=1}^{N}\prod_{m=0}^{M} \xi_{n,m}(P_i),$$

where $\ell(k,j) := (j+1)N-(k+j+kj)$. ■

V. CONSERVED QUANTITIES OF gPBBS

Using the results in Sec. IV, we construct the conserved quantities of the gPBBS.

For $k=0,1,\ldots,N$ and $j=0,1,\ldots,[(N-k)(M+1)/N]$, the ultradiscrete limit of $e_k^{[j]}$ is

$$ue_k^{[j]} := -\lim_{\epsilon\to+0}\epsilon\log((-1)^{\ell(k,j)}e_k^{[j]}) = -\lim_{\epsilon\to+0}\epsilon\log\left(\Delta^j \sum_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}} \sum_{\substack{(P_1,\ldots,P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1,\ldots,d_{N-k})}} \prod_{i=1}^{N-k}\prod_{n=1}^{N}\prod_{m=0}^{M} \xi_{n,m}(P_i)\right).$$

Since θ_n is the capacity of the nth box,

$$\lim_{\epsilon\to+0}\epsilon\log\Delta^j = j\cdot\lim_{\epsilon\to+0}\epsilon\log\prod_{j=1}^{N}(1+e^{-\theta_j/\epsilon}) = j\cdot\sum_{j=1}^{N}\max[0,-\theta_j] = 0.$$

Therefore, from Theorem IV.1, $ue_k^{[j]}$ is given by the following.

Theorem V.1: *Set*

$$x_{n,0} = \kappa_n^s, \quad x_{n,m} = u_n^{s-M+m}\ (m\neq 0)$$

in (4). Then, for $k=0,1,\ldots,N$ *and* $j=0,1,\ldots,[(N-k)(M+1)/N]$, *it holds that*

$$ue_k^{[j]} = \min_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}}\left[\min_{\substack{(P_1,\ldots,P_{N-k}) \\ \in \mathcal{P}^{(j)}(d_1,\ldots,d_{N-k})}}\left[\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)\right]\right].$$

■

Remark V.1: The conserved quantity $ue_k^{[0]}$ $(0\le k\le N)$ *is trivial. Since* $j=0$, *all paths are vertical lines. Hence we have*

$$ue_k^{[0]} = \min_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}}\left[\sum_{i=1}^{N-k}\left(\kappa_{d_i}^s + \sum_{m=1}^{M}u_{d_i}^{s-M+m}\right)\right] = \min_{\substack{1\le d_1<d_2<\cdots \\ \cdots<d_{N-k}\le N}}\left[\sum_{i=1}^{N-k}\theta_{d_i}\right].$$

As θ_n *is the capacity of the nth box,* $ue_k^{[0]}$ *does not depend on the time steps. So we are not interested in them.*

Remark V.2: Once we obtain all quantities that are conserved in variable s, *we are to have all quantities that are conserved in the original time variable* t. *The reasoning is as follows: Assume that* A_1 *is a conserved quantity of the gPBBS; this means* $A_1=A_1(s)$ *has period* M *in* s. [*Since equation of motion (8) is* M*th order in* s, $A_1(s)$ *is written as*

$$A_1(s) = F(u_1^s,\ldots,u_N^s,u_1^{s-1},\ldots,u_N^{s-1},\ldots,u_1^{s-M+1},\ldots,u_N^{s-M+1})$$

by some function F.] *Let* $A_j(s)=A_1(s+j-1)$ $(j=2,3,\ldots)$; *they all have period* M *in* s. *By definition* $A_j(s+1)=A_{j+1}(s)$ $(j=1,2,\ldots)$, *and, since* $A_1(s)$ *has period* M, $A_M(s+1)=A_1((s+1)+M-1)=A_1(s+M)=A_1(s)$; *hence, symmetric polynomials of* $A_1(s),\ldots,A_M(s)$ *are conserved in* s. *More explicitly, let* $S_k(s)$ *be the kth elementary symmetric polynomial of* $A_1(s),\ldots,A_M(s)$; *then, we have* M *quantities* $S_1(s),\ldots,S_M(s)$ *that are conserved in* s. *Conversely, once we know the elementary symmetric polynomials* $S_1(s),\ldots,S_M(s)$ *of* $A_1(s),\ldots,A_M(s)$, *we can obtain* $A_1(s),\ldots,A_M(s)$. *Therefore, the statement follows.*

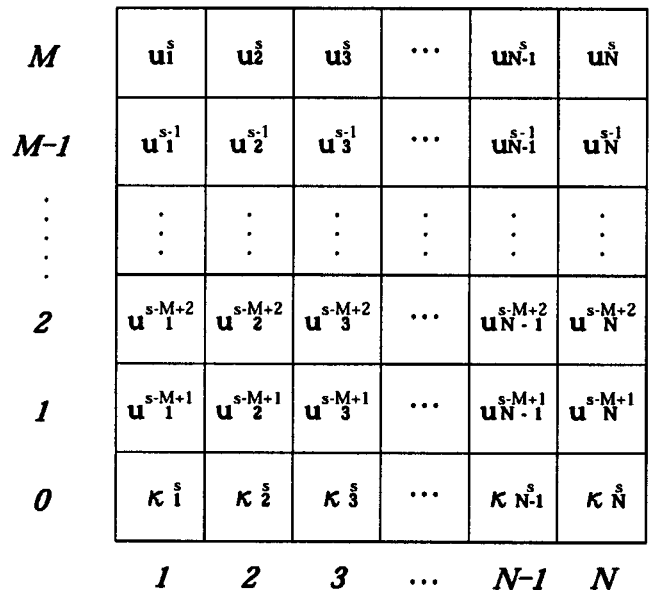

FIG. 10. Associate values $\kappa_n^s, u_n^s, \ldots, u_n^{s-M+1}$ with boxes of $C_{N,M+1}$.

An easy way to read off $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$ is as follows: Associate values κ_n^s, $u_n^s, \ldots, u_n^{s-M+1}$ with boxes of $C_{N,M+1}$ as shown in Fig. 10. For $(P_1, \ldots, P_{N-k}) \in \mathcal{P}^{(j)}(d_1, \ldots, d_{N-k})$, summing up the values corresponding to the vertical lines of the paths, we get the value $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$.

Example V.1: For a state in Fig. 11 $(N=10, M=5)$, *we obtain a table in Fig. 12. For paths shown in Fig. 13,*

$$\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i) = (0+0+1)+(0+1+0+0)+(0+0+0+0)+(0+1+0)+(0+0+1)$$

$$+(0+1+0)+(1+0)=6.$$

Occasionally these paths minimize $\sum_{i=1}^{N-k}\sum_{n=1}^{N}\sum_{m=0}^{M}\xi_{n,m}(P_i)$; *thus,* $ue_3^{[2]}$ *is* 6.

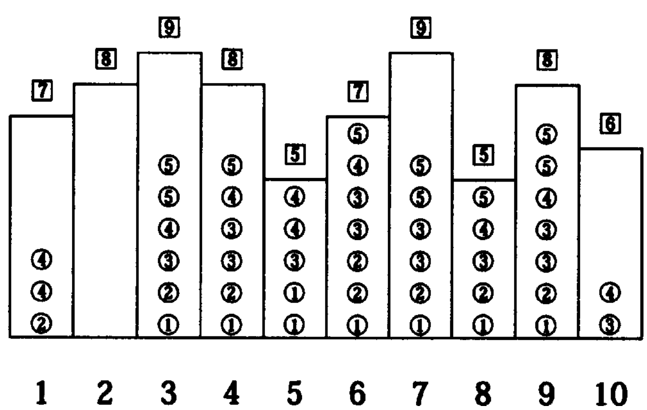

FIG. 11. A state of the gPBBS.

5	0	0	2	1	0	1	2	1	2	0
4	2	0	1	1	2	1	0	1	1	1
3	0	0	1	2	1	2	2	1	2	1
2	1	0	1	1	0	2	1	1	1	0
1	0	0	1	1	2	1	1	1	1	0
0	4	8	3	2	0	0	3	0	1	4
	1	2	3	4	5	6	7	8	9	10

FIG. 12. A table obtained for a state in Fig. 11 ($N=10, M=5$).

VI. DISCUSSION

In this section we discuss some algebraic aspects of the gPBBS. The time evolution of the gPBBS is decomposed into a product of transformations, each of which is a representation of the generators of the affine Weyl group $\widetilde{W}(A_M^{(1)})$. Furthermore a state of the gPBBS is naturally identified with a vector of a tensor product of the crystals $U_q'(A_M^{(1)})$, and a time evolution pattern is interpreted as twisted lattices of the crystals $U_q'(A_{N-1}^{(1)})$ whose Boltzmann weights are determined by the combinatorial R matrices.

A. Affine Weyl group and gPBBS

Let $\mathcal{T}$ be the set of $N\times(M+1)$ rectangular tableaux with integer entries, and $s_\ell(\ell\in\mathbb{Z}/(M+1)\mathbb{Z})$ and π be mappings: $\mathcal{T}\to\mathcal{T}$. For a tableau

$$Y=\begin{array}{|c|c|c|c|}\hline y_{1,M} & y_{2,M} & \cdots & y_{N,M}\\ \hline \vdots & \vdots & \cdots & \vdots\\ \hline y_{1,1} & y_{2,1} & \cdots & y_{N,1}\\ \hline y_{1,0} & y_{2,0} & \cdots & y_{N,0}\\ \hline\end{array}\,,$$

these mappings are given as

$$s_\ell(Y)=\begin{array}{|c|c|c|c|}\hline s_\ell(y_{1,M}) & s_\ell(y_{2,M}) & \cdots & s_\ell(y_{N,M})\\ \hline \vdots & \vdots & \cdots & \vdots\\ \hline s_\ell(y_{1,1}) & s_\ell(y_{2,1}) & \cdots & s_\ell(y_{N,1})\\ \hline s_\ell(y_{1,0}) & s_\ell(y_{2,0}) & \cdots & s_\ell(y_{N,0})\\ \hline\end{array}\,,$$

$$\pi(Y)=\begin{array}{|c|c|c|c|}\hline \pi(y_{1,M}) & \pi(y_{2,M}) & \cdots & \pi(y_{N,M})\\ \hline \vdots & \vdots & \cdots & \vdots\\ \hline \pi(y_{1,1}) & \pi(y_{2,1}) & \cdots & \pi(y_{N,1})\\ \hline \pi(y_{1,0}) & \pi(y_{2,0}) & \cdots & \pi(y_{N,0})\\ \hline\end{array}\,,$$

where

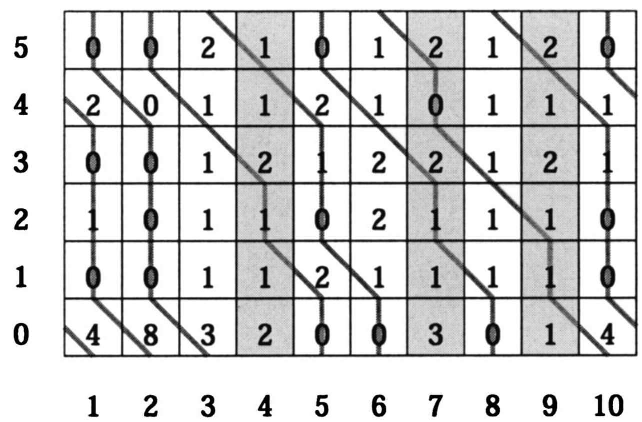

FIG. 13. Paths (see text).

$$s_\ell(y_{n,m}) = y_{n,m+1} + Q_{n,m} - Q_{n-1,m} \quad (m \equiv \ell \bmod M+1),$$

$$s_\ell(y_{n,m+1}) = y_{n,m} + Q_{n-1,m} - Q_{n,m} \quad (m \equiv \ell \bmod M+1),$$

$$s_\ell(y_{n,m}) = y_{n,m} \quad (m \neq \ell, \ell+1 \bmod M+1),$$

$$\pi(y_{n,m}) = y_{n,m+1},$$

and

$$Q_{n,m} = \max_{1 \leqslant h \leqslant N} \left[\sum_{k=1}^{h-1} y_{n+k,m+1} + \sum_{k=h+1}^{N} y_{n+k,m} \right].$$

Here we extend the indices n,m of $y_{n,m}$ for $n,m \in \mathbb{Z}$ by the condition $y_{n+N,m}=y_{n,m+M+1}=y_{n,m}$.

The following theorem is proved by direct calculations.

Theorem VI.1 (Refs. 14 and 15): *The mappings s_ℓ ($\ell \in \mathbb{Z}/(M+1)\mathbb{Z}$) and π defined as above give a realization of the affine Weyl group $\widetilde{W}(A_M^{(1)})$.* ■

Remark VI.1: The affine Weyl group $\widetilde{W}(A_{n-1}^{(1)})$ is defined as the group generated by the simple reflections $s_0, s_1, \ldots, s_{n-1}$ and diagram rotation π subject to the fundamental relations

$$s_i^2 = 1,$$

$$s_i s_j = s_j s_i \quad (j \neq i, i \pm 1 \bmod n),$$

$$s_i s_j s_i = s_j s_i s_j \quad (j \equiv i \pm 1 \bmod n),$$

$$\pi s_i = s_{i+1} \pi,$$

where we understand the indices for s_i as elements of $\mathbb{Z}/(M+1)\mathbb{Z}$.

When we set

$$y_{n,0} = \kappa_n^s, \quad y_{n,m} = u_n^{s-M+m} \quad (m \neq 0), \tag{16}$$

we get the following theorem which gives a relation between the gPBBS and the affine Weyl group.

Theorem VI.2: $\pi s_{M-1} s_{M-2} \cdots s_0$ *gives the time evolution which concerns the original time* *variable t, i.e.,*

$$\pi s_{M-1}s_{M-2}\cdots s_0\left(\begin{array}{|c|c|c|c|}\hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s-M+1} & u_2^{s-M+1} & \cdots & u_N^{s-M+1} \\ \hline \kappa_1^s & \kappa_2^s & \cdots & \kappa_N^s \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|}\hline u_1^{s+M} & u_2^{s+M} & \cdots & u_N^{s+M} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s+1} & u_2^{s+1} & \cdots & u_N^{s+1} \\ \hline \kappa_1^{s+M} & \kappa_2^{s+M} & \cdots & \kappa_N^{s+M} \\ \hline \end{array}$$

. ■

The proof goes as follows.

The equation of motion (8) is

$$u_n^{s+1} = u_n^{s-M+1} + Q_{n-1,0} - Q_{n,0}.$$

This means

$$s_0(\kappa_n^s) = u_n^{s+1},$$

and

$$s_0(u_n^{s-M+1}) = \kappa_n^s + Q_{n-1,0} - Q_{n,0} = \kappa_n^s + u_n^{s+1} - u_n^{s-M+1};$$

in the gPBBS, κ_n^s denotes the number of spaces of the nth box at s, and u_n^{s+1}, u_n^{s-M+1} denote the numbers of balls which come in the nth box and get out the nth box from time step s to $s+1$, respectively; hence

$$s_0(u_n^{s-M+1}) = \kappa_n^{s+1}.$$

Therefore

$$s_0\left(\begin{array}{|c|c|c|c|}\hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s-M+1} & u_2^{s-M+1} & \cdots & u_N^{s-M+1} \\ \hline \kappa_1^s & \kappa_2^s & \cdots & \kappa_N^s \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|}\hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline s_0(u_1^{s-M+1}) & s_0(u_2^{s-M+1}) & \cdots & s_0(u_N^{s-M+1}) \\ \hline s_0(\kappa_1^s) & s_0(\kappa_2^s) & \cdots & s_0(\kappa_N^s) \\ \hline \end{array}$$

$$= \begin{array}{|c|c|c|c|}\hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline \kappa_1^{s+1} & \kappa_2^{s+1} & \cdots & \kappa_N^{s+1} \\ \hline u_1^{s+1} & u_2^{s+1} & \cdots & u_N^{s+1} \\ \hline \end{array}.$$

Repeating the above procedure, we obtain

$$s_{M-1}s_{M-2}\cdots s_0\left(\begin{array}{|c|c|c|c|}\hline u_1^s & u_2^s & \cdots & u_N^s \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s-M+1} & u_2^{s-M+1} & \cdots & u_N^{s-M+1} \\ \hline \kappa_1^s & \kappa_2^s & \cdots & \kappa_N^s \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|}\hline \kappa_1^{s+M} & \kappa_2^{s+M} & \cdots & \kappa_N^{s+M} \\ \hline u_1^{s+M} & u_2^{s+M} & \cdots & u_N^{s+M} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline u_1^{s+1} & u_2^{s+1} & \cdots & u_N^{s+1} \\ \hline \end{array}.$$

Finally, applying π upon it immediately gives Theorem VI.2.

B. gPBBS as twisted crystal lattice

The BBSs can be reformulated as integrable lattice models at temperature zero from viewpoint of the crystal theory and the combinatorial R matrix.[16,17] The PBBS with one kind of ball and box capacity one has also been reformulated into two types of lattice models, a periodic $A_1^{(1)}$ crystal lattice and a twisted $A_{N-1}^{(1)}$ crystal chain, where N denotes the number of the boxes in the system.[8] It is straightforward to extend this result to the case of the gPBBS. In this section, we will briefly show how the gPBBS is reinterpreted as some integrable lattice systems. Since the proofs for the statements below are almost the same as those in Ref. 8, we will omit them here.

Let B_k be the classical crystal of $U_q'(A_M^{(1)})$ corresponding to the k-fold symmetric tensor representation of $U_q(A_M)$. As a set it consists of the single row semistandard tableaux of length k on letters $\{1,2,\ldots,M+1\}$,

$$B_k := \left\{ \boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_k} \;\middle|\; 1 \le i_1 \le i_2 \le \cdots \le i_k \le M+1 \right\}.$$

An element b

$$b = \boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_k} \;\in\; B_k$$

is also denoted as a series of $M+1$ integers $b \equiv (x^{(M+1)}, x^{(M)}, \ldots, x^{(2)}, x^{(1)})$, where $x^{(j)}$ is the number of letters j in b. A state $|\psi\rangle_t$ of the gPBBS is naturally identified with

$$|\psi\rangle_t \cong b_1^t \otimes b_2^t \otimes \cdots \otimes b_N^t \in B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N},$$

where

$$b_n^t = (\kappa_n^s, u_n^{s-M+1}, u_n^{s-M+2}, \ldots, u_n^s) \quad (n = 1,2,\ldots,N).$$

For the BBS without the periodic boundary condition, time evolution is given by the isomorphism induced by the combinatorial R matrices,

$$\mathcal{T}: B_\infty \otimes (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \to (B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N}) \otimes B_\infty,$$

$$\mathcal{T}: |\{0\}\rangle \otimes |\psi\rangle_t \to |\psi\rangle_{t+1} \otimes |\{0\}\rangle,$$

where $|\{0\}\rangle$ is the highest weight vector of B_∞. For the gPBBS, by taking the trace of the auxiliary state in B_∞, $T := \mathrm{Tr}_{B\infty}\,\mathcal{T}$, we have the time evolution

$$T: B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N} \to B_{\theta_1} \otimes B_{\theta_2} \otimes \cdots \otimes B_{\theta_N},$$

$$T: |\psi\rangle_t \to |\psi\rangle_{t+1}.$$

As the $A_1^{(1)}$ crystal, the operator T maps $|\psi\rangle_t$ to the unique tensor product of $A_M^{(1)}$ crystal that exactly corresponds to the state of the gPBBS at $t+1$.

The gPBBS is also reformulated as a twisted lattice of M vertical axes in terms of $A_{N-1}^{(1)}$ crystals. In this case, a state $|\psi\rangle_t$ is identified

$$|\psi\rangle_t \cong b_\kappa^t \otimes (b_{u_1}^t \otimes b_{u_2}^t \otimes \cdots \otimes b_{u_M}^t) \in B_k \otimes (B_{n_1} \otimes B_{n_2} \otimes \cdots \otimes B_{n_M}),$$

where

$$b_\kappa^t = (\kappa_N^s, \kappa_{N-1}^s, \ldots, \kappa_1^s),$$

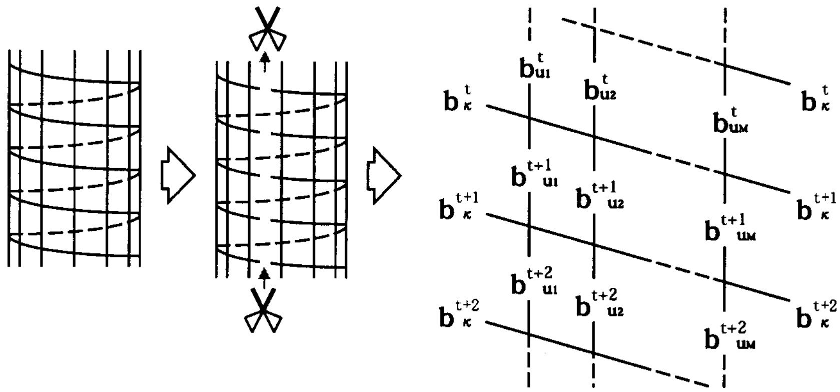

FIG. 14. The twisted crystal lattice associated with the gPBBS.

$$b_{u_j}^t = (u_N^{s-M+j}, u_{N-1}^{s-M+j}, \ldots, u_1^{s-M+j}) \quad (j = 1, 2, \ldots, M),$$

$k := \sum_{n=1}^N \kappa_n^s$ and $n_j := \sum_{n=1}^N u_n^{s-M+j}$. The time evolution is determined by the isomorphism induced by the combinatorial R-matrix for $A_{N-1}^{(1)}$ crystal,

$$b_\kappa^t \otimes (b_{u_1}^t \otimes b_{u_2}^t \otimes \cdots \otimes b_{u_M}^t) \cong (b_{u_1}^{t+1} \otimes b_{u_2}^{t+1} \otimes \cdots \otimes b_{u_M}^{t+1}) \otimes b_\kappa^{t+1}.$$

In Fig. 14, we schematically show the twisted crystal lattice associated with the gPBBS.

VII. CONCLUDING REMARKS

In this paper, using a path description of the characteristic polynomial of particular matrices and an algorithm to construct the conserved quantities using the Lax representation of the ndKP equation, we showed explicit form of the conserved quantities of the gPBBS. Relations to the affine Weyl group action and the crystal theory were also clarified. An advantage to reformulate the PBBS as crystal lattices is that we can extend it to the crystals associated with other root systems.

Since the gPBBS is composed of a finite number of boxes and balls, it can only take on a finite number of patterns. Hence its trajectory is always periodic and a fundamental cycle, i.e., the shortest period of the periodic motion, exists for any given initial state. In the case where the box capacity is one everywhere and only one kind of ball exists, the formula used to calculate the fundamental cycle is explicitly obtained using the conserved quantities and some rescaling properties of the states.[18] Hence, using the results in this paper, we may get the formula to calculate the fundamental cycle for the gPBBS, which is a problem we wish to address in the future.

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[1] D. Takahashi and J. Satsuma, J. Phys. Soc. Jpn. **59**, 3514 (1990).
[2] D. Takahashi, "One some soliton systems defined by using boxes and balls," Proceedings of the International Symposium on nonlinear theory and its applications, NOLTA'93, 1993, pp. 555–558.
[3] T. Miwa, M. Jimbo, and E. Date, *Solitons-Differential Equations, Symmetries and Infinite Dimensional Algebras* (Cambridge University Press, Cambridge, 2000).
[4] R. Hirota, "Chokusetsuhou ni yoru soliton no suuri" (in Japanese), Iwanami, 1992.
[5] T. Tokihiro, D. Takahashi, J. Matsukidaira, and J. Satsuma, Phys. Rev. Lett. **76**, 3247 (1996).
[6] R. Hirota and S. Tsujimoto, J. Phys. Soc. Jpn. **64**, 3125 (1995).
[7] A. Nagai, D. Takahashi, and T. Tokihiro, Phys. Lett. A **255**, 265 (1999).
[8] F. Yura and T. Tokihiro, J. Phys. A **35**, 3787 (2002).

[9] J. Mada, M. Idzumi, and T. Tokihiro, J. Phys. A **37**, 6531 (2004).
[10] R. Willox, T. Tokihiro, and J. Satsuma, J. Math. Phys. **38**, 6455 (1997).
[11] M. Torii, D. Takahashi, and J. Satsuma, Physica D **92**, 209 (1996).
[12] M. Sato and Y. Sato, "Soliton equations as dynamical systems on infinite dimensional Grassmann manifold," *Nonlinear PDE in Applied Science. U.S.-Japan Seminar, Tokyo, 1982* [Lect. Notes Num. Appl. Anal. **5**, 259 (1982)].
[13] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, "Transformation groups for soliton equations," in *Proceedings of RIMS Symposium on Non-linear Integrable Systems-Classical Theory and Quantum Theory, Kyoto, Japan, 1981*, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983), pp. 39–119.
[14] K. Kajiwara, M. Noumi, and Y. Yamada, Lett. Math. Phys. **60**, 211 (2002).
[15] M. Noumi and Y. Yamada, "Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions," Advanced Studies in Pure Mathematics Vol. 40, 2004, math-ph/0203030 (unpublished).
[16] K. Fukuda, M. Okado, and Y. Yamada, Int. J. Mod. Phys. A **15**, 1379 (2000).
[17] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi, and T. Tokihiro, J. Math. Phys. **42**, 274 (2001).
[18] D. Yoshihara, F. Yu, and T. Tokihiro, J. Phys. A **36**, 99 (2003).