REU Day 7:

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DEF'N: A simplicial complex on a set V is a subset Δ of 2^V such that $F \in \Delta$, $F' \subseteq F$
$\Longrightarrow F' \in \Delta$

EXAMPLES:

(1) $\Delta = 2^V$

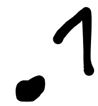
(2) $V = \{1,2,3\}$, $\Delta = \{\overbrace{12, 13}^{\text{facets}}, 1, 2, 3, \emptyset\}$

Elements of Δ are called <u>faces</u> and maximal faces are called <u>facets</u>.

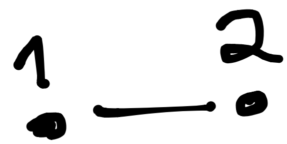
Note that the facets of Δ determine Δ completely, since $\Delta = \bigcup_{\text{facets } F} 2^F$.

Draw a face of size $n+1$ as an <u>n-simplex</u>:

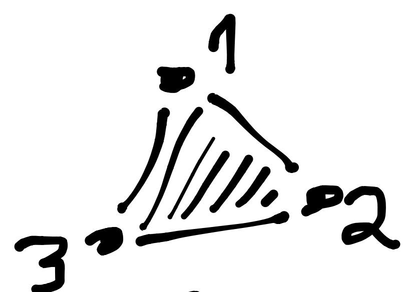
$F = \{1\} \rightsquigarrow$



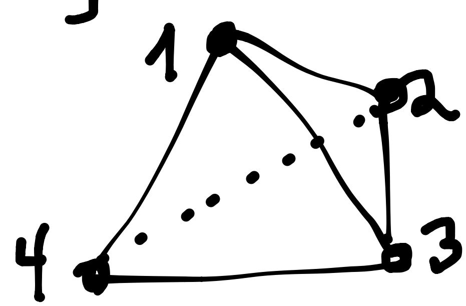
$F = \{1,2\} \rightsquigarrow$



$F = \{1,2,3\} \rightsquigarrow$

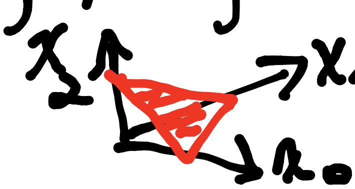


$F = \{1,2,3,4\} \rightsquigarrow$

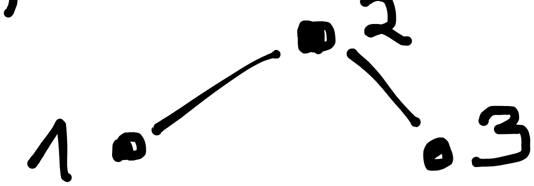


In general,

$F = \{1,2,\dots,n+1\} \rightsquigarrow \{(x_0,x_1,\dots,x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, x_0+\dots+x_n = 1\}$

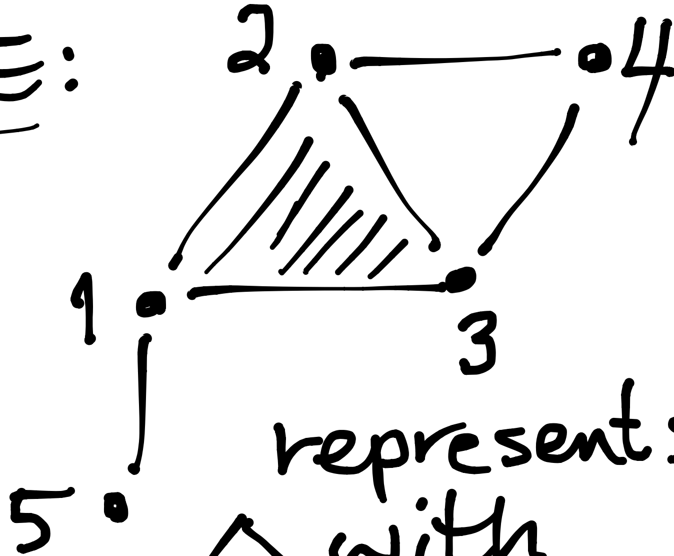


EXAMPLE: Δ having facets 12, 23 will be drawn



Note that faces of an n-simplex $\leftrightarrow$ subsets of $\{1, 2, \ldots, n+1\}$

EXAMPLE:



represents Δ with facets 123, 24, 34, 15.

DEF'N: Say that Q is a word of length ℓ in alphabet $1, 2, 3, \ldots$, and $\mathcal{Q}$ is a set of distinguished words. Then the subword complex is

$$\Delta(Q, \mathcal{Q}) := \{F \subseteq \{1, 2, \ldots, \ell\} : \text{the subword of } Q \text{ in positions } F^c \text{ contains a member of } \mathcal{Q} \text{ as a subword}\}.$$

complement of F within $\{1, 2, \ldots, \ell\}$

EXAMPLE: $Q = 1\,2\,1\,2\,1\,2$
$\qquad\quad (1)(2)(3)(4)(5)(6)$

$$\mathcal{Q} = \{121, 212\}$$

$\rightsquigarrow V = \{1, 2, 3, 4, 5, 6\}$

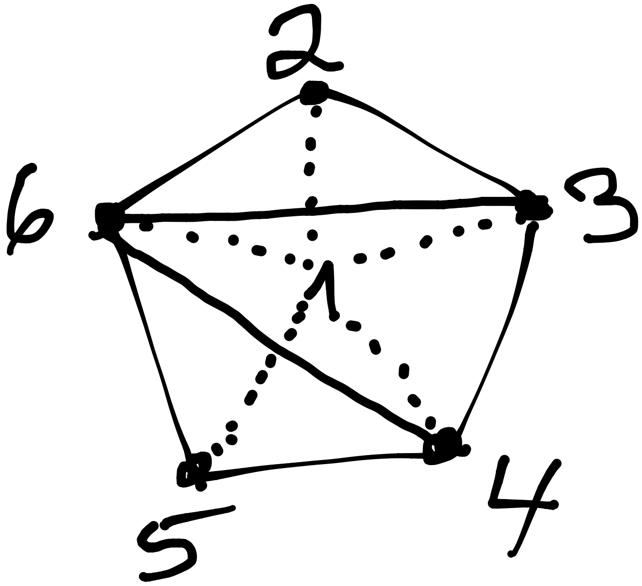
$14 \in \Delta(Q, \mathcal{Q})$ since $\underline{1}\,\underline{2}\underline{1}\underline{2}\,\underline{1}\,\underline{2}$
$\qquad\qquad\qquad\qquad\qquad\qquad (1) \qquad (4)$

$\rightsquigarrow 2122$ contains 212

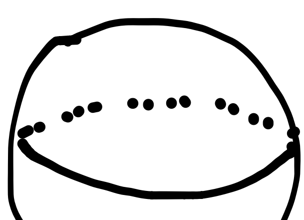
Facets of $\Delta(Q, \mathcal{Q})$:

123 156 145 346
456 126 236
134

Here is $\Delta(Q, \mathcal{Q})$:



Note that it is homeomorphic (topologically equivalent) to a 2-dimensional sphere S^2



Where to get some interesting $\mathcal{Q}$ & $\mathcal{R}$'s?

<u>DEF'N:</u> A <u>Coxeter group</u> is a group W with a presentation

$$W = \langle s \in S \mid s^2 = 1,\ (ss')^{m(s,s')} = 1 \rangle$$

where $m(s,s') \in \{2,3,4,\ldots\} \cup \{\infty\}$

<u>EXAMPLE</u>: $W = S_n$ = symmetric group on n letters

$W = S_n$ is generated by

$S = \{s_1, s_2, \dots, s_{n-1}\}$

where $s_i = (i \;\; i{+}1)$ for $i = 1, 2, \dots, n-1$

with relations: $\underline{s_i^2 = 1 \;\; \forall i}$

$s_i s_j = s_j s_i$ if $|i-j| > 1$ e.g. $s_1 s_3 = s_3 s_1 = (12)(34)$

$\Updownarrow$

$\underline{(s_i s_j)^2 = 1}$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

$\Updownarrow$

$(s_i s_{i+1})^3 = 1$

Coxeter relations

EXAMPLE:

Signed permutations

EXAMPLE:

Dihedral groups

Say $w \in W$ a Coxeter group with generators S.

The minimal ℓ such that $w = s_1 s_2 \cdots s_\ell$ with $s_i \in S$ is called the length $\ell(w)$. Any Such expression for w is called a reduced expression for w.

EXAMPLE:

$$
\begin{aligned}
w = {} & 3421 \\
& \Big\downarrow s_2 \ \text{(multiply on right)} \\
& 3241 \\
& \Big\downarrow s_1 \\
& 2341 \\
& \Big\downarrow s_3 \\
& 2314 \\
& \Big\downarrow s_2 \\
& 2134 \\
& \Big\downarrow s_1 \\
& 1234
\end{aligned}
$$

$\ell(w) = 5$

$= \#$ of <u>inversion pairs</u> $i<j$ with $w(i)>w(j)$

$\Rightarrow s_1 s_2 s_3 s_1 s_3$ is a reduced expression for w

Call 12312 a <u>reduced</u> <u>word</u> for w.

Let $\mathcal{R}(w) := \left\{\begin{matrix}\text{all reduced} \\ \text{words for } w\end{matrix}\right\}$.

THEOREM (Matsumoto-Tits)
Any two reduced expressions for $w \in W$ are connected by a sequence of Coxeter relations in this form

$$\underbrace{ss'ss'\cdots\cdots}_{\substack{m(s,s') \\ \text{letters}}} = \underbrace{s'ss's\cdots\cdots}_{\substack{m(s,s') \\ \text{letters}}}$$

EXAMPLE:

$$\mathcal{R}(3421) =$$

$$\left\{ \begin{tikzcd} 12312 \arrow[r, dash] & 12132 \arrow[d, dash] \\ & 21232 \arrow[d, dash] \\ 23123 \arrow[r, dash] & 21323 \end{tikzcd} \right\}.$$

Say $\mathcal{Q}$ is a word in S and $w \in W$. The associated subword complex is

$$\Delta(\mathcal{Q}, \mathcal{R}(w))$$

EXAMPLE:

$$\mathcal{Q} = 121212$$

$$\mathcal{R} = \{121, 212\} = \mathcal{R}(321)$$

THEOREM (Knutson-Miller)

$$\Delta(Q, \mathcal{Q}(w)) \cong \begin{cases} \text{sphere if } Q \text{ has } \underline{\text{Demazure}}\ \underline{\text{product}}\ w \\ \text{ball otherwise} \end{cases}$$

where the Demazure product is defined by

$$\underset{\underset{W}{\cap}}{w} \cdot \underset{\underset{S}{\cap}}{s} := \begin{cases} ws & \text{if } \ell(ws) > \ell(w) \\ w & \text{if } \ell(ws) < \ell(w) \end{cases}.$$

EXAMPLE:

The Demazure product of 121212 in S_3 is

$$((((s_1 s_2) s_1) s_2) s_1) s_2 = 321$$

$s_1 s_2 = 231$, $231 \circ s_1 = 321$

$\ell(s_1) = 1$ $\quad$ $\ell(231) = 2$

$321 \circ s_2 = 321$

$\vdots$

This explains why $\Delta(121212, \mathcal{R}(321))$ was a sphere.

Vertex-decomposability

DEF'N: Given $v \in V$, the *deletion* of v (from Δ) is

$$\mathrm{del}(\Delta, v) := \{F \in \Delta : v \notin F\}.$$

DEF'N: The *link* of v (in Δ) is

$$\mathrm{link}(\Delta, v) := \{F \setminus \{v\} : v \in F \in \Delta\}.$$

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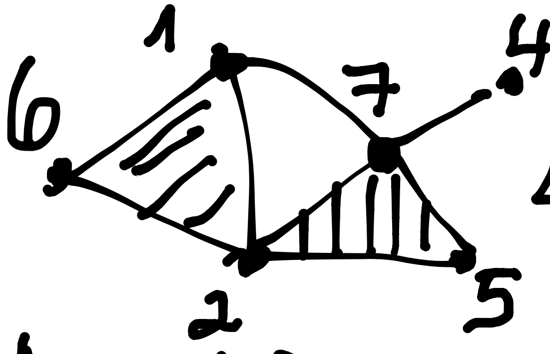
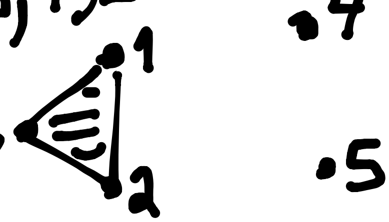
e.g.



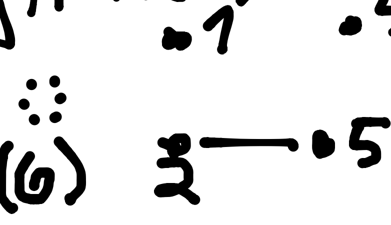
Δ

Δ on $V = \{1,2,3,4,5,6,7\}$

$\mathrm{del}(\Delta, 7) =$



$\mathrm{link}(\Delta, 7)$

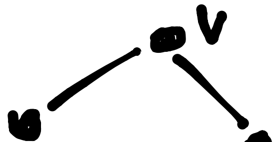


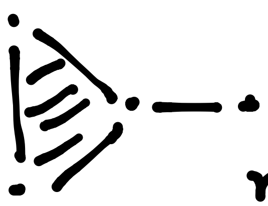
DEF'N: Δ is pure (of dimension d) if all facets of Δ have $d+1$ vertices.

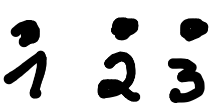
DEF'N: Δ is vertex-decomposable if it is pure, and in addition,

either 1) $\exists v \in V$ with $\mathrm{del}(\Delta, v)$, $\mathrm{link}(\Delta, v)$ both vertex-decomposable

or 2) $\Delta = \{\emptyset\}$.

EXAMPLE: ①  is vertex-decomposable

②  is not pure, hence not vertex-decomposable

③  on $V = \{1,2,3\}$ is vertex-decomposable

but  on $V = \{1,2,3\}$ is not.

EXERCISE 20:

Show that any $w \in W$
$\Delta(Q, \mathcal{Q}(w))$ is always
vertex-decomposable
(for general Coxeter groups W,
or just for $W = S_n$)

EXERCISE 21:

Characterize which $w \in W$
have a palindromic reduced
word.

reads same
backward & forward.

Involutions in Coxeter groups

Let W be a Coxeter groups with generators S, and

$$I(W) = \{w \in W : w^2 = 1\}.$$

For $w \in I(W)$ and $s \in S$, define

$$w \ltimes s := \begin{cases} ws & \text{if } ws = sw, \\ sws & \text{else.} \end{cases}$$

<u>Note</u> that $w \ltimes s \in I(W)$.

Given $w \in W = S_n$, say that $a = a_1 a_2 \cdots a_p$ is a (reduced) involution word for $w \in I(W)$ if

$$w = (\cdots((\mathrm{id} \ltimes s_{a_1}) \ltimes s_{a_2}) \cdots) \ltimes s_{a_p}$$

and p is the minimum length for such an expression.

Say $\hat{\ell}(w) := p$ in this situation, and let

$$\hat{\mathcal{R}}(w) := \left\{ \begin{array}{c} \text{all involution} \\ \text{words for } w \end{array} \right\}$$

EXAMPLE: $W = S_4$

$a = 3121 \in \widehat{\mathcal{Q}}(4321)$

since

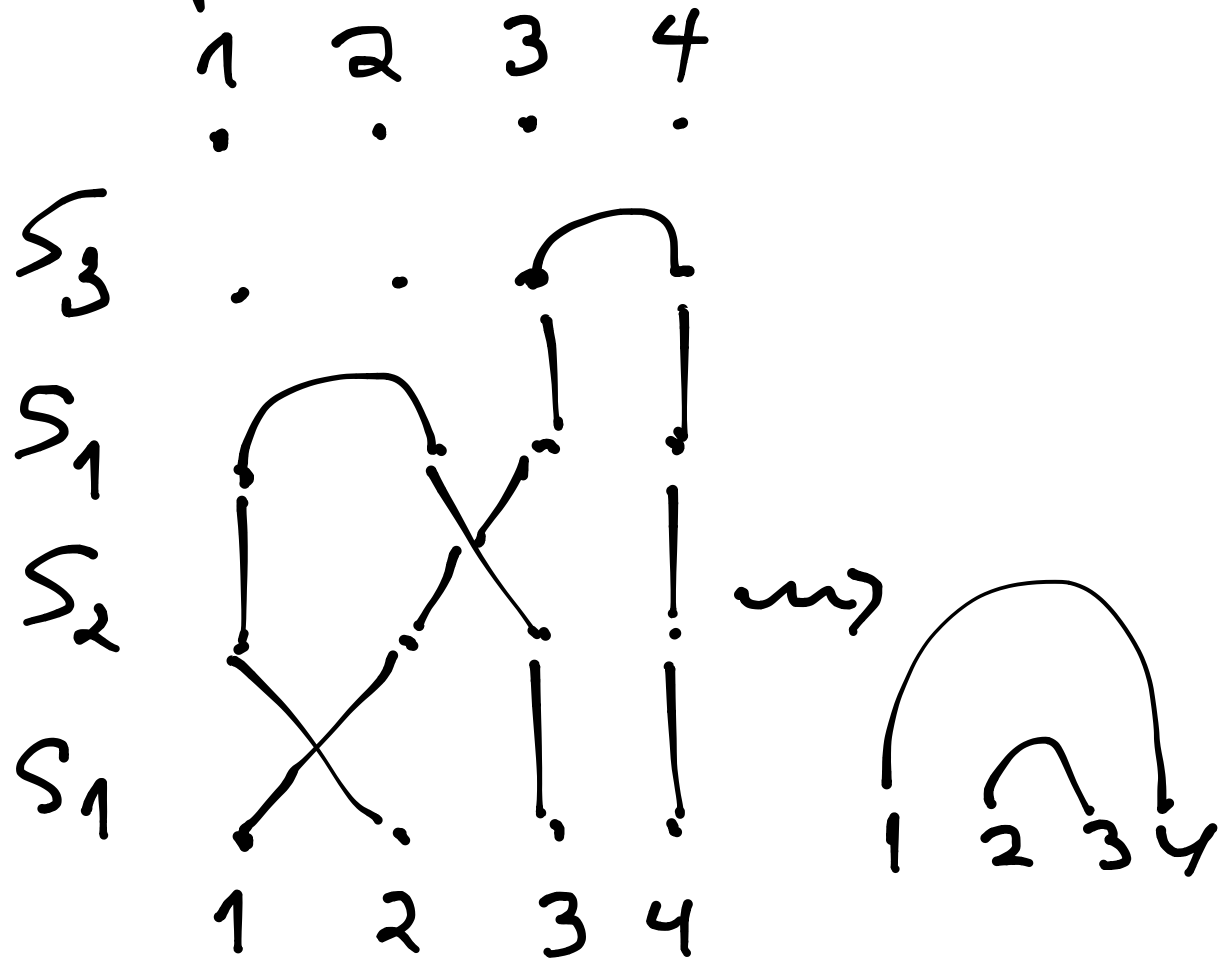
$id \rtimes s_3 = s_3 = 1243 = (34)(1)(2)$

$s_3 \rtimes s_1 = 2143 = (12)(34)$

$(s_3 \rtimes s_1) \rtimes s_2 = 3412 = (13)(24)$

$((s_3 \rtimes s_1) \rtimes s_2) \rtimes s_1 = 4321 = (14)(23)$

One can visualize this as building up partial matchings corresponding to the aj denotations:



FACT (not obvious):
Every involution word $a \in \hat{\mathcal{R}}(w)$ is a reduced word (in the previous sense), that is $a \in \mathcal{R}(v)$ for some $v \in S_n$.

THM (Richardson-Springer '89)
If $w \in I(W)$, $a \in \hat{\mathcal{R}}(w)$ and $a \in \mathcal{R}(v)$ for some $v \in W$, then in fact

$$\mathcal{R}(v) \subseteq \hat{\mathcal{R}}(w).$$

In this case, we call v an <u>atom</u> of w. Denote by $\mathcal{A}(w)$ the set of atoms of w.

Then $$\hat{\mathcal{Q}}(w) = \bigsqcup_{v \in \mathcal{A}(w)} \mathcal{Q}(v)$$

<u>REU Problem 7(a):</u>

Let Q be a word in the alphabet S of Coxeter generators for $w \in I(W)$. Show $\Delta(Q, \hat{\mathcal{Q}}(w))$ is a ball or sphere.

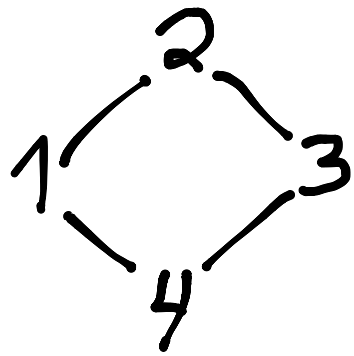
Characterize when the two cases (ball/sphere) occur.

CONJECTURE:

$\Delta(Q, \hat{Q}(w))$ is a sphere exactly when the (involutional) Demazure product of Q is w.

EXAMPLE: $\hat{Q}(321) = \{12, 21\}$

$\Delta(\underset{1\,2\,3\,4}{2121}, \hat{Q}(321)) =$



EXERCISE 22:

Find all facets in
$\Delta(123452343, \hat{\mathcal{Q}}(54321))$

<u>THM</u> (H.-Marberg-P. 15+):
The set $\hat{\mathcal{Q}}(w)$ is connected
by the Coxeter relations
$ss's\ldots = s'ss'\ldots$
from before, together with

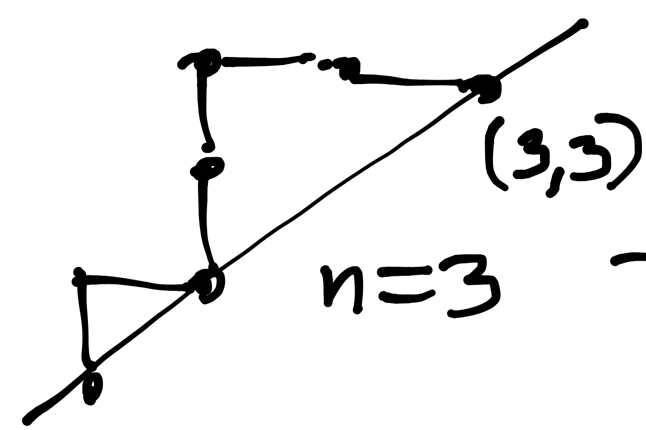
$$\underline{a_1 a_2} a_3 a_4 \ldots\ldots \quad \updownarrow \quad \underline{a_2 a_1} a_3 a_4 \ldots\ldots$$

(only for $W = S_n$)

Enumeration & Subword Complexes

$C_n := \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.

They count, for example, lattice paths $(0,0) \to (n,n)$ weakly above the line $y=x$:



Dyck paths

THM (Woo '04)

Let $c_{(n)} = 1\,2 \cdots (n-1)$

$w_0^{(n)}(c) = c_{(n)} c_{(n-1)} \cdots c_{(2)}$

Then the number of facets in

$$\Delta\big(c_{(n)} w_0^{(n)}(c),\ e(n\ n{-}1 \cdots 2\,1)\big)$$

is the Catalan number C_n

THM (Serrano-Stump '12)

There is an easy bijection from facets to Dyck paths.

EXAMPLE: $n=3 \quad c=12$

$$Q = c_{(3)}\, c_{(3)}\, c_{(2)}$$

$$= \underbrace{1\,2}_{\text{chunk 1}} \quad \underbrace{1\,2}_{\text{chunk 2}} \quad \underbrace{1}_{\text{chunk 3}}$$

Given a facet

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 1 \\ & 2 & 1 & 2 & \end{array}$$

do Edelman-Greene insertion

$$212 \longmapsto \boxed{\begin{array}{cc} 1 & 2 \\ 2 & \end{array} \; P \qquad \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \; Q} \longrightarrow \begin{array}{cc} 1 & 2 \\ 2 & \end{array}$$

replace with label of chunk in Q

subtract i from row i

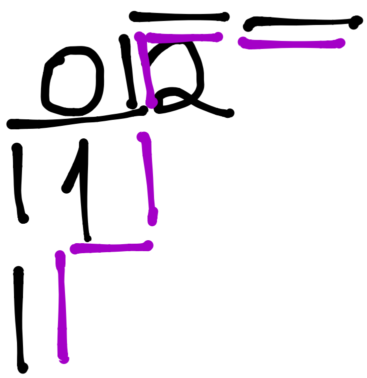
$$\begin{array}{cc} 0 & 1 \\ 0 & \end{array}$$

Dyck path in 0-1 boundary

k-fans of Dyck paths

= collections of k non-intersecting Dyck paths

e.g. k=2



Same bijection with facets of

$$\Delta\Big(\, Q \,,\ \hat{Q}(n\ n{-}1 \cdots 2\ 1)\Big)$$

$$Q = \underbrace{c_{(n)}\, c_{(n)} \cdots c_{(n)}}_{k}\, w_0^{(n)}(c)$$

NOTE: The counts extend to other Coxeter groups.

REU Problem 7b:

Find the # of facets in

$$\Delta\left(\overset{n=3}{\underset{1\,2\,3\,4\,5\,6}{1\,2\cdots(2n)}}\;\underset{2\,3\,4\,5}{2\,3\cdots(2n-1)}\cdots\underset{3\,4}{n\;n+1},\;\hat{Q}\left(\underset{6\,5\,4\,3\,2\,1}{2n\;2n-1\cdots2\,1}\right)\right)$$

$$\Delta\left(\overset{n=3}{\underset{1\,2\,3\,4\,5}{1\,2\cdots(2n-1)}}\;\underset{2\,3\,4}{2\,3\cdots2n-2}\cdots\underset{3}{n},\;\hat{Q}\left(\underset{5\,4\,3\,2\,1}{2n-1\;2n-2\cdots2\,1}\right)\right)$$

Catalan C_n?

$$\Delta\left(c^{k}_{(n)}w^{(n)}_{0}(c),\;\hat{Q}(n\;n-1\cdots2\,1)\right)$$

no conjecture

It appears that

$$\Delta\Big(1\,2\cdots(2n+k-1)\;2\,3\cdots(2n+k-2)\cdots,\;\hat{Q}\big(2n\,(2n-1)\cdots 2\,1\big)\Big)$$

or is it $(2n-1)(2n-2)\cdots 2\,1$?

corresponds to k-fans of Dyck paths.

For signed permutations, one also gets Catalan #'s. (Type A)