Applications of graphical condensation for enumerating matchings and tilings

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Abstract

A technique called graphical condensation is used to prove various combinatorial identities among numbers of (perfect) matchings of planar bipartite graphs and tilings of regions. Graphical condensation involves superimposing matchings of a graph onto matchings of a smaller subgraph, and then re-partitioning the united matching (actually a multigraph) into matchings of two other subgraphs, in one of two possible ways. This technique can be used to enumerate perfect matchings of a wide variety of planar bipartite graphs. Applications include domino tilings of Aztec diamonds and rectangles, diabolo tilings of fortresses, plane partitions, and transpose complement plane partitions.
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1. Introduction

The *Aztec diamond* of order n is defined as the union of all unit squares whose corners are lattice points which lie within the region $\{(x, y): |x| + |y| \leqslant n + 1\}$. A *domino* is simply a 1-by-2 or 2-by-1 rectangle whose corners are lattice points. A *domino tiling* of a region R is a set of non-overlapping dominoes whose union is R. Fig. 1 shows an Aztec diamond of order 4 and a sample domino tiling.

In [6], it was conjectured that the number of tilings for the order-n Aztec diamond is $2^{n(n+1)/2}$. The conjecture was proved in [4]. As the author went about trying to enumerate domino tilings for a similar region, he discovered a new technique called *graphical condensation*. This technique has some far-reaching applications for proving
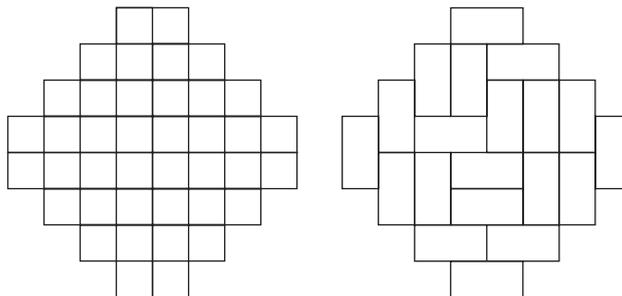
Fig. 1. Order 4 Aztec diamond and a sample domino tiling.

various combinatorial identities. These identities usually take the form

$$T(R_1)T(R_2) = T(R_3)T(R_4) + T(R_5)T(R_6),$$

where $T(R_i)$ stands for the number of tilings for a region R_i. In our applications, the regions R_i are complexes built out of vertices, edges, and faces, and the legal tiles correspond to pairs of faces that share an edge; a collection of such tiles constitutes a tiling if each face of R_i belongs to exactly one tile in the collection (such tilings are sometimes called diform tilings). Each region R_i could be represented by its dual graph G_i. The number of tilings for R_i would equal the number of perfect matchings of G_i. Thus we could replace each term $T(R_i)$ in the identity with $M(G_i)$, which stands for the number of perfect matchings of G_i. (Hereafter, it will be understood that any use of the term "matching" refers to a perfect matching.)

Graphical condensation involves superimposing a matching of one graph onto a matching of another, and then partitioning that union into matchings of two other graphs. The phrase *graphical condensation* comes from the striking resemblance between Dodgson condensation of determinants and graphical condensation of Aztec diamonds. A proof of Dodgson condensation which illustrates this striking resemblance can be found in [14].

This article describes how graphical condensation can be used to prove bilinear relations among numbers of matchings of planar bipartite graphs or diform tilings of regions. Among the applications are domino tilings of Aztec diamonds (as well as some variant regions with holes in them), and rhombus (or lozenge) tilings of semiregular hexagons (equivalent to plane partitions), with or without the requirement of bilateral symmetry. The main result extends to weighted enumeration of matchings of edge-weighted graphs, and this extension gives us a simple way to apply the method to count domino tilings of rectangles and diabolo tilings of fortresses.

2. Enumerative relations among matchings of planar bipartite graphs

Before we state our enumerative relations, let us introduce some notation. We will be working with a bipartite graph $G = (V_1, V_2, E)$ in which V_1 and V_2 are disjoint sets
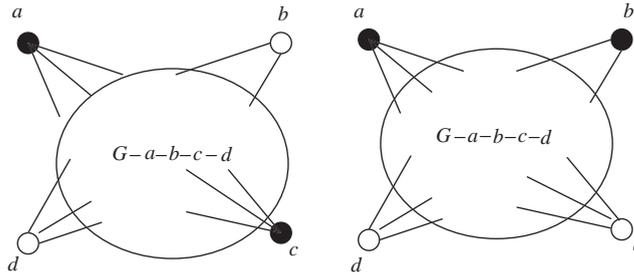
Fig. 2. Left: Graph for Theorem 2.1. Right: Graph for Theorem 2.3.

of vertices in G and every edge in E connects a vertex in V_1 to a vertex in V_2. If U is a subset of vertices in G, then $G - U$ is the subgraph of G obtained by deleting the vertices in U and all edges incident to those vertices. If a is a vertex in G, then $G - a = G - \{a\}$. Finally, we will let $M(G)$ be the number of perfect matchings of G, and $\mathcal{M}(G)$ be the set of all perfect matchings of G.

In order to state the enumerative relations, we must first embed G into the plane $\mathbb{R}^2$. The *plane* graph G divides $\mathbb{R}^2$ into faces, one of which is unbounded.

Theorem 2.1. *Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c, and d appear in a cyclic order on a face of G. (See* Fig. 2, *left. Note a, b, c, d lie on the unbounded face.) If $a, c \in V_1$ and $b, d \in V_2$, then*

$$M(G)M(G - \{a, b, c, d\})$$

$$= M(G - \{a, b\})M(G - \{c, d\}) + M(G - \{a, d\})M(G - \{b, c\}).$$

Proof. To prove this relation, we would like to establish that the two sets $\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})$ and $(\mathcal{M}(G - \{a, b\}) \times \mathcal{M}(G - \{c, d\})) \cup (\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\}))$ have the same cardinality. Consider superimposing a matching of $G - \{a, b, c, d\}$ onto a matching of G. Whenever both matchings share a common edge, we retain both edges and place a doubled edge in the united matching. Thus in the united matching (strictly speaking a multigraph, since some edges may belong with multiplicity 2), each vertex has degree 2 except for a, b, c, and d, which have degree 1.

Now consider superimposing a matching of $G - \{a, b\}$ onto a matching of $G - \{c, d\}$. Each vertex in the resulting graph has degree 2 except for a, b, c, and d, which have degree 1. The same type of graph results from superimposing a matching of $G - \{a, d\}$ onto a matching of $G - \{b, c\}$.

We define $\mathcal{H}$ to be the set of multigraphs on the vertices of G in which vertices a, b, c, and d have degree 1, and all remaining vertices have degree 2. The edges of G form cycles, doubled edges, and two paths whose endpoints are a, b, c, and d. Each pair of graphs in $\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})$, $\mathcal{M}(G - \{a, b\}) \times \mathcal{M}(G - \{c, d\})$, and $\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\})$ can be merged to form a multigraph in $\mathcal{H}$. (Hereafter, we shall drop the prefix "multi-" and refer to the elements of $\mathcal{H}$ as simply graphs.)

Let H be a graph in $\mathcal{H}$. From a, we can trace a path through H until we hit another vertex of degree 1. No vertex can be visited twice by this path since each vertex has degree at most two. Eventually we must end at one of the other vertices of degree 1. If one path connects a to b, then the path from c must end at the remaining degree-1 vertex d. Otherwise if a connects to d, then b must connect to c. And since $a, b, c,$ and d occur in cyclic order around a face of G, it is impossible for one path to connect a to c and the other path to connect b to d. If such paths existed, then they would have to intersect, forcing some other vertex to have a degree greater than 2.

We now show that H can be partitioned into a matching M_1 of G and a matching M_2 of $G - \{a, b, c, d\}$ in 2^k ways, where k is the number of cycles in H. Since H is bipartite, each cycle has even length. We partition each cycle in H so that adjacent edges go into different matchings; each vertex in a cycle is incident to one edge from each matching. Each doubled edge is split and shared between both matchings. Since the paths connect a to b (or d) and c to d (or b), one end of each path must belong to V_1 and the other end must be in V_2. Thus each path has odd length (as measured by the number of edges), so we may assign the edges at the ends of each path to M_1. The remaining edges in the paths are assigned to M_1 and M_2, and thus it is always possible to partition H into matchings M_1 and M_2. Since there are two choices for distributing edges in each cycle of H into matchings M_1 and M_2, there are 2^k possible ways to partition H into matchings of G and $G - \{a, b, c, d\}$.

Next, we show that H can always be partitioned into either matchings of $G - \{a, b\}$ and $G - \{c, d\}$, or matchings of $G - \{a, d\}$ and $G - \{b, c\}$, but never both. Once again, the cycles and doubled edges are split between the matchings as described earlier. Without loss of generality, assume that paths connect a to b and c to d. As shown earlier, the edge incident to a must be in the same matching as the edge incident to b. A matching of $G - \{c, d\}$ may contain both of those edges, but matchings of $G - \{a, d\}, G - \{b, c\}$, and $G - \{a, b\}$ cannot. Likewise, the edges incident to c and d can both belong only to a matching of $G - \{a, b\}$. Thus it is possible for H to be partitioned into matchings of $G - \{a, b\}$ and $G - \{c, d\}$, but not into matchings of $G - \{a, d\}$ and $G - \{b, c\}$. And just as in the previous paragraph, the partitioning can be done in 2^k ways (where k is the number of cycles in G). Thus the number of partitions of H into matchings of G and $G - \{a, b, c, d\}$ is equal to the number of partitions into matchings of $G - \{a, b\}$ and $G - \{c, d\}$, or of $G - \{a, d\}$ and $G - \{b, c\}$.

Thus we can partition $\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})$ and $(\mathcal{M}(G - \{a, b\}) \times \mathcal{M}(G - \{c, d\})) \cup (\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\}))$ into subsets such that the union of each pair of graphs within the same subset forms the same graph in $\mathcal{H}$. Each graph $H \in \mathcal{H}$ corresponds to one subset from each of $\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})$ and $(\mathcal{M}(G - \{a, b\}) \times \mathcal{M}(G - \{c, d\})) \cup (\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\}))$, and those subsets have equal size. Thus $\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})$ and $(\mathcal{M}(G - \{a, b\}) \times \mathcal{M}(G - \{c, d\})) \cup (\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\}))$ have the same cardinality, so the relation is proved. $\square$

Before Theorem 2.1 was known, James Propp proved a special case in which $a, b, c,$ and d form a 4-cycle in G; see [10].

Corollary 2.2. *Let a, b, c, d be four vertices forming a 4-cycle face in a plane bipartite graph G, joined by edges that we will denote by ab, bc, cd, and da. Then the proportion P of matchings of G that have an alternating cycle at this face (i.e., the proportion of matchings of G that either contain edges ab and cd or contain edges bc and da) is*

$$P = 2(p(ab)p(cd) + p(bc)p(da)),$$

where $p(uv)$ denotes the proportion of matchings of G that contain the specified edge uv.

Proof. We note that for each edge uv in G,

$$p(uv) = \frac{M(G - \{u, v\})}{M(G)}.$$

The number of matchings of G that contain the alternating cycle at $abcd$ is twice the number of matchings of $G - \{a, b, c, d\}$. Thus

$$P = \frac{2M(G - \{a, b, c, d\})}{M(G)}.$$

Then after multiplying the relation in Theorem 2.1 by $2/M(G)^2$, we get our result. $\square$

With this same technique, we can prove similar theorems in which we alter the membership of a, b, c, and d in V_1 and V_2.

Theorem 2.3. *Let $G = (V_1, V_2, E)$ be a plane bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c, and d appear in a cyclic order on a face of G (as in Fig. 2, right). If $a, b \in V_1$ and $c, d \in V_2$, then*

$$M(G - \{a, d\})M(G - \{b, c\})$$

$$= M(G)M(G - \{a, b, c, d\}) + M(G - \{a, c\})M(G - \{b, d\}).$$

Proof. The proof of this relation is similar to that of Theorem 2.1 with several differences. In this case, we show that $\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\})$ and $(\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})) \cup (\mathcal{M}(G - \{a, c\}) \times \mathcal{M}(G - \{b, d\}))$ have the same cardinality. The combination of a pair of matchings from either set produces a graph in the set $\mathcal{H}$ of graphs on the vertices of G in which all vertices have degree 2 except for a, b, c, and d, which have degree 1. Now consider a graph $H \in \mathcal{H}$. If paths connect a to b and c to d, then each path has even length. The edges at the ends of each path must go into different matchings. Thus H can be partitioned into matchings of $G - \{a, d\}$ and $G - \{b, c\}$, or into matchings of $G - \{a, c\}$ and $G - \{b, d\}$. Otherwise, if a is connected to d and b to c, then each path has odd length. Then H can be partitioned into matchings of $G - \{a, d\}$ and $G - \{b, c\}$, or into matchings of G and $G - \{a, b, c, d\}$.

No matter which ways the path connect, H can always be partitioned into matchings of $G - \{a, d\}$ and $G - \{b, c\}$. We can also partition H into either matchings of
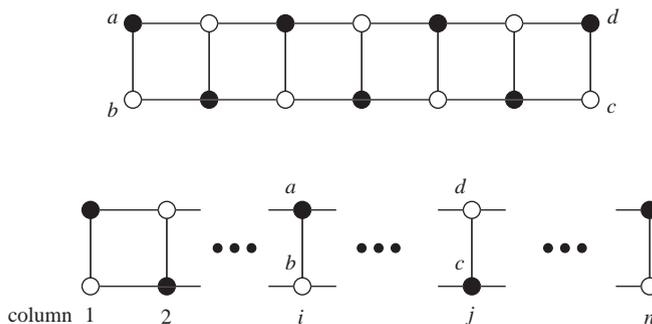
Fig. 3. Top: $2 \times n$ grid with corners a, b, c, d. Bottom: Same rectangle, only with a, b, c, d in the middle.

$G - \{a, c\}$ and $G - \{b, d\}$, or matchings of G and $G - \{a, b, c, d\}$, but not both. Moreover, the number of partitions of H into matchings of $G - \{a, d\}$ and $G - \{b, c\}$ is equal to the number of partitions into matchings of $G - \{a, c\}$ and $G - \{b, d\}$, or of G and $G - \{a, b, c, d\}$. Thus $\mathcal{M}(G - \{a, d\}) \times \mathcal{M}(G - \{b, c\})$ and $(\mathcal{M}(G) \times \mathcal{M}(G - \{a, b, c, d\})) \cup (\mathcal{M}(G - \{a, c\}) \times \mathcal{M}(G - \{b, d\}))$ have the same cardinality. $\quad \square$

We show a simple application of Theorems 2.1 and 2.3 in which our graphs are $2 \times n$ grids. From elementary combinatorics, the number of matchings of a $2 \times n$ grid is F_{n+1}, where $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. These theorems lead to a straightforward derivation for some bilinear relations among the Fibonacci numbers. Consider a $2 \times n$ rectangle with a, b, c and d being the four corners, as shown in Fig. 3. Theorems 2.1 and 2.3 produce the relations

$$F_{n+1}F_{n-1} = \begin{cases} F_n^2 + 1 \cdot 1 & \text{for even } n, \\ F_n^2 - 1 \cdot 1 & \text{for odd } n, \end{cases}$$

which could be simplified to $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$. This is also known as Cassini's identity. Another combinatorial proof for this relation is found in [11].

We could go a step further by letting a, b, c, d be somewhere in the middle of the $2 \times n$ grid. If a, b are in column i, and c, d are in column $j > i$ (see Fig. 3, bottom), then the relations become

$$F_{n+1}F_iF_{j-i}F_{n-j+1} = F_iF_{n-i+1}F_jF_{n-j+1} + (-1)^{j-i-1}(F_iF_{n-j+1})^2,$$

which simplifies to

$$F_{n+1}F_{j-i} = F_{n-i+1}F_j + (-1)^{j-i-1}F_iF_{n-j+1}.$$

We close this section with two additional relations applicable in situations in which V_1 and V_2 have different size.

Theorem 2.4. *Let* $G = (V_1, V_2, E)$ *be a plane bipartite graph in which* $|V_1| = |V_2| + 1$. *Let vertices a, b, c, and d appear cyclically on a face of G. If* $a, b, c \in V_1$ *and* $d \in V_2$, *then*

$$M(G - b)M(G - \{a, c, d\})$$

$$= M(G - a)M(G - \{b, c, d\}) + M(G - c)M(G - \{a, b, d\}).$$

Theorem 2.5. *Let* $G = (V_1, V_2, E)$ *be a plane bipartite graph in which* $|V_1| = |V_2| + 2$. *Let vertices a, b, c, and d appear cyclically on a face of G, and* $a, b, c, d \in V_1$. *Then*

$$M(G - \{a, c\})M(G - \{b, d\})$$

$$= M(G - \{a, b\})M(G - \{c, d\}) + M(G - \{a, d\})M(G - \{b, c\}).$$

The proofs for these relations are similar to the proofs for Theorems 2.1 and 2.3.

3. Proof of Aztec diamond theorem

The order-*n* *Aztec diamond graph* refers to the graph dual of the order-*n* Aztec diamond. Throughout this proof, an *Aztec matching* will mean a matching of an Aztec diamond graph. Fig. 4 shows the order-4 Aztec diamond graph and an order-4 Aztec matching. Thus counting tilings for an Aztec diamond of order *n* is the same as counting Aztec matchings of order *n*.

To prove that the number of Aztec matchings of order *n* is $2^{n(n+1)/2}$, we need the following recurrence relation.

Proposition 3.1. *Let* $T(n)$ *represent the number of Aztec matchings of order n. Then*
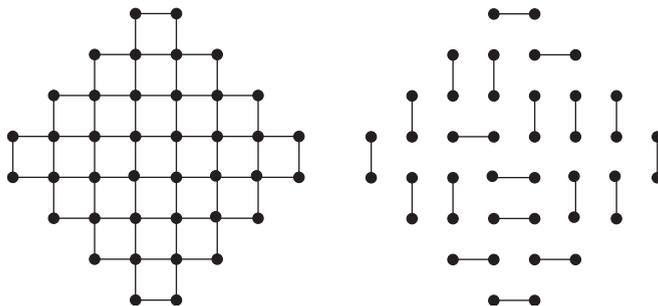
$$T(n) = \frac{2(T(n-1))^2}{T(n-2)}.$$

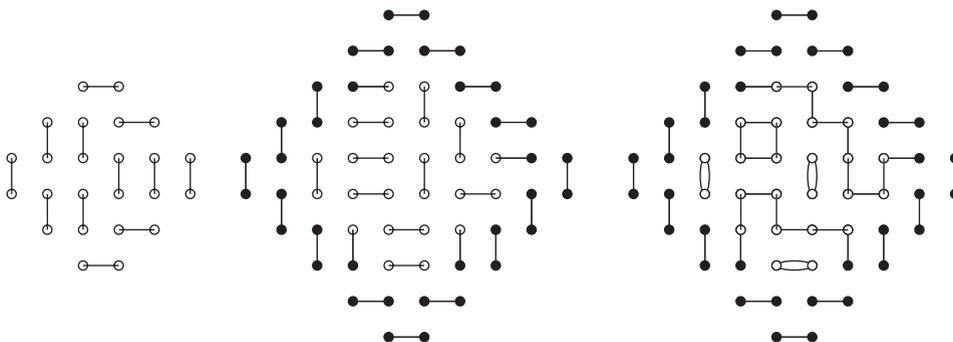Fig. 4. Left: Order-4 Aztec diamond graph. Right: Order-4 Aztec matching.

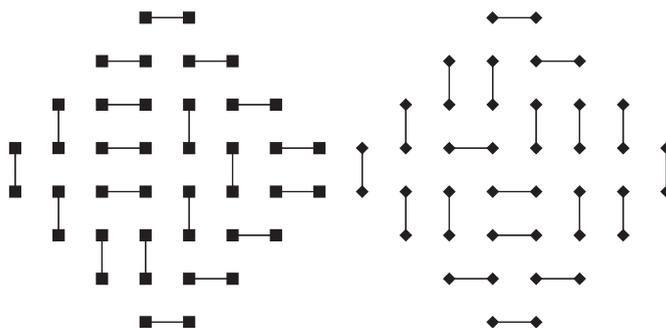Fig. 5. Left: Order-3 Aztec matching. Center: Order-5 Aztec matching. Right: The two matchings combined.

Fig. 6. Aztec matchings C and D of order 4.

Proof. It is sufficient to show that

$$T(n)T(n-2) = 2(T(n-1))^2.$$

To prove this relation, we show that the number of ordered pairs (A,B) is twice the number of ordered pairs (C,D), where A, B, C, and D are Aztec matchings of orders n, $n-2$, $n-1$, and $n-1$, respectively.

We superimpose an Aztec matching B of order $n-2$ with an order-n Aztec matching A so that the matchings are concentric. Fig. 5 shows Aztec matchings of orders 3 and 5, and the result of superimposing the two matchings. In the combined graph, the white vertices are shared by both the order-3 and order-5 matchings. The black vertices are from the order-5 matching only. Note that some edges are shared by both matchings. Note also that each black vertex has degree 1 in the combined graph, whereas each white vertex has degree 2.

Now consider the two Aztec matchings of order $n-1$ shown in Fig. 6. Let us call the first and second matchings C and D, respectively. Fig. 7 shows the two possible resulting graphs by superimposing matchings C and D and adding two extra segments. The left graph was made by fitting matching C to the top and matching D
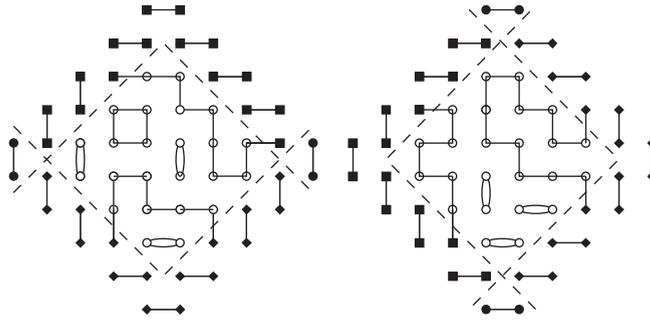
Fig. 7. The two possible combinations of the matchings C and D.

to the bottom of the order-5 diamond, and then adding two side edges. The graph on the right was made by fitting matching C to the left and matching D to the right of the order-5 diamond, and then adding the top and bottom edges. In both cases, each of the center vertices has degree 2, and all other vertices have degree 1. The graphs resemble order-3 Aztec matchings on top of order-5 Aztec matchings.

In general, we are given a graph G on the vertices of the order-n Aztec diamond graph with the following properties:
(1) The inner vertices in G that form an order-$(n-2)$ Aztec diamond have degree 2.
(2) The remaining outer vertices in G have degree 1.
(3) The edges of G form cycles, doubled edges, single edges, and lattice paths of length greater than 1.
Let us call a graph with such properties a *doubled Aztec graph*. For each superimposition we have described so far, the result is a doubled Aztec graph G. We want to show that the number of partitions of G into two Aztec matchings A and B of orders n and $n-2$ is equal to the number of partitions of G into two order-$(n-1)$ Aztec matchings C and D (along with two line segments). We will show that this number is 2^k, where k is the number of cycles in G. Since G is bipartite, all cycles have even length. These cycles are contained in the middle common vertices, as they are the only vertices with degree 2. Each cycle can then be partitioned so that every other edge will go to the same subgraph; adjacent edges go to different subgraphs. For each cycle, there are two ways to decide which half of the cycle goes to A or B. Similarly, there are two ways to decide which half goes to C or D. All doubled edges in G are split and shared by each subgraph. It remains to show that the other edges must be partitioned uniquely.

We now label G as shown in Fig. 8. The vertices whose degree is 2 are labeled O. The degree-one vertices surrounding the O-vertices are labeled T, U, V, and W such that each side is assigned a different label. Every vertex on the outer boundary of G is labeled Y, except for four vertices, one on each corner. Those four exceptions are assigned the label ($T, U, V,$ or W) of the vertices on the same diagonal. We have labeled the vertices such that each vertex labeled Y will match with exactly one vertex labeled $T, U, V,$ or W. For each label $T, U, V,$ and W, exactly one vertex
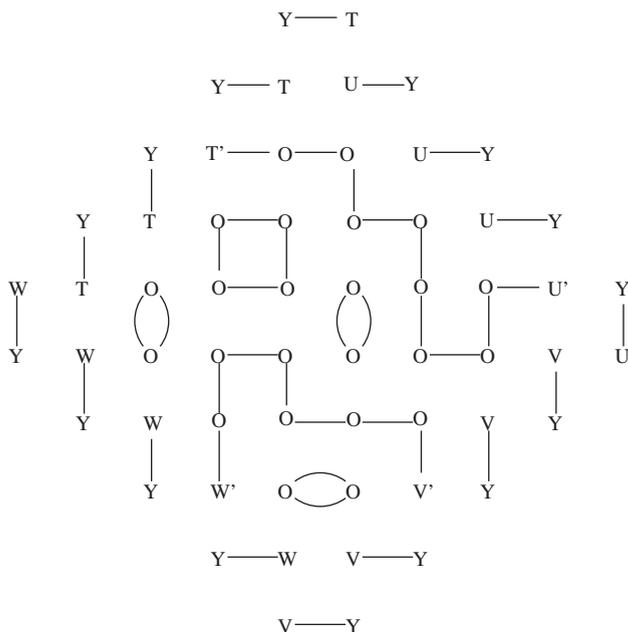
Fig. 8. Specially labeled graph of doubled Aztec graph G.

will not be connected to a Y-vertex. We denote these special vertices T', U', V', and W'.

In a doubled Aztec graph, there must be paths joining T' to U' and V' to W', or paths joining T' to W' and U' to V'. However, we cannot have paths going from T' to V' and from U' to W'. If such paths existed, then both paths would have to travel through the O-vertices and intersect, thus forcing the degree of some O-vertex to be more than 2.

Let us show that the segments from both ends of a path must belong to the same subgraph in any partition of G. Let us 2-color the vertices of G black and white so that black vertices are adjacent to white vertices and vice versa. The T- and V-vertices must be the same color; let us color all the T- and V-vertices white. Then the U- and W-vertices must be of the other color, which is black. Therefore, any path from T to U, from U to V, from V to W, or from W to T must have odd length since the path goes from a black to a white vertex. Thus the segments from both ends of a path must belong to the same subgraph in any partition of G.

Thus, when we partition G into matchings A and B of orders n and $n-2$, we must always place the ending segments into A and determine the rest of the partition thereafter. Such a partition always exists.

Next we show that G can be partitioned into two matchings C and D of order $n-1$ along with two additional side edges. There are two possible ways this partition could be done. The first is top–bottom: the top diamond contains the T- and U-vertices, and the bottom diamond contains the V- and W-vertices. The second is left–right: the left

diamond contains the T- and W-vertices, and the right diamond contains the U- and V-vertices.

Without loss of generality, let the paths in G connect T' to U' and V' to W'. When G is partitioned into two matchings C and D, both of order $n - 1$, one matching (say C) must have both T' and U', as they are the ends of the same lattice path. Thus C is the top Aztec matching containing all T- and U-vertices (except for one U-vertex on the far right corner). Vertices V' and W' must belong to the other matching D. The paths are then partitioned uniquely. Thus we can partition G into two order-$(n - 1)$ Aztec matchings placed top–bottom (plus two edges on the sides). However, it is not possible to partition G into two side-by-side Aztec matchings of order $n - 1$ such that one contains the T- and W-vertices, and the other contains the U- and V-vertices. The reason is that since the left matching has W', it would then contain V'. The latter cannot happen, since V' is in the other matching.

Hence each doubled Aztec graph can be partitioned into two order-$(n - 1)$ Aztec matchings in one way (top–bottom) or the other (left–right), but never both. The partition of the paths is uniquely determined.

The number of ways to combine Aztec matchings of orders n and $n - 2$ is $T(n)T(n - 2)$, while the number of ways to combine two order-$(n - 1)$ matchings is $2T(n - 1)^2$. Each combination becomes a doubled Aztec graph, so the relation is proved. $\square$

There are 2 ways to tile an order-1 Aztec diamond, and 8 ways to tile an order-2 Aztec diamond. Having proved the recurrence relation, we can now compute the number of tilings of an Aztec diamond of order n. The following result is easily proved by induction on n:

Theorem 3.2 (Aztec diamond theorem). *The number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$.*

4. Regions with holes

4.1. Placement probabilities

We can use graphical condensation to derive recurrence relations for placement probabilities of dominoes in tilings of Aztec diamonds. Let domino D be a specified pair of adjacent squares in an Aztec diamond. The *placement probability* of D in an order-n Aztec diamond is the probability that D will appear in a tiling of the order-n Aztec diamond, given that all tilings are equally likely.

Placement probabilities are of interest in the study of random tilings. If we look at a random tiling of an Aztec diamond of large order, we notice four regions in which the dominoes form a brickwork pattern, and a central circular region where dominoes are mixed up. The placement probability of any domino at the center of the diamond will be near 1/4. However, in the top corner, dominoes which conform to the brickwork will have probabilities near 1. All other dominoes in this corner would have probabilities near 0. For proofs of these assertions, see [3].
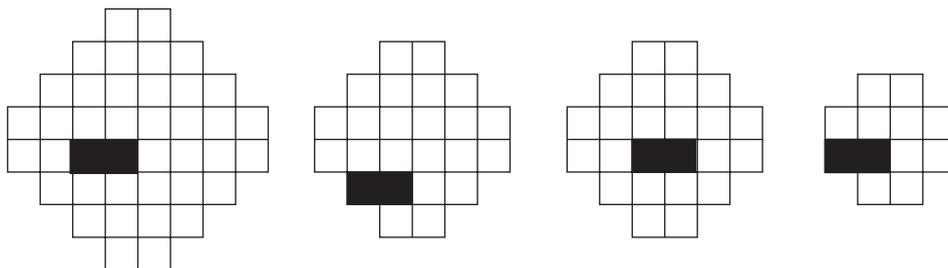
Fig. 9. Regions from left: $A_n - D$, $A_{n-1} - D_{\text{down}}$, $A_{n-1} - D_{\text{right}}$, $A_{n-2} - D$.

We could calculate the placement probability of a domino with the following steps. First, we replace the domino with a two-square hole in the Aztec diamond. Then we compute the number of tilings of that diamond with the hole. Finally, we divide it by the number of tilings of the (complete) Aztec diamond.

We can express the number of tilings of the order-n Aztec diamond with the hole at D in terms of tilings of lower-order Aztec diamonds with holes. But first, let us introduce some notation. We will let $A_n - D$ stand for the order-n Aztec diamond with domino D missing. The dominoes D_{up}, D_{down}, D_{left}, and D_{right} will represent dominoes shifted up, down, left, and right by a square relative to D in the Aztec diamond. Then $A_{n-1} - D_{\text{up}}$ is the order-$(n-1)$ diamond such that when it is placed concentrically with $A_n - D$, the hole of $A_{n-1} - D_{\text{up}}$ will match up with D_{up}. The regions $A_{n-1} - D_{\text{down}}$ and so forth represent similar Aztec diamonds with domino holes. Finally, $A_{n-2} - D$ is the order-$(n-2)$ Aztec diamond such that when $A_{n-2} - D$ is placed directly over $A_n - D$, domino D is missing. See Fig. 9 for examples. (In case D_{up}, etc. lies outside A_{n-1}, the region $A_{n-1} - D_{\text{up}}$ will not be defined.)

By using graphical condensation, we can relate the number of tilings of these Aztec diamonds with holes:

$$T(A_n - D)T(A_{n-2} - D)$$
$$= T(A_{n-1} - D_{\text{up}})T(A_{n-1} - D_{\text{down}}) + T(A_{n-1} - D_{\text{left}})T(A_{n-1} - D_{\text{right}}).$$

We also have the following relation, which relates numbers of tilings of Aztec diamonds:

$$T(A_n)T(A_{n-2}) = 2T(A_{n-1})T(A_{n-1}).$$

We can then derive a relation among placement probabilities of dominoes in Aztec diamonds of orders n, $n-1$, and $n-2$. When we divide the first relation by the second, we get

$$P(A_n, D)P(A_{n-2}, D)$$
$$= \frac{P(A_{n-1}, D_{\text{up}})P(A_{n-1}, D_{\text{down}}) + P(A_{n-1}, D_{\text{left}})P(A_{n-1}, D_{\text{right}})}{2},$$
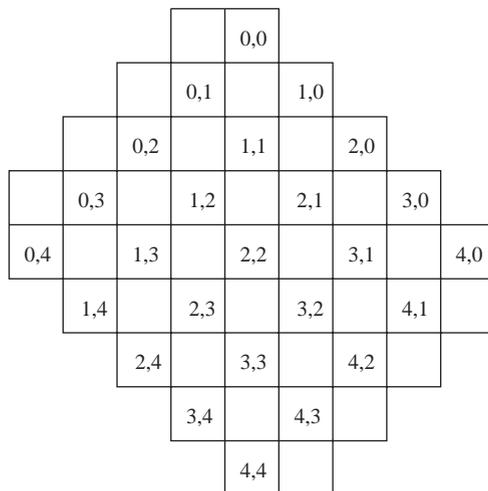
Fig. 10. Labeling of squares in a holey Aztec rectangle. By removing square (a, b) we obtain $R_{4,a,b}$.

where $P(R, D)$ is the placement probability on domino D in region R. The probability $P(R, D)$ was computed by dividing $T(R - D)$ by $T(R)$.

4.2. Holey Aztec rectangles

Another application of graphical condensation deals with regions called holey Aztec rectangles. A *holey Aztec rectangle* is a region similar to an Aztec diamond, except that the boundary of an n-by-$(n + 1)$ holey Aztec rectangle consists of diagonals of length n, $n + 1$, n, and $n + 1$. In addition, to maintain the balance of squares of different parity so that the region can be tiled, a square is removed from its interior. Problems 9 and 10 in [9] ask to enumerate tilings of a holey Aztec rectangle with a square removed in the center or adjacent to the center square, depending on the parity of n.

Let us label some of the squares in an Aztec rectangle as shown in Fig. 10. We label a square only if the region becomes tileable after deleting that square. We let $R_{n,a,b}$ represent the n-by-$(n - 1)$ Aztec rectangle whose square (a, b) has been deleted. Then we can apply our technique and come up with a theorem which relates the numbers of tilings among holey Aztec rectangles.

Theorem 4.1. *Let $T(R)$ stand for the number of tilings of a region R. Then for a, b between 1 and $n - 1$, the number of tilings of $R_{n,a,b}$ is expressed in the following relation*:

$$T(R_{n,a,b}) = \frac{T(R_{n-1,a,b-1})T(R_{n-1,a-1,b}) + T(R_{n-1,a,b})T(R_{n-1,a-1,b-1})}{T(R_{n-2,a-1,b-1})}.$$

Proof. The proof is very similar to that of Proposition 3.1. Instead of superimposing an order-n Aztec matching on an order-$(n - 2)$ Aztec matching, we superimpose
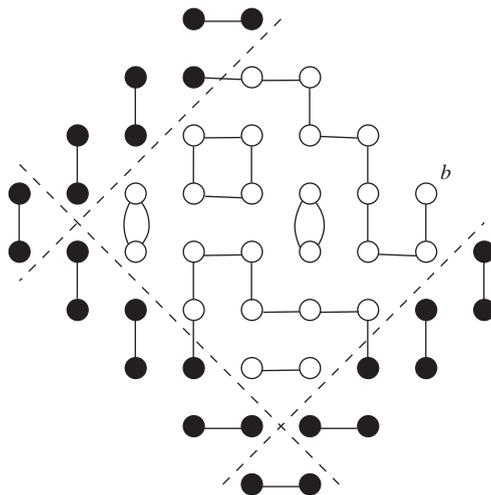
Fig. 11. Combined matchings of an order-3 Aztec diamond graph and $R_{4,3,0}$.

$R_{n-2,a-1,b-1}$ on top of $R_{n,a,b}$ so that the holes align to the same spot. Given the graph G resulting from the superimposition, we can partition it into two $(n-1)$-by-n holey Aztec rectangles. The partition can be done either left–right or top–bottom, but only one or the other. The left–right rectangles are isomorphic to $R_{n-1,a-1,b}$ and $R_{n-1,a,b-1}$. The top–bottom rectangles are isomorphic to $R_{n-1,a,b}$ and $R_{n-1,a-1,b-1}$. $\square$

Another relation can be proven for the case in which the hole is on the edge of the rectangle:

Theorem 4.2. *If* $1 \leqslant a \leqslant n$, *then*

$$T(R_{n,a,0}) = \frac{T(R_{n-1,a,0})T(A_n) + T(R_{n-1,a-1,0})T(A_n)}{T(A_{n-1})}$$

where A_n *is the Aztec diamond of order* n.

Proof. As an example, Fig. 11 shows a matching of an order-3 Aztec diamond graph (which is shown in white vertices) on a matching of $R_{4,3,0}$ (which is missing the vertex b). The relation is derived in a manner analogous to Theorem 2.4. $\square$

4.3. "Pythagorean" regions

We can derive one more relation as a corollary to Theorem 2.5. Let R_n be an $n \times (n+1)$ Aztec rectangle, where n is even. Let t_1, t_2, and t_3 be (overlapping) trominoes in R_n, each of which contain the center square and two squares adjacent to it. Trominoes t_1 and t_2 are L-shaped, while t_3 is straight. Let t_1 point to a side of length n, and t_2
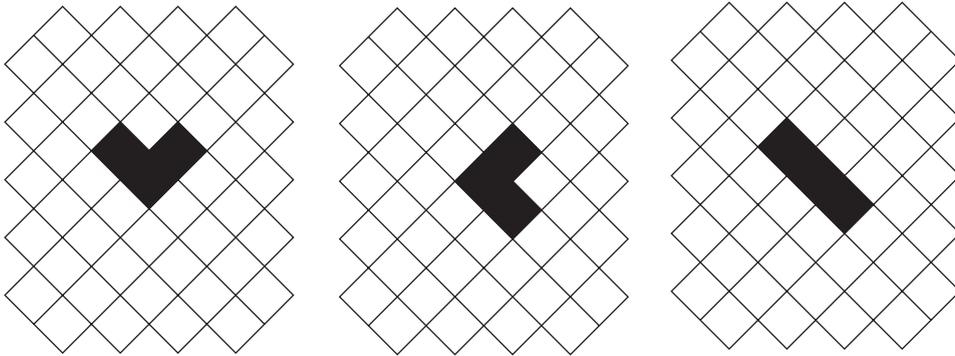
Fig. 12. From left to right: $R_4 - t_1$, $R_4 - t_2$, and $R_4 - t_3$.
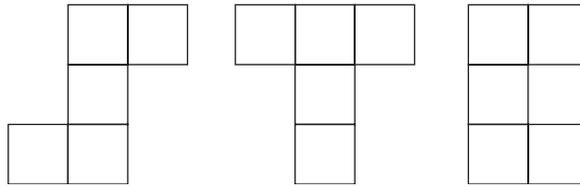

Fig. 13. Pentominoes missing from rectangular regions.

point to a side of length $n + 1$. (See Fig. 12.) Then

$$T(R_n - t_1)^2 + T(R_n - t_2)^2 = T(R_n - t_3)^2.$$

In other words, we have a Pythagorean relation among the number of tilings of these regions! The proof of this relation is to set G to be R_n minus the center square, let a, b, c, d be squares adjacent to the center, and then apply Theorem 2.5.

The reader may also like to puzzle over a similar "Pythagorean" relation among the numbers of tilings of rectangular (*not* Aztec rectangular) regions in which each region has a pentomino hole in its center. The pentominoes are shown in Fig. 13.

5. Weighted matchings of planar bipartite graphs and Aztec diamonds

5.1. Weighted planar bipartite graphs

We can generalize the enumerative relations proved in Section 2 to cover weighted planar bipartite graphs. Given a graph G, we can assign a weight to each edge to form a *weighted graph*. The *weight* of any subgraph H of G is the product of the weights of all the edges in G (in the case where H is a multigraph, each edge-weight contributes with exponent equal to the multiplicity of the associated edge in H); e.g., the weight of a matching of G is the product of the weights of each edge in that matching. We

denote the weight of G itself by $w(G)$. We also define the *weighted sum* $W(G)$ of G to be the sum of the weights of all possible matchings on G.

We can now state and prove a weighted version of Theorem 2.1:

Theorem 5.1. *Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c, and d appear on a face of G, in that order. If $a, c \in V_1$ and $b, d \in V_2$, then*

$$W(G)W(G - \{a, b, c, d\})$$

$$= W(G - \{a, b\})W(G - \{c, d\}) + W(G - \{a, d\})W(G - \{b, c\}).$$

Proof. The proof essentially follows that of Theorem 2.1, except that we must now account for the weights. Let $\mathscr{H}$ be the set of graphs on the vertices of G in which vertices a, b, c, and d have degree 1, all other vertices have degree 2, and doubled edges are permitted. Let H be a graph in $\mathscr{H}$. As before, H may be partitioned into two matchings M_1 and M_2 with these possibilities:
(1) $(M_1, M_2) \in \mathscr{M}(G) \times \mathscr{M}(G - \{a, b, c, d\})$.
(2) $(M_1, M_2) \in \mathscr{M}(G - \{a, b\}) \times \mathscr{M}(G - \{c, d\})$.
(3) $(M_1, M_2) \in \mathscr{M}(G - \{a, d\}) \times \mathscr{M}(G - \{b, c\})$.
As we have seen before, H can always be partitioned in choice 1, and also in either choice 2 or choice 3 (but not both). The number of possible partitions is 2^k, where k is the number of cycles in H. So

$$W(G)W(G - \{a, b, c, d\}) = \sum_{H \in \mathscr{H}} 2^{k(H)} w(H)$$

$$= W(G - \{a, b\})W(G - \{c, d\})$$

$$+ W(G - \{a, d\})W(G - \{b, c\}),$$

where $k(H)$ is the number of cycles in H. $\square$

Similar relations can be generalized from Theorems 2.3, 2.4, and 2.5:

Theorem 5.2. *Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2|$. Let vertices a, b, c, and d appear on a face of G, in that order (as in Fig. 2, right). If $a, b \in V_1$ and $c, d \in V_2$, then*

$$W(G - \{a, d\})W(G - \{b, c\})$$

$$= W(G)W(G - \{a, b, c, d\}) + W(G - \{a, c\})W(G - \{b, d\}).$$

Theorem 5.3. *Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2| + 1$. Let vertices a, b, c, and d appear on a face of G, in that order. If $a, b, c \in V_1$*

and $d \in V_2$, then

$$W(G - b)W(G - \{a, c, d\})$$

$$= W(G - a)W(G - \{b, c, d\}) + W(G - c)W(G - \{a, b, d\}).$$

Theorem 5.4. *Let $G = (V_1, V_2, E)$ be a weighted plane bipartite graph in which $|V_1| = |V_2| + 2$. Let vertices a, b, c, and d appear on a face of G, in that order, and $a, b, c, d \in V_1$. Then*

$$W(G - \{a, c\})W(G - \{b, d\})$$

$$= W(G - \{a, b\})W(G - \{c, d\}) + W(G - \{a, d\})W(G - \{b, c\}).$$

5.2. Weighted Aztec diamonds

Consider a weighted Aztec diamond graph A of order n. Define A_{top} to be the upper order-$(n - 1)$ Aztec sub-diamond along with its corresponding edge weights in A. Similarly, we can refer to the bottom, left, and right subgraphs of A as A_{bottom}, A_{left}, and A_{right}, which are all order $n - 1$ Aztec sub-diamonds. Finally, let A_{middle} be the inner order-$(n - 2)$ Aztec diamond within A. Fig. 14 shows an Aztec diamond and its five sub-diamond graphs.

It turns out that the superimposition technique can also be used to establish an identity for weighted Aztec diamond graphs. The following theorem shows how the weighted sum of a weighted Aztec diamond can be expressed in terms of the weighted sums of the subdiamonds and a few edge weights.

Theorem 5.5. *Let A be a weighted Aztec diamond of order n. Also let t, b, l, and r be the weights of the top, bottom, left, and right edges of A, respectively. Then*

$$W(A) = \frac{l \cdot r \cdot W(A_{\text{top}}) \cdot W(A_{\text{bottom}}) + t \cdot b \cdot W(A_{\text{left}}) \cdot W(A_{\text{right}})}{W(A_{\text{middle}})}.$$

Proof. This proof is very similar to Proposition 3.1, except that we must fill in the details concerning the weights. Indeed, we want to show that

$$W(A) \cdot W(A_{\text{middle}})$$

$$= l \cdot r \cdot W(A_{\text{top}}) \cdot W(A_{\text{bottom}}) + t \cdot b \cdot W(A_{\text{left}}) \cdot W(A_{\text{right}}). \tag{1}$$

We have seen how a doubled Aztec graph G of order n can be decomposed into subgraphs in two of three following ways:
(1) (Big–small) Two Aztec matchings of orders n and $n - 2$.
(2) (Top–bottom) Top and bottom Aztec matchings of order $n - 1$, plus the left and right edges.
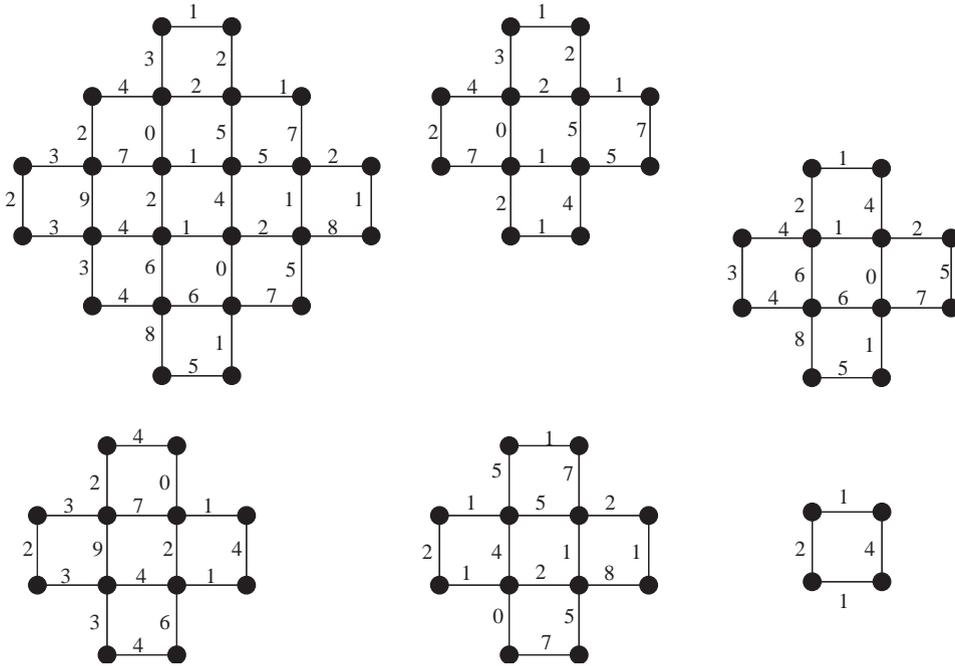(3) (Left–right) Left and right Aztec matchings of order $n - 1$, plus the top and bottom edges.

Fig. 14. Top row: A weighted Aztec diamond graph A, followed by A_{top} and A_{bottom}. Bottom row: A_{left}, A_{right}, and A_{middle}.

As we know, G can always be decomposed via Big–small and by either Top–bottom or Left–right (but not both). The number of possible decompositions by either method is 2^k, where k is the number of cycles in G.

Each edge in G becomes a part of exactly one of the subgraphs. Therefore, the product of the weights of the subgraphs will always equal to the weight of G, since each edge weight is multiplied once.

Recall that $W(A)$ is the sum of the weights of all possible matchings on A. Then

$$W(A) \cdot W(A_{\text{middle}}) = \sum_G 2^{k(G)} w(G)$$

where G ranges over all doubled Aztec graphs of order n, and $k(G)$ is the number of cycles in G. Each term in the sum represents the weight of G multiplied by the number of ways to partition G via Big–small. Each partition is accounted for in $W(A)W(A_{\text{middle}})$. Similarly, we also have

$$l \cdot r \cdot W(A_{\text{top}}) \cdot W(A_{\text{bottom}}) + t \cdot b \cdot W(A_{\text{left}}) \cdot W(A_{\text{right}}) = \sum_G 2^{k(G)} w(G).$$
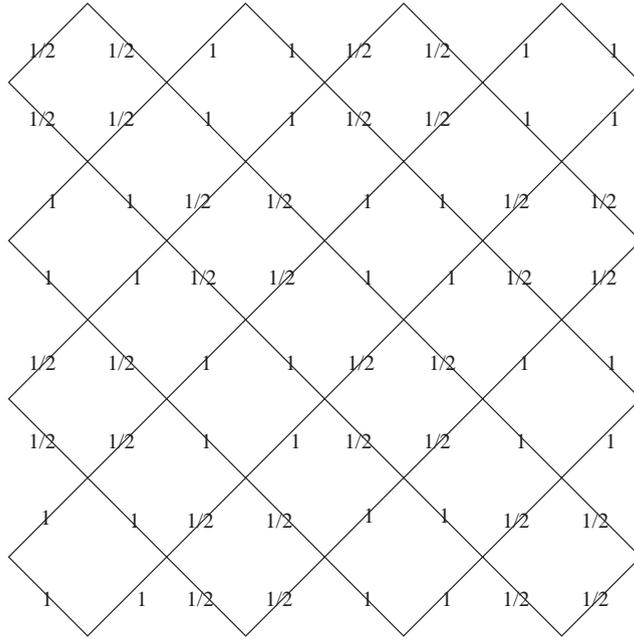
Fig. 15. A fortress-weighted Aztec diamond, rotated by 45°.

Thus both sides of Eq. (1) are equal to a common third quantity, so the relation is proved. □

Theorem 5.5 may be used to find the weighted sum of a *fortress-weighted Aztec diamond*. Imagine rotating an Aztec diamond graph by 45° and then partitioning the edges of the graph into *cells*, or sets of four edges forming a cycle. In a fortress-weighted Aztec diamond, there are two types of cells: (1) cells whose edges are weight 1, and (2) cells whose edges are weight 1/2. Cells with edge-weights of 1/2 are adjacent to cells with edge-weights of 1. (See Fig. 15.)

There are three kinds of fortress-weighted Aztec diamonds:
(1) The order *n* is odd, and all edges in the corner cells have weight 1.
(2) The order *n* is odd, and all edges in the corner cells have weight 1/2.
(3) The order *n* is even, and two opposite corners have edges weighted 1/2, and the other two corners have edges weighted 1.

Let A_n, B_n, and C_n stand for the weighted sums of the these diamonds, respectively. We then use Theorem 5.5 to establish relations among A_n, B_n, and C_n. They are

$$A_{2k+1} = \frac{1 \cdot 1 \cdot C_{2k} \cdot C_{2k} + 1 \cdot 1 \cdot C_{2k} \cdot C_{2k}}{A_{2k-1}} = \frac{2C_{2k}^2}{A_{2k-1}},$$

$$B_{2k+1} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot C_{2k} \cdot C_{2k} + \frac{1}{2} \cdot \frac{1}{2} \cdot C_{2k} \cdot C_{2k}}{B_{2k-1}} = \frac{\frac{1}{2} C_{2k}^2}{B_{2k-1}},$$
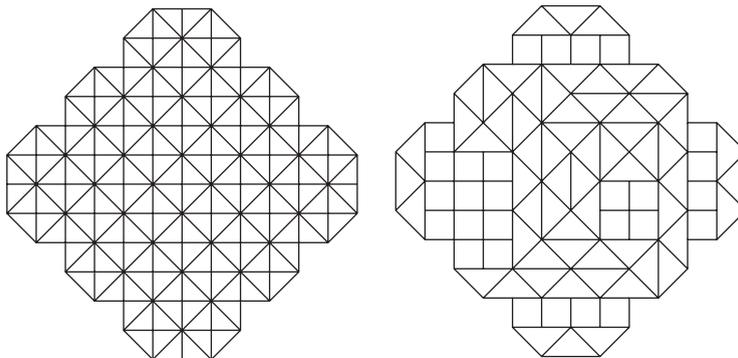
Fig. 16. A fortress and sample diabolo tiling.

$$C_{2k} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot A_{2k-1} \cdot A_{2k-1} + 1 \cdot 1 \cdot B_{2k-1} \cdot B_{2k-1}}{C_{2k-2}} = \frac{\frac{1}{4} A_{2k-1}^2 + B_{2k-1}^2}{C_{2k-2}}.$$

From these relations, we can easily prove by induction that for odd k,

$$A_{2k+1} = (5/4)^{k(k+1)},$$

$$B_{2k+1} = (5/4)^{k(k+1)},$$

$$C_{2k} = (5/4)^{k^2}.$$

For even k,

$$A_{2k+1} = 2(5/4)^{k(k+1)},$$

$$B_{2k+1} = \frac{(5/4)^{k(k+1)}}{2},$$

$$C_{2k} = (5/4)^{k^2}.$$

The importance of the fortress-weighted Aztec diamond comes from the problem of computing the number of diabolo tilings for a fortress. A *diabolo* is either an isosceles right triangle or a square, formed by joining two smaller isosceles right triangles. A *fortress* is a diamond shaped region that is made up of isosceles right triangles and can be tiled by diabolos. A fortress and a sample tiling by diabolos are shown in Fig. 16.

To transform a fortress graph into a weighted Aztec diamond graph, we must use a method called *urban renewal*. This technique is explained in [10] along with proofs and applications. In [10], the transformation is described for the fortress, and the number of tilings for the fortress would be the weighted sum of the fortress-weighted diamond times some power of 2. Thus, graphical condensation, in combination with this

known result about enumeration of fortresses, provides a very simple way to derive the formulas for the number of fortress tilings, first proven by Bo-Yin Yang [12].

A different sort of weighting scheme allows us to apply graphical condensation to count domino tilings of ordinary (non-Aztec!) rectangles. Every rectangle R of even area can be imbedded in some Aztec diamond A of order n (with n sufficiently large) in such a fashion that the complement $A \backslash R$ (the portion of A that is not covered by R) can be tiled by dominoes $d_1, \ldots, d_M$. For any such tiling of $A \backslash R$, we can define a weighting of the Aztec diamond graph of order n with the property that each matching of A has weight 1 if the associated tiling of A contains all the dominoes $d_1, \ldots, d_M$ and weight 0 otherwise. (Specifically, assign weight 1 to every edge that corresponds to one of the dominoes $d_1, \ldots, d_M$ or to a domino that lies entirely inside R, and weight 0 to every other edge.) Then the sum of the weights of the matchings of the weighted Aztec diamond graph equals the number of tilings of R.

6. Plane partitions

A plane partition is a finite array of integers such that each row and column is a weakly decreasing sequence of nonnegative integers. If we represent each integer n in the plane partition as a stack of n cubes, then the plane partition is a collection of cubes pushed into the corner of a box. When this collection of cubes is viewed at a certain angle, these cubes will appear as a rhombus tiling of a hexagon.

In 1912, Percy MacMahon [7] published a proof of a generating function that enumerates plane partitions that fit in a box $\mathcal{B}(r, s, t)$ with dimensions $r \times s \times t$.

Theorem 6.1. *Define $P(r, s, t)$ as the generating function for plane partitions that fit in $\mathcal{B}(r, s, t)$. Then*

$$P(r, s, t) = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j-1}}.$$

Other proofs of this theorem have been published by Carlitz [2], and Gessel and Viennot [5].

In this section, we will prove MacMahon's formula with the help of graphical condensation. Using graphical condensation, we derive a relation that enables us to prove MacMahon's formula by induction on $r + s + t$.

Theorem 6.2.

$$P(r + 1, s + 1, t)P(r, s, t)$$

$$= q^t P(r, s + 1, t)P(r + 1, s, t) + P(r + 1, s + 1, t - 1)P(r, s, t + 1).$$

Proof. Let us take the dual graph of a hexagonal region of triangles in which r is the length of the bottom right side, s is the length of the bottom left side, and t is
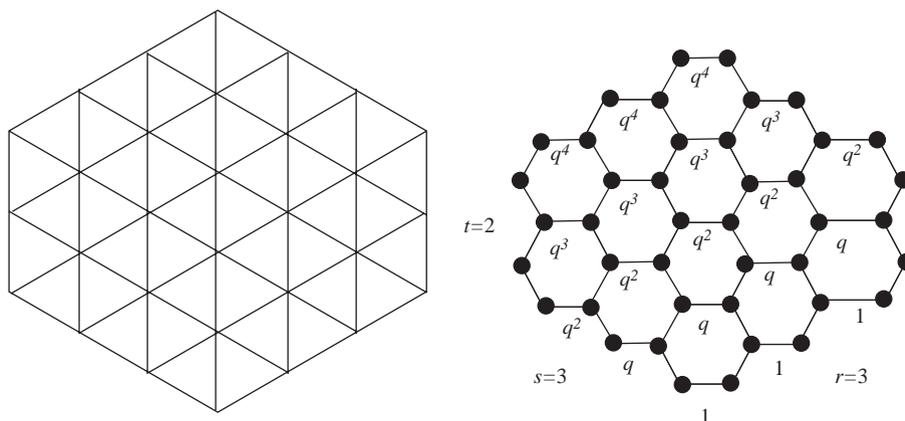
Fig. 17. Weighting scheme for $H(3,3,2)$. All unlabeled edges have weight 1.

the height of the vertical side. In this dual graph, all edges that are not horizontal are weighted 1. The horizontal edges are weighted as follows: the r edges along the bottom right diagonal are each weighted 1. On the next diagonal higher up, each edge is weighted q, and the weights of the edges on each subsequent diagonal are q times the weights of the previous diagonal. Thus the range of weights should be from 1 to q^{s+t-1}. (See Fig. 17.) Call this weighted graph $H(r,s,t)$.

This weighting scheme is specifically designed so that, if a matching consists of the bottom edge (weighted q^i) and two other edges of a 6-cycle, then by replacing those edges with the other three edges, we have dropped the q^i-weighted edge in favor of the q^{i+1}-weighted edge. (See Fig. 18.) The matching would then gain a factor of q, resembling the action of adding a new block (weighted q) to a plane partition. The minimum weight of a matching of this graph is $q^{rs(s-1)/2}$, corresponding to the rs horizontal edges that would make up the "floor" of the empty plane partition. The weighted sum of the graph is therefore $q^{rs(s-1)/2}P(r,s,t)$.

Now the proof of this relation is very similar to the proofs of Theorem 2.1 and Proposition 3.1. We superimpose the two weighted hexagonal graphs $H(r+1,s+1,t)$ and $H(r,s,t)$ such that the bottom edge common to sides r and s of $H(r,s,t)$ coincides with the bottom edge of $H_{r-1,s-1,t}$. The two hexagons completely overlap except for four outer strips of triangles from $H(r+1,s+1,t)$. Let us number these strips 1, 2, 3, and 4. (See Fig. 19.) When we superimpose the two matchings in the manner described above, we get once again a collection of cycles, doubled edges, single edges, and two paths. Each vertex inside $H(r,s,t)$ has degree 2, and each vertex in the four outer strips has degree 1.

Within each strip, all but one of the vertices are matched with each other. Those four unmatched vertices are the endpoints of the two paths. If one path runs between vertices on strips 1 and 2, and the other runs between vertices on strips 3 and 4, then the collection can be partitioned into matchings of the duals of $H(r,s+1,t)$ and $H(r+1,s,t)$, plus the edge on the corner of strips 2 and 3 (of weight q^{s+t-1}). The
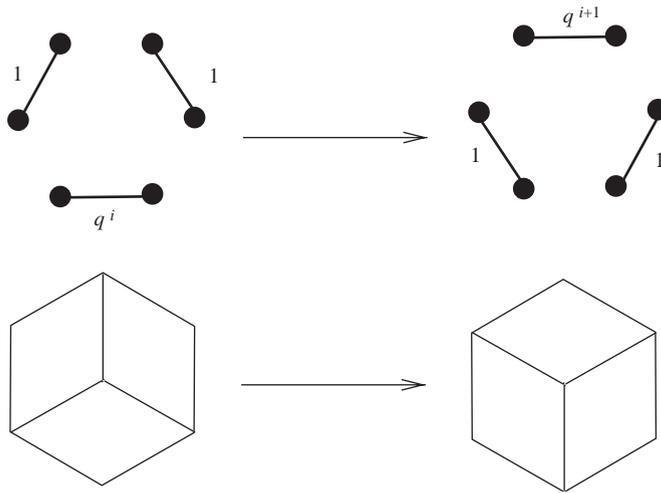
Fig. 18. A transition representing the addition of another cube to a Young diagram.
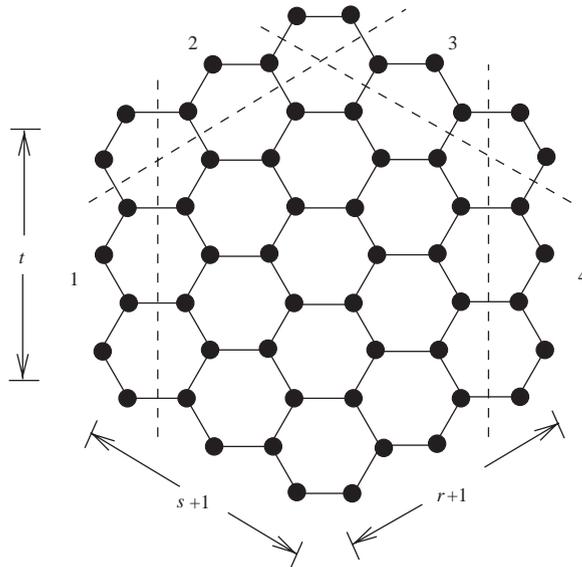

Fig. 19. Graphs $H(r+1, s+1, t)$ and $H(r, s, t)$ overlapping. The outer strips are numbered 1,2,3,4.

graph $H(r, s+1, t)$ lacks strips 3 and 4, while $H(r+1, s, t)$ is the graph without strips 1 and 2. Alternatively, if the paths run from strip 1 to strip 4, and from strip 2 to strip 3, then the collection can be partitioned into matchings of $H(r, s, t+1)$ (the graph without strips 1 and 4) and $H(r+1, s+1, t-1)$ (the graph without strips 2 and 3),

plus two additional corner edges. In both cases, it is possible to partition the collection into matchings of $H(r+1, s+1, t)$ and $H(r, s, t)$. Finally, it is impossible for the paths to run from strips 1 to strip 3 and from strip 2 to strip 4 without intersecting. Thus

$$q^{(r+1)(s+1)s/2}P(r+1, s+1, t) \cdot q^{rs(s-1)/2}P(r, s, t)$$

$$= q^{s+t} \cdot q^{r(s+1)s/2}P(r, s+1, t) \cdot q^{(r+1)s(s-1)/2}P(r+1, s, t)$$

$$+ q^{(r+1)(s+1)s/2}P(r+1, s+1, t-1) \cdot q^{rs(s-1)/2}P(r, s, t+1).$$

Note how the factor of q^{s+t} in the right-hand side comes from the edge of weight q^{s+t} that was not covered by either subgraph $H(r, s+1, t)$ or $H(r+1, s, t)$.

We simplify this relation by dividing through by $q^{(r+1)(s+1)s/2+rs(s-1)/2}$ to get the desired relation:

$$P(r+1, s+1, t)P(r, s, t)$$

$$= q^t P(r, s+1, t)P(r+1, s, t) + P(r+1, s+1, t-1)P(r, s, t+1). \qquad \square$$

Now we can prove MacMahon's formula for $P(a, b, c)$ by induction on $a+b+c$. When any of a, b, or c are 0, $P(a, b, c) = 1$. Now suppose MacMahon's formula holds for all a, b, c such that $a+b+c \leqslant r+s+t+1$. We show MacMahon's formula holds for $(a, b, c) = (r+1, s+1, t)$:

$$P(r, s+1, t)P(r+1, s, t)$$

$$= \left(\prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}} \right)^2 \left(\prod_{j=1}^{s} \frac{1-q^{j+r+t}}{1-q^{j+r}} \right) \left(\prod_{i=1}^{r} \frac{1-q^{i+s+t}}{1-q^{i+s}} \right)$$

$$= P(r, s, t) \cdot \frac{1-q^{r+s-1}}{1-q^{r+s+t-1}} \prod_{i=1}^{r+1} \prod_{j=1}^{s+1} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}}$$

$$P(r+1, s+1, t-1)P(r, s, t+1)$$

$$= \frac{\prod_{i=1}^{r+1} \prod_{j=1}^{s+1} (1-q^{i+j+t-2}) \prod_{i=1}^{r} \prod_{j=1}^{s} (1-q^{i+j+t})}{\prod_{i=1}^{r+1} \prod_{j=1}^{s+1} (1-q^{i+j-1}) \prod_{i=1}^{r} \prod_{j=1}^{s} (1-q^{i+j-1})}$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}} \frac{(1-q^t) \prod_{i=1}^{r+1} \prod_{j=1}^{s} (1-q^{i+j+t-1}) \prod_{j=1}^{r} (1-q^{j+s+t})}{\prod_{i=1}^{r} \prod_{j=1}^{s} 1-q^{i+j-1}}$$

$$= P(r, s, t) \cdot \frac{1-q^t}{1-q^{r+s+t-1}} \prod_{i=1}^{r+1} \prod_{j=1}^{s+1} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}},$$
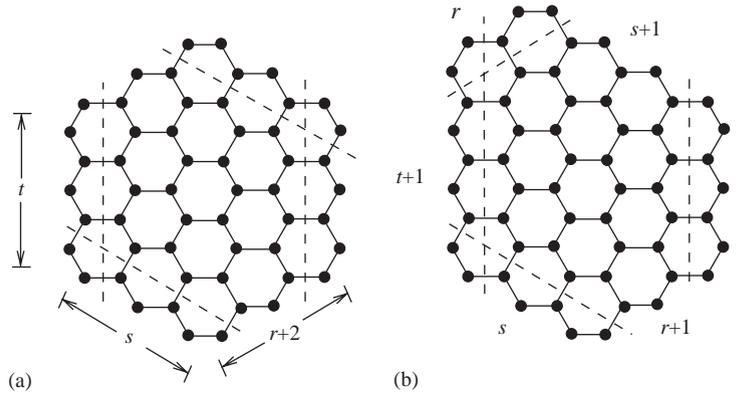
Fig. 20. Graphs for proving other bilinear relations. (a) $r = 1$, $s = t = 3$. (b) $r = 2$, $s = 3$, $t = 3$.

$$P(r + 1, s + 1, t)P(r, s, t)$$

$$= q^t P(r, s + 1, t)P(r + 1, s, t) + P(r + 1, s + 1, t - 1)P(r, s, t + 1)$$

$$= \frac{q^t(1 - q^{r+s-1}) + (1 - q^t)}{1 - q^{r+s+t-1}} P(r, s, t) \prod_{i=1}^{r+1} \prod_{j=1}^{s+1} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j-1}}$$

$$= P(r, s, t) \prod_{i=1}^{r+1} \prod_{j=1}^{s+1} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j-1}}.$$

Thus

$$P(r + 1, s + 1, t) = \prod_{i=1}^{r+1} \prod_{j=1}^{s+1} \frac{1 - q^{i+j+t-1}}{1 - q^{i+j-1}}.$$

It is interesting to note a similarity between Theorems 2.1 and 6.2. In the proof of each theorem, the two paths always run between vertices of opposite parity. We can find additional bilinear relations with MacMahon's formula that are analogous to Theorems 2.3 and 2.4. For instance, if we partition a hexagonal graph as shown in Fig. 20(a), we get

$$q^{(r+2)s(s-1)/2}P(r + 2, s, t) \cdot q^{rs(s-1)/2}P(r, s, t)$$

$$= (q^{(r+1)s(s-1)/2}P(r + 1, s, t))^2$$

$$- q^{(r+1)(s-1)(s-2)/2}P(r + 1, s - 1, t + 1) \cdot q^{(r+1)(s+1)s/2}P(r + 1, s + 1, t - 1).$$

After dividing through by $q^{(r+1)s(s-1)}$, we get

$$P(r+2,s,t)P(r,s,t)$$
$$= P(r+1,s,t)^2 - q^{r+1}P(r+1,s-1,t+1)P(r+1,s+1,t-1).$$

The relation analogous to Theorem 2.4 is:

$$q^{rs(s-1)/2}P(r,s,t+1) \cdot q^{rs(s-1)/2}P(r,s,t)$$
$$= q^{(r+1)s(s-1)/2}P(r+1,s,t) \cdot q^{(r-1)s(s-1)/2}P(r-1,s,t+1)$$
$$+ q^{r(s+1)s/2}P(r,s+1,t) \cdot q^{r(s-1)(s-2)/2}P(r,s-1,t+1),$$

which simplifies to

$$P(r,s,t+1)P(r,s,t)$$
$$= P(r+1,s,t)P(r-1,s,t+1) + q^r P(r,s+1,t)P(r,s-1,t+1).$$

Fig. 20(b) shows how to prove this relation. The graph has sides $r,s+1,t,r+1,s,t+1$. For each pair of hexagons, one is missing one of the strips along the sides of length r, t, or s, and the other hexagon is missing the other three strips (but contains the strip that the first hexagon is missing).

By taking the limit as $q \to 1$, we derive relations among the numbers of plane partitions fitting in $\mathcal{B}(r,s,t)$. These numbers also enumerate rhombus tilings of semiregular hexagons with sides r,s,t,r,s,t. In particular, the following relation was proven by Doron Zeilberger in [13]:

$$N(r,s,t+1)N(r,s,t)$$
$$= N(r+1,s,t)N(r-1,s,t) + N(r,s+1,t)N(r,s-1,t+1),$$

where $N(r,s,t)$ is $\lim_{q\to1} P(r,s,t)$.

7. Transpose complement plane partitions

If we view a plane partition as a collection of stacks of cubes, certain plane partitions will exhibit some symmetry. Such symmetry classes are outlined in [1]. The *complement* of a plane partition π in the box of dimensions $r \times s \times t$ is the set of cubes in the box that are not in π, reflected through the center of the box. A *transpose complement plane partition* (*TCPP*) π is one for which the complement is the same as the reflection of π in the plane $y = x$. If we visualize a TCPP as a rhombus tiling of a hexagon, the line of symmetry goes through the midpoints of two sides of the hexagon. Note that the sides of the hexagon must be of the form $r,r,2t,r,r,2t$, and the line of symmetry goes through the sides of length $2t$. The following theorem about the number of TCPPs was proved in [8]:
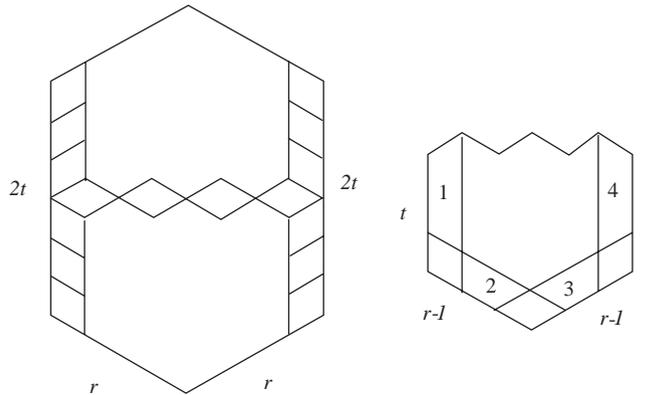
Fig. 21. Left: The forced rhombi in a TCPP. Right: The four strips along the sides of a (stripped) semi-hexagon.

Theorem 7.1. *The number of TCPPs in an* $r \times r \times 2t$ *box is*

$$\binom{t+r-1}{r-1} \prod_{1 \leqslant i \leqslant j \leqslant r-2} \frac{2t+i+j+1}{i+j+1}.$$

Let $N(r, r, 2t)$ be the number of TCPPs in an $r \times r \times 2t$ box.

Proposition 7.2. *If* $r \geqslant 2$ *and* $t \geqslant 1$, *then*

$$N(r, r, 2t)N(r-2, r-2, 2t)$$

$$= N(r-1, r-1, 2t)^2 + N(r, r, 2t-2)N(r-2, r-2, 2t+2).$$

Proof. Because of the symmetry of a TCPP, we only need to consider the number of ways to tile one half of an $(r, r, 2t, r, r, 2t)$-hexagon. Also note that the triangles that lie on the line of symmetry must join to form rhombi. We can cut the hexagon in half to form an (r, t)-semihexagon. We can strip this semihexagon even further since rhombi are forced along the sides of length t. (This also shortens the sides of length r by one.) See Fig. 21, left. Let us label the four strips of triangles 1, 2, 3, and 4, so that strips 1 and 4 are along the sides of length t, and strips 2 and 3 are along the sides of length $r-1$ (see Fig. 21, right). Removing all four strips would produce an $(r-2, t)$-semihexagon. Removing only strips 1 and 2 (or only strips 3 and 4) would produce a region with the same number of tilings as an $(r-1, t)$-semihexagon. If we remove only strips 2 and 3, we shorten t by one to form an $(r, t-1)$-semihexagon (with its outer edges stripped). Finally, if we remove only strips 1 and 4, we get an $(r-1, t+1)$-semihexagon. The relation follows from graphical condensation.   □

This relation was noted by Michael Somos in a private communication.

We can now prove Theorem 7.1 by induction on $r + t$. The relevant base cases are $N(r, r, 0) = 1$ for all r, and $N(0, 0, 2t) = 1$ and $N(1, 1, 2t) = 1$ for all t. These cases are trivially established. We now use Proposition 7.2 to prove the inductive step. Given that the formula holds for $N(r - 2, r - 2, 2t), N(r - 1, r - 1, 2t), N(r, r, 2t - 2)$, and $N(r - 2, r - 2, 2t + 2)$, we show that it holds also for $N(r, r, 2t)$. We need to verify that

$$
\binom{t + r - 1}{r - 1} \binom{t + r - 3}{r - 3} \prod_{1 \leqslant i \leqslant j \leqslant r-2} \frac{2t + i + j + 1}{i + j + 1} \prod_{1 \leqslant i \leqslant j \leqslant r-4} \frac{2t + i + j + 1}{i + j + 1}
$$

$$
= \binom{t + r - 2}{r - 2}^2 \left(\prod_{1 \leqslant i \leqslant j \leqslant r-3} \frac{2t + i + j + 1}{i + j + 1} \right)^2
$$

$$
+ \binom{t + r - 2}{r - 1} \binom{t + r - 2}{r - 3} \prod_{1 \leqslant i \leqslant j \leqslant r-2} \frac{2t - 2 + i + j + 1}{i + j + 1}
$$

$$
\times \prod_{1 \leqslant i \leqslant j \leqslant r-4} \frac{2t + 2 + i + j + 1}{i + j + 1}.
$$

We divide the right-hand side by the left hand side to obtain

$$
\frac{r - 1}{t + r - 1} \cdot \frac{t + r - 2}{r - 2} \prod_{1 \leqslant i \leqslant r-2} \frac{i + (r - 2) + 1}{2t + i + (r - 2) + 1} \prod_{1 \leqslant i \leqslant r-3} \frac{2t + i + (r - 3) + 1}{i + (r - 3) + 1}
$$

$$
+ \frac{t}{t + r - 1} \cdot \frac{t + r - 2}{t + 1} \prod_{1 \leqslant i \leqslant j \leqslant r-2} \frac{2t + i + j - 1}{2t + i + j + 1} \prod_{1 \leqslant i \leqslant j \leqslant r-4} \frac{2t + i + j + 3}{2t + i + j + 1}.
$$

After a heavy dose of cancellations, this expression simplifies to

$$
\frac{(r - 1)(t + r - 2)(2r - 3)(2r - 4)(2t + r - 1)}{(t + r - 1)(r - 2)(r - 1)(2t + 2r - 3)(2t + 2r - 4)}
$$

$$
+ \frac{t(t + r - 2)(2t + 1)(2t + 2)}{(t + r - 1)(t + 1)(2t + r - 3)(2t + 2r - 4)}
$$

$$
= \frac{(2r - 3)(2t + r - 1)}{(t + r - 1)(2t + 2r - 3)} + \frac{t(2t + 1)}{(t + r - 1)(2t + 2r - 3)}
$$

$$
= \frac{(4rt + 2r^2 - 5r - 6t + 3) + (2t^2 + t)}{2t^2 + 2r^2 + 4rt - 5t - 5r + 3} = 1
$$

and thus the inductive step and Theorem 7.1 follows.

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