Jacobi-Trudi Determinants Over Finite Fields

Shuli Chen and Jesse Kim

Based on work with Ben Anzis, Yibo Gao, and Zhaoqi Li

August 19, 2016

Outline

Basic Definitions

Definition (e_k and h_k)

For any positive integer k, the elementary symmetric function e_k is defined as

$$e_k(x_1, \cdots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

The complete homogeneous symmetric function h_k is defined as

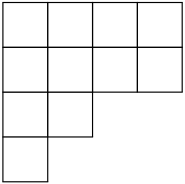
$$h_k(x_1, \cdots, x_n) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$$

For example, $e_2(x_1, x_2) = x_1 x_2$, while $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$.

Basic Definitions

A **partition** λ of a positive integer n is a sequence of weakly decreasing positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ that sum to n. For each i, the integer λ_i is called the i^{th} part of λ. We call n the size of λ, and denote by $|\lambda| = n$. We call k the length of λ.

$\lambda = (4, 4, 2, 1)$ is a partition of 11. We can represent it by a Young diagram:



Basic Definitions

A semi-standard Young tableau (SSYT) of shape λ and size n is a filling of the boxes of λ with positive integers such that the entries weakly increase across rows and strictly increase down columns. To each SSYT T of shape λ and size n we associate a monomial x^T given by

$$x^T = \prod_{i \in \mathbb{N}^+} x_i^{m_i},$$

where m_i is the number of times the integer i appears as an entry in T.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

$$x^T = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6$$

Basic Definitions

Definition (Schur Function)

The Schur function s_λ is defined as

$$s_\lambda = \sum_T x^T,$$

where the sum is across all semi-standard Young tableaux of shape λ.

Theorem (Jacobi-Trudi Identity)

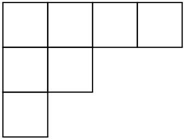
For any partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ *and its transpose* λ', *we have*

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^{k},$$

$$s_{\lambda'} = \det\left(e_{\lambda_i - i + j}\right)_{i,j=1}^{k}.$$

where $h_0 = e_0 = 1$ *and* $h_m = e_m = 0$ *for* $m < 0$.

For example, let $\lambda = (4, 2, 1)$.



$$s_\lambda = \begin{vmatrix} h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_3 & e_4 & e_5 & e_6 \\ e_1 & e_2 & e_3 & e_4 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{vmatrix}$$

Problem Statement

Main Question

If we assign the h_i's to numbers in some finite field $\mathbb{F}_q$ randomly, then for an arbitrary λ, what is the probability that $s_\lambda \mapsto 0$?

Besides, we also investigate when the probabilities are independent and what is the probability $P(s_\lambda \mapsto a)$ for some nonzero $a \in \mathbb{F}_q$.

Equivalence of Assigning e_i's and h_i's

For any positive integer k, Look at the single row partition $\lambda = (k)$. We have

$$s_\lambda = h_k = \begin{vmatrix} e_1 & e_2 & \cdots & e_k \\ 1 & e_1 & \cdots & e_{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & e_1 \end{vmatrix}.$$

Calculating the determinant from expansion across the first row we get $h_k = (-1)^{k+1}e_k + P(e_1, \cdots, e_{k-1})$.

Hence each assignment of $h_1, \cdots, h_k$ corresponds to exactly one assignment of $e_1, \cdots, e_k$ that results in the same value for s_λ, and vice versa.

Equivalence of Assigning e_i's and h_i's

We thus have

Theorem

For any partition λ, the value distribution of s_λ from assigning the h_i's is the same as the value distribution from assigning the e_i's.
Or equivalently, for any $a \in \mathbb{F}_q$, $P(s_\lambda \mapsto a) = P(s_{\lambda'} \mapsto a)$, where λ' is the transpose of λ.

Generally Bad Behavior

Theorem

$P(s_\lambda \mapsto 0)$ *is not always a rational function in* q.

Counterexample: $\lambda_1 = (4, 4, 2, 2)$
However, we have proved that

$$P(s_{\lambda_1} \mapsto 0) = \begin{cases} \frac{q^4+(q-1)(q^2-q)}{q^5} & \text{if} \quad q \equiv 0 \mod 2 \\ \frac{q^4+(q-1)(q^2-q+1)}{q^5} & \text{if} \quad q \equiv 1 \mod 2 \end{cases}$$

Other counterexamples we find are $\lambda_2 = (4, 4, 3, 2)$ and $\lambda_3 = (4, 4, 3, 3)$.

Generally Bad Behavior

Theorem

$P(s_\lambda \mapsto 0)$ is not always a rational function in q.

Counterexample: $\lambda_1 = (4,4,2,2)$

However, we have proved that

$$P(s_{\lambda_1} \mapsto 0) = \begin{cases} \frac{q^4+(q-1)(q^2-q)}{q^5} & \text{if} \quad q \equiv 0 \mod 2 \\ \frac{q^4+(q-1)(q^2-q+1)}{q^5} & \text{if} \quad q \equiv 1 \mod 2 \end{cases}$$

Other counterexamples we find are $\lambda_2 = (4,4,3,2)$ and $\lambda_3 = (4,4,3,3)$.

Conjecture

For a partition λ, $P(s_\lambda \mapsto 0)$ is always a quasi-rational function depending on the residue class of q modulo some integer.

Lower Bound on the Probability

Definition

Let M be a square matrix of size n with m free variables $x_1, \cdots, x_m$. We call it a **general Schur matrix** if

1. The 0's forms a (possibly empty) upside-down partition shape on the lowerleft corner.
2. Each of the other entries is either a nonzero constant in $\mathbb{F}_q$ (in which case we call the entry has label 0) or a polynomial in the form $x_k - f_{k-1}$ where $k \in [m]$ and f_{k-1} is a polynomial in $x_1, \cdots, x_{k-1}$, and in this case we call the entry has label k.
3. The labels of the nonzero entries are strictly increasing across rows and strictly decreasing across columns. So in particular, the label of the upperright entry is the largest.

Lower Bound on the Probability

Definition

Let M be a general Schur matrix of size n with m free variables $x_1, \cdots, x_m$. It is called a **reduced general Schur matrix** if it has the additional property that no entry is a nonzero constant.

Notice if we use each of the 1's in a Jacobi-Trudi matrix as a pivot to zero out all the other entries in its column and row and then delete these rows and columns, we obtain a reduced general Schur matrix M'. And we have $P(s_\lambda \mapsto 0) = P(\det M' \mapsto 0)$.

Lower Bound on the Probability

Theorem (Lower Bound)

For any λ, we have $P(s_\lambda \mapsto 0) \geq \frac{1}{q}$.

Idea of proof: We show $P(\det M \mapsto 0) \geq 1/q$ for an arbitrary reduced general Schur matrix M using induction on the number of free variables.

Asymptotic Bound on the Probability

Lemma

For a reduced general Schur matrix M of size n with 0's strictly below the main diagonal, we have $P(det(M) \mapsto 0) \leq \frac{n}{q}$.

Asymptotic Bound on the Probability

Lemma

For a reduced general Schur matrix M of size n with 0's strictly below the main diagonal, we have $P(det(M) \mapsto 0) \leq \frac{n}{q}$.

Lemma

Let M be a reduced general Schur matrix of size $n \geq 2$ with 0's strictly below the $(n-1)^{th}$ diagonal. Let M' be the $(n-1) \times (n-1)$ minor on its lower left corner. Then $P(\det M \mapsto 0 \ \& \ \det M' \mapsto 0) \leq \frac{n(n-1)}{q^2}$.

Asymptotic Bound on the Probability

Theorem (Asymptotic Bound)

For any λ, as $q \to \infty$, we have $P(s_\lambda \mapsto 0) \to \frac{1}{q}$.

Idea of proof:
Reduce to a reduced general Schur matrix.
Use conditional probability on whether its minor has zero determinant.
Get an upper bound $1/q + n(n-1)/q^2$ for the probability from the lemmas.

General Case and Conjecture on the Upper Bound

Proposition

Fix k. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i - \lambda_{i+1} \geq k - 1$ and $\lambda_k \geq k$. Then

$$P(s_\lambda \mapsto 0) = 1 - \frac{|GL(k,q)|}{q^{k^2}} = \frac{1}{q^{k^2}} \left(q^{k^2} - \prod_{j=0}^{k-1} (q^k - q^j) \right),$$

where $|GL(k,q)|$ denote the number of invertible matrices of size k with entries in $\mathbb{F}_q$.

Conjecture (Upper Bound)

For any partition λ with k parts, the above probability gives a tight upper bound for $P(s_\lambda \mapsto 0)$.

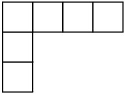
Achieving $\frac{1}{q}$

Partition shapes that achieve $\frac{1}{q}$ can be completely characterized.

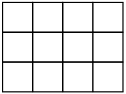
Theorem

$P(s_\lambda \mapsto 0) = \frac{1}{q} \iff \lambda$ *is a hook, rectangle or staircase.*

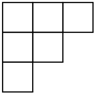
Hook shapes: $\lambda = (a, 1^n)$



Rectangle shapes: $\lambda = (a^n)$



and Staircase shapes: $\lambda = (a, a-1, a-2, ..., 1)$



Hooks

Hook shapes have very nice Jacobi-Trudi matrices:

$$s_{(a,1^n)} = \begin{vmatrix} h_a & h_{a+1} & \cdots & & h_{a+n} \\ 1 & h_1 & & & \\ 0 & 1 & h_1 & & \\ & & & \ddots & \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}$$

Hooks

Hook shapes have very nice Jacobi-Trudi matrices:

$$s_{(a,1^n)} = \begin{vmatrix} h_a & h_{a+1} & \cdots & & h_{a+n} \\ 1 & h_1 & & & \\ 0 & 1 & h_1 & & \\ & & & \ddots & \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}$$

$$s_{(a,1^n)} = \pm h_{a+n} + p(h_1, h_2, ..., h_{a+n-1})$$

Hooks

Hook shapes have very nice Jacobi-Trudi matrices:

$$s_{(a,1^n)} = \begin{vmatrix} h_a & h_{a+1} & \cdots & & h_{a+n} \\ 1 & h_1 & & & \\ 0 & 1 & h_1 & & \\ & & & \ddots & \\ 0 & \cdots & 0 & 1 & h_1 \end{vmatrix}$$

$$s_{(a,1^n)} = \pm h_{a+n} + p(h_1, h_2, ..., h_{a+n-1})$$

$$P(s_{(a,1^n)} \mapsto 0) = \frac{1}{q}$$

Rectangles

Rectangle shapes also have nice Jacobi-trudi matrices:

$$s_{(a^a)} = \begin{vmatrix} h_a & h_{a+1} & h_{a+2} & \cdots & h_{2a-1} \\ \vdots & & & \iddots & \vdots \\ h_3 & h_4 & h_5 & & h_{a+2} \\ h_2 & h_3 & h_4 & & h_{a+1} \\ h_1 & h_2 & h_3 & \cdots & h_a \end{vmatrix}$$

Rectangles

Rectangle shapes also have nice Jacobi-trudi matrices:

$$s_{(a^a)} = \begin{vmatrix} h_a & h_{a+1} & h_{a+2} & \cdots & h_{2a-1} \\ \vdots & & & \iddots & \vdots \\ h_3 & h_4 & h_5 & & h_{a+2} \\ h_2 & h_3 & h_4 & & h_{a+1} \\ h_1 & h_2 & h_3 & \cdots & h_a \end{vmatrix}$$

Idea of proof: Assign h_i's in order until it is clear that the determinant is 0 with probability $\frac{1}{q}$

Rectangles

Definition

Let M be a general Schur matrix. Define an operation ψ from general Schur matrices to reduced general Schur matrices by:
(a) If M has no nonzero constant entries, $\psi(M) = M$
(b) Otherwise, take each nonzero entry in M and zero out its row and column, then delete its row and column. $\psi(M)$ is the resulting matrix

Example:

$$M = \begin{bmatrix} 0 & 2x_2 & x_4 & x_5 \\ 0 & 1 & 4x_3 & x_4 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 2x_2 & x_4 & x_5 \\ 0 & 1 & 4x_3 & x_4 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 2x_2 & x_4 & x_5 \\ 0 & 1 & 4x_3 & x_4 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 - x_2 \\ 0 & 0 & 0 & x_2 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & x_4 - 8x_2x_3 & x_5 - 2x_2x_4 \\ 0 & x_1 & x_3 - x_2 \\ 0 & 0 & x_2 \end{bmatrix} = \psi(M)$$

Rectangles

Definition

Let M be a general Schur matrix. Define φ that takes general Schur matrices and a set of assignments to reduced general Schur matrices by:

(a) $\varphi(M; x_1 = a_1) = \psi(M(x_1 = a_1))$

(b) $\varphi(M; x_1 = a_1, x_2 = a_2, ..., x_i = a_i)$
$= \varphi(\varphi(M; x_1 = a_1, ...x_{i-1} = a_{i-1}); x_i = a_i)$

Rectangles

Example:

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

Rectangles

Example:

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \xrightarrow{\varphi(x_1=1)} \begin{bmatrix} x_5 - x_2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - x_2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - x_2^2 & x_4 - x_2x_3 & x_5 - x_2x_4 \end{bmatrix}$$

Rectangles

Example:

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \xrightarrow{\varphi(x_1=1)} \begin{bmatrix} x_5 - x_2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - x_2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - x_2^2 & x_4 - x_2x_3 & x_5 - x_2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_2=2)} \begin{bmatrix} x_5 - 2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - 2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - 4 & x_4 - 2x_3 & x_5 - 2x_4 \end{bmatrix}$$

Rectangles

Example:

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \xrightarrow{\varphi(x_1=1)} \begin{bmatrix} x_5 - x_2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - x_2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - x_2^2 & x_4 - x_2x_3 & x_5 - x_2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_2=2)} \begin{bmatrix} x_5 - 2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - 2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - 4 & x_4 - 2x_3 & x_5 - 2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_3=4)} \begin{bmatrix} x_5 - 2x_4 & x_6 - 4x_4 & x_7 - x_4^2 \\ x_4 - 8 & x_5 - 16 & x_6 - 4x_4 \\ 0 & x_4 - 8 & x_5 - 2x_4 \end{bmatrix}$$

Rectangles

Example:

$$A = \begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \xrightarrow{\varphi(x_1=1)} \begin{bmatrix} x_5 - x_2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - x_2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - x_2^2 & x_4 - x_2x_3 & x_5 - x_2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_2=2)} \begin{bmatrix} x_5 - 2x_4 & x_6 - x_3x_4 & x_7 - x_4^2 \\ x_4 - 2x_3 & x_5 - x_3^2 & x_6 - x_3x_4 \\ x_3 - 4 & x_4 - 2x_3 & x_5 - 2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_3=4)} \begin{bmatrix} x_5 - 2x_4 & x_6 - 4x_4 & x_7 - x_4^2 \\ x_4 - 8 & x_5 - 16 & x_6 - 4x_4 \\ 0 & x_4 - 8 & x_5 - 2x_4 \end{bmatrix}$$

$$\xrightarrow{\varphi(x_4=8)} \begin{bmatrix} x_5 - 16 & x_6 - 32 & x_7 - 64 \\ 0 & x_5 - 16 & x_6 - 32 \\ 0 & 0 & x_5 - 16 \end{bmatrix}$$

Rectangles

Lemma

Let A be a matrix corresponding to a rectangle partition shape, i.e. $A = (x_{j-i+n})_{1 \le i,j \le n}$.
Then the lowest nonzero diagonal of $\varphi(A; x_1 = a_1, ..., x_r = a_r)$ has all entries the same for any $a_1, ..., a_r$.

In particular, if $\varphi(A; x_1 = a_1, ..., x_r = a_r)$ is upper triangular with variables on the main diagonal, the probability it has determinant 0 is $\frac{1}{q}$

Rectangles

We can now divide assignments of the h_i's into disjoint sets based on the first time φ gives an upper triangular matrix: If two assignments are the same up until this point, they are put in the same set.

Each set will have $\frac{1}{q}$ of its members with determinant 0, so $P(s_{a^n} \mapsto 0) = \frac{1}{q}$

Independence of Schur functions

A natural continuation of the question of when some Schur function is sent to 0 is whether two Schur functions are sent to 0 independently.

In general this is hard to determine, beyond the trivial case where the two Jacobi-Trudi matrices contain no e_i or h_i in common.

Independence of Hooks

Theorem

Let $\Lambda := \{\lambda^{(k)}\}_{k\in\mathbb{N}}$ *be a collection of hook shapes such that* $|\lambda^{(k)}| = k$ *for all* k*. Then the distributions of values of the collection* $\{s_{\lambda^{(k)}}\}_k$ *is uniform and independent of each other.*

Independence of Hooks

Theorem

Let $\Lambda := \{\lambda^{(k)}\}_{k\in\mathbb{N}}$ *be a collection of hook shapes such that* $|\lambda^{(k)}| = k$ *for all* k*. Then the distributions of values of the collection* $\{s_{\lambda^{(k)}}\}_k$ *is uniform and independent of each other.*

$$s_{(a,1^n)} = \pm h_{a+n} + p(h_1, h_2, \dots, h_{a+n-1})$$

Independence of Rectangles

Focusing on rectangles, we can find multiple families of rectangles whose probabilities of being 0 are all independent of one another.

Theorem

Let $c \in \mathbb{N}$ be arbitrary. Then the events $\{s_{a^n} \mapsto 0 | a + n = c\}$ are setwise independent.

Theorem

Let $c \in \mathbb{N}$ be arbitrary. Then the events $\{s_{a^n} \mapsto 0 | a - n = c\}$ are setwise independent.

Independence of Rectangles

Results from independence come from the structure of the relevant matrices. We can find one of the Jacobi-Trudi matrices of two rectangles in the same family as a minor of the other:

$$\begin{bmatrix} x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \text{contains} \begin{bmatrix} x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 \end{bmatrix} \text{and} \begin{bmatrix} x_5 & x_6 & x_7 \\ x_4 & x_5 & x_6 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

Nonzero values of Schur functions

Another natural continuation lies in values of $\mathbb{F}_q$ other than 0, and finding the probability some Schur function is sent to one of these values.

Proposition

Let $a, x \in \mathbb{F}_q$ with $x \neq 0$, and let λ be a partition of size n. Then $P(s_\lambda \mapsto a) = P(s_\lambda \mapsto x^n a)$

s_λ is homogeneous of degree n, and each h_i is homogeneous with degree i. Thus if $h_1 = a_1, h_2 = a_2, ...h_n = a_n$ sends s_λ to a, $h_1 = xa_1, h_2 = x^2 a_2, ...h_n = x^n a_n$ will send s_λ to $x^n a$. This is a bijection since x is nonzero, so the two probabilities are equal.

Corollary

Let λ be a partition of size n, and let q be a prime power such that $\gcd(n, q-1) = 1$. Then $P(s_\lambda \mapsto a) = P(s_\lambda \mapsto b)$ for any nonzero $a, b \in \mathbb{F}_q$.

Nonzero values of rectangles

Theorem

$$P(S_{a^n} \mapsto b) = \sum_{d \mid \gcd(q-1,a)} \frac{f_b(d)}{q^{a(d-1)/d+1}}$$

where

$$f_b(d) = \sum_{e \mid d} \mu(e) g_b(\frac{d}{e})$$

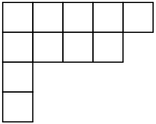
is the Möbius inverse of

$$g_b(d) = \begin{cases} 0 & d \nmid \frac{q-1}{ord(b)} \\ d & d \mid \frac{q-1}{ord(b)} \end{cases}$$

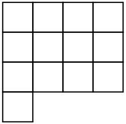
Shapes with Probability $(q^2+q-1)/q^3$

Two hook-like shapes:

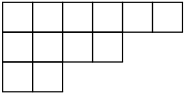
- $\lambda = (a, b, 1^m)$, where $b \geq 2$ and $a \neq b + m$.



- $\lambda = (a^m, 1^n)$ where $a, m > 1$.



(Conjecture) 2-staircases: $\lambda = (2k, \cdots, 4, 2)$



Relaxing the Condition of General Shape

Let $\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_i - \lambda_{i+1} \geq k - 1$ and $\lambda_k < k$, then

$$P(s_\lambda \mapsto 0) = 1 - \frac{GL(k-1, q)}{q^{(k-1)^2}} = \frac{1}{q^{(k-1)^2}} \left(q^{(k-1)^2} - \prod_{j=0}^{k-2} (q^{k-1} - q^j) \right)$$

Let $\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_j - \lambda_{j+1} = k - 2$ for some $j < k$, $\lambda_i - \lambda_{i+1} \geq k - 1$ for all $i < k, i \neq j$ and $\lambda_k \geq k$. Then

$$P(s_\lambda \mapsto 0) = 1 - \frac{q^{2k-2} - q^{k-1} - q^{k-2} + 1}{q^{k^2-2k+2}} \prod_{i=0}^{k-3} (q^{k-2} - q^i).$$

The End