Shards and noncrossing tree partitions

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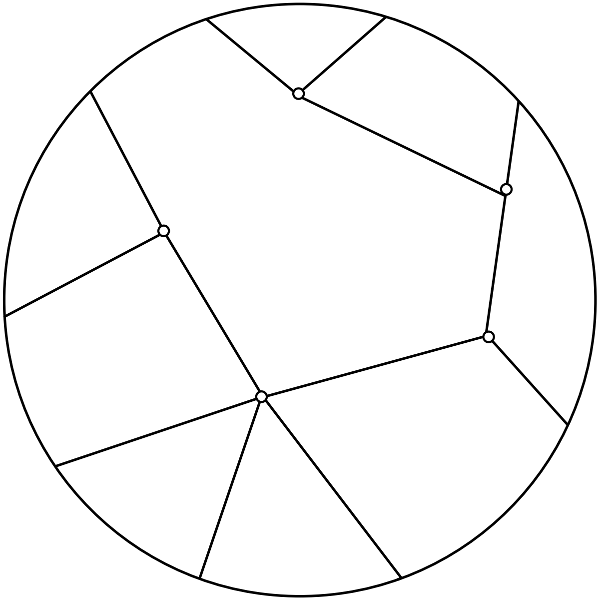
Outline

Broad Overview

Fix a tree T embedded in a disk with exactly its leaves on the boundary and whose interior vertices (the vertices not on the boundary) have degree at least 3.

$T =$ 

Broad Overview

We obtain the following diagram of posets defined from T:

$$
\begin{array}{ccccc}
\mathrm{Bic}(T) & \xrightarrow{\psi} & \Psi(\mathrm{Bic}(T)) & & \\
\uparrow \phi & & \uparrow ? & & \\
\overrightarrow{FG}(T) & \xrightarrow{\psi} & \Psi(\overrightarrow{FG}(T)) & \xrightarrow{\sim} & \mathrm{NCP}(T)
\end{array}
$$

Goal: Understand the combinatorics of $\mathrm{NCP}(T)$

Noncrossing tree partitions: Overview

The poset NCP(T) is called the *noncrossing tree partitions* of T. In this part of the talk, we will discuss our research of the following properties of NCP(T):

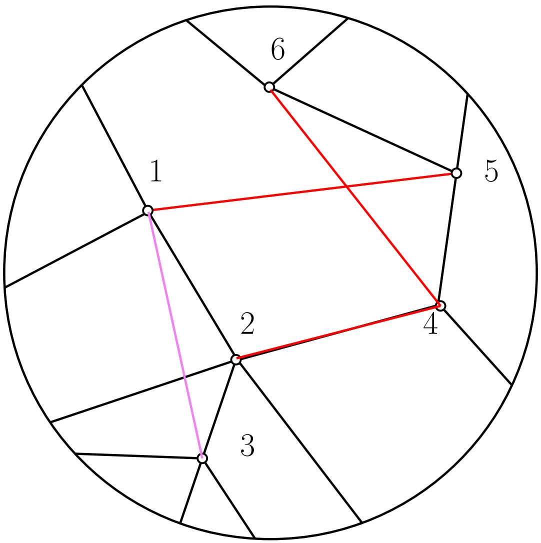
1. NCP(T) is a lattice
2. NCP(T) is graded (conjecture)
3. NCP(T) is not self-dual
4. How to count the maximal chains in NCP(T)

What is NCP(T)?

For a tree T, a *segment* $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geqslant 1$ is a sequence of interior vertices of T that takes a "sharp" turn at each v_i. In particular, the interior vertices of T are not segments.

Example

In the tree below, $(1, 5)$ and $(2, 4, 6)$ are segments. The sequence $(1, 3)$ is not a segment.



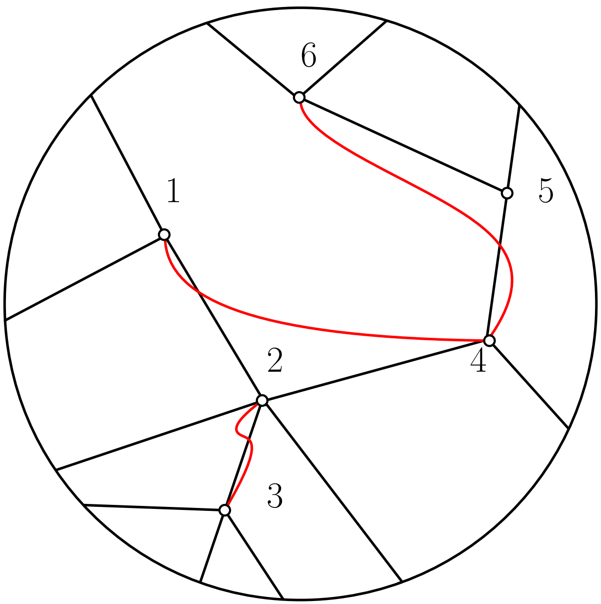
A *noncrossing partition* $\mathbf{B} = (B_1, \ldots, B_k)$ is a set partition of the interior vertices of T where

- the vertices in B_i can be connected by *red admissible curves* (i.e. curves whose endpoints define segments of T and leave their endpoints to the right), where any pair of such curves can only agree at their endpoints, and
- red admissible curves connecting vertices of B_i do not cross those of B_j for $i \neq j$.

We let $\mathrm{NCP}(T)$ denote the poset of noncrossing tree partitions ordered by refinement.

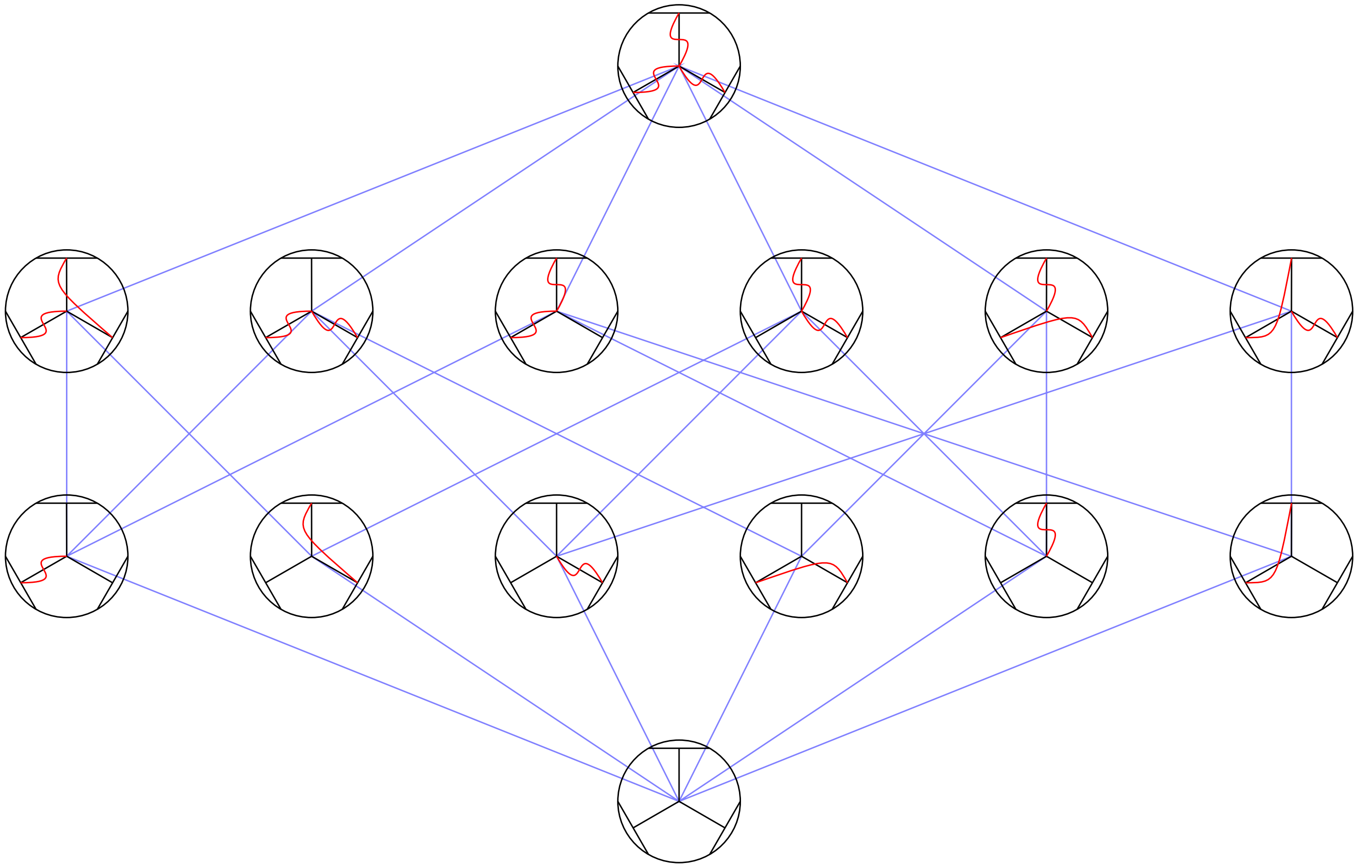
Example

$\mathbf{B} = \{\{1,4,6\},\{2,3\},\{5\}\}$ is an element of $\mathrm{NCP}(T)$.



Theorem (Garver-McConville)

The poset NCP(T) *is a lattice.*



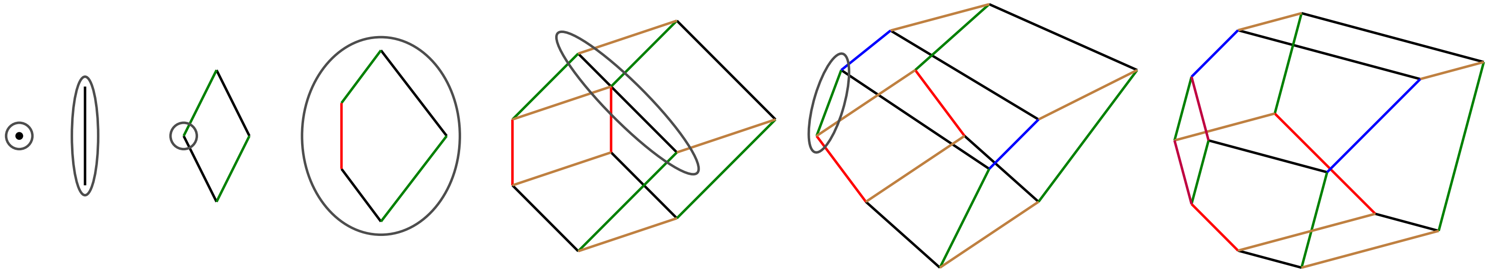
Lattice Theory

Before we talk about the structural properties of NCP(T), we need to discuss the relevant lattice theory.

Definition

A lattice is called *congruence-uniform* if it can be constructed from a single point using interval doublings.

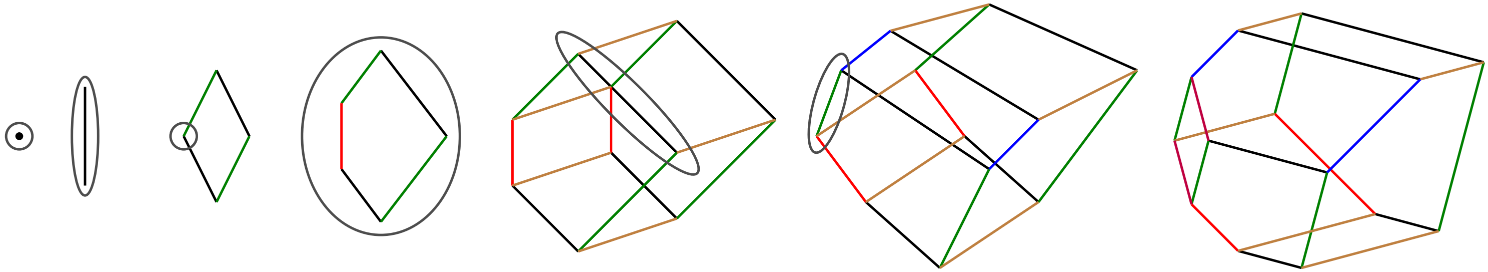
Here is an example of a lattice constructed from interval doublings:



Lattice Theory

Theorem

A lattice is congruence-uniform if and only if it admits an edge labeling known as a ***CU-labeling****.*



In fact, the colors on the edges of the picture above form a CU-labeling, where the color set is ordered $s \leqslant t$ if the color s appears before t in the sequence of doublings.

Lattice Theory

L a lattice
λ a CU-labeling of L
$x \in L$

$\rightsquigarrow$

$\Psi(L)$
Shard intersection order

$\Psi(L)$ consists of sets

$$\psi(x) = \{\text{labels appearing between } \bigwedge_{i=1}^{k} y_i \text{ and } x\}$$

where $\{y_i\}_{i=1}^{k}$ is the set of elements immediately below x in L. The partial ordering on $\Psi(L)$ is inclusion. We call the interval $[\bigwedge_{i=1}^{k} y_i, x]$ the *facial interval* corresponding to x.

Back to NCP(T)

Theorem (Garver-McConville)

For a tree T, NCP(T) is isomorphic to $\Psi(\overrightarrow{FG}(T))$.

This brings us to one of the main objects in our project:

Conjecture

The lattice NCP(T) is graded by the number of blocks in a partition.

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The lattice $\mathrm{NCP}(T)$ is graded by the number of blocks in a partition.

How we want to prove this conjecture:

- Show that every covering relation in $\mathrm{NCP}(T)$ is given by merging two blocks of a partition (which is what happens with $\mathrm{NC}(n)$).
- To do this, it suffices to show that if we can merge m blocks of $\mathbf{B}$, $m \geqslant 3$, then we can merge $m-1$ blocks.
- To show the above, we work with $\overrightarrow{FG}(T)$. We know that $\mathbf{B}$ corresponds to a facial interval in $\overrightarrow{FG}(T)$. We want to show that it is contained in a facial interval "one dimension lower" than the entire lattice.

$$\mathbf{B} \mapsto \psi(x) \sim [a, x] \subsetneq [a', x'] \subsetneq \overrightarrow{FG}(T)$$

Corollaries of conjecture and further structure

Garver and McConville defined a bijection NCP(T) called the *Kreweras Complement*. The Kreweras complement sends a partition with m blocks to a partition with $\#V^o(T) + 1 - m$ blocks. A corollary of this map and the previous conjecture is the following:

Corollary

The lattice NCP(T) *is rank-symmetric.*

The above property is shared by NC(n). A natural question to ask is: How many of the nice properties of NC(n) carry over to NCP(T)? We provide a partial answer here:

Theorem

In general, NCP(T) *is not self-dual.*

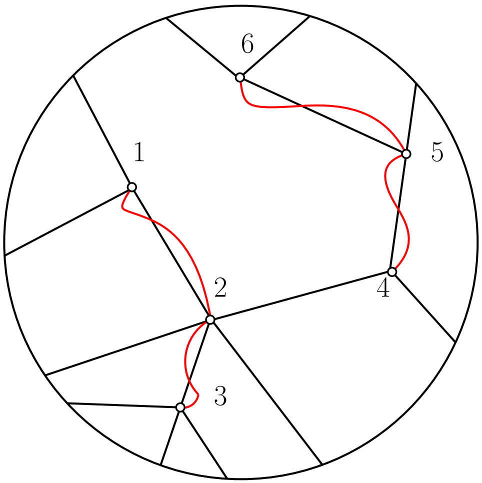
We conclude our discussion of NCP(T) with a method of calculating the number of maximal chains, denoted mc(T).

We will exploit the following fact in order to obtain recursions: let $\{a_i\}_{i=1}^{n}$ be the set of coatoms of NCP(T); then

$$\mathrm{mc}(T) = \sum_{i=1}^{n} \mathrm{mc}([\hat{0}, a_i]).$$

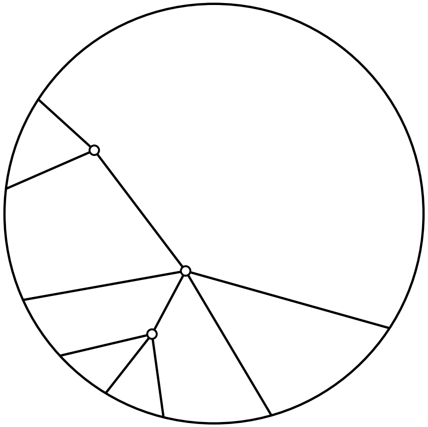
From here, we can note that $[\hat{0}, a_i]$ is isomorphic to the product of two noncrossing tree partitions of smaller trees, as shown by the following picture:

$a_i =$

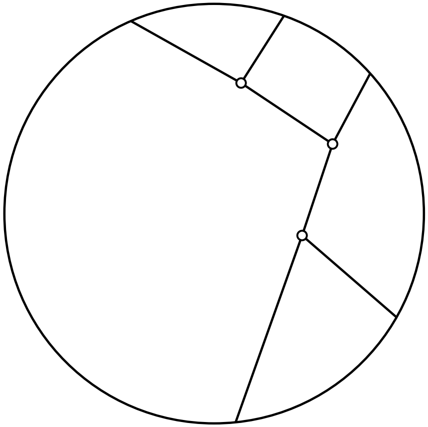


We have that $[\hat{0}, a_i] \cong \text{NCP}(T_1) \times \text{NCP}(T_2)$, where

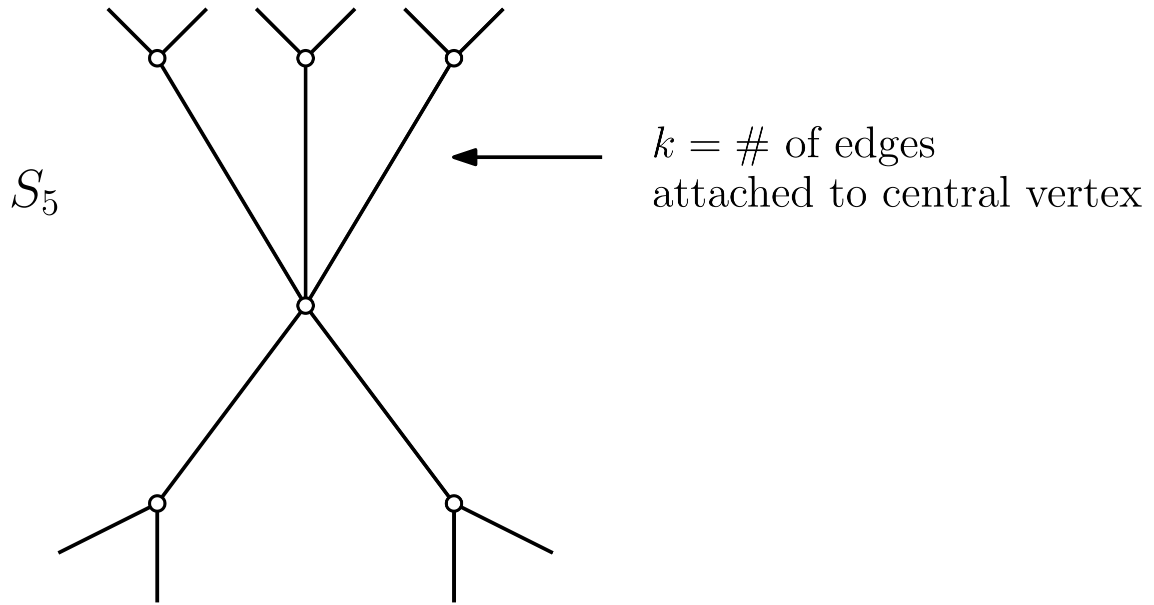
$T_1 =$



$T_2 =$

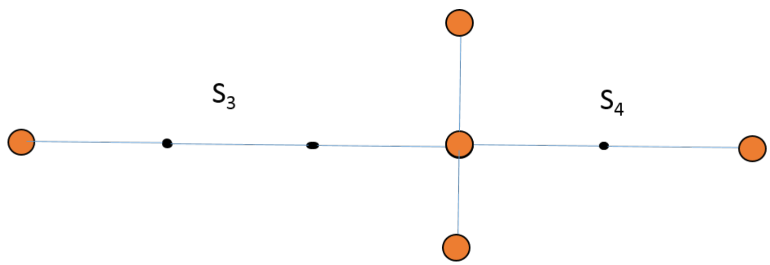


Using this method, we can count the maximal chains of the *kth star-graph*, denoted S_k, which is the family of trees of the following form:



We get that $\mathrm{mc}(S_k) = \frac{k!F_{k+1}}{2}$, where F_{k+1} is the $(k+1)$th fibonacci number.

Segments $S_1, S_2 \in Seg(T)$ whose composition is also in $Seg(T)$ are **composable**.



A subset $B \subset Seg(T)$ is **closed** if for any composable $S_1, S_2 \in B$, we have $S_1 \dot{S}_2 \in B$.

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$Bic(T)$ is a poset who elements are biclosed sets $B \subset Seg(T)$, partially ordered by inclusion.

We will explicitly demonstrate the CU-labeling for $Bic(T)$.

For a segment $[a, c]$ with vertex b in between, we say that $[a, b]$ and $[b, c]$ constitute a **break** of $[a, c]$

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Each of $[a, b]$ and $[b, c]$ is a **split** of $[a, c]$ corresponding to that break.

Recall that a CU-labeling is a map $\lambda : \{$covering relations of $Bic(T)\} \to P$ for some poset P of labels.

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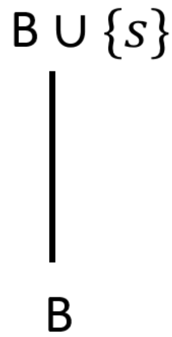
We choose P with elements of the form S_Δ where $S \in Seg(T)$ and Δ is a set of splits of S. The partial ordering is given by $S_\Delta \geqslant Q_\mu$ if S contains Q.

Covering relations in $Bic(T)$ look like:



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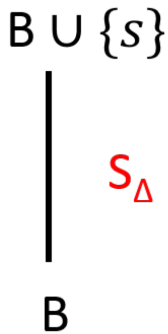

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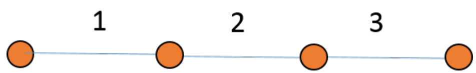

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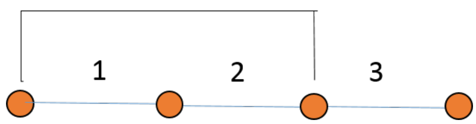

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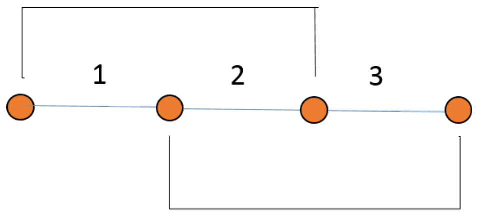

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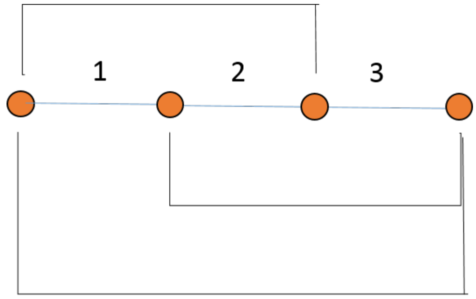

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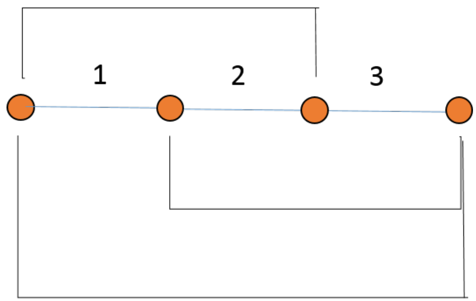

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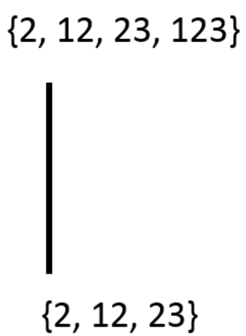

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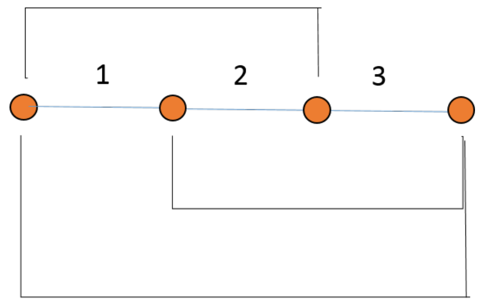

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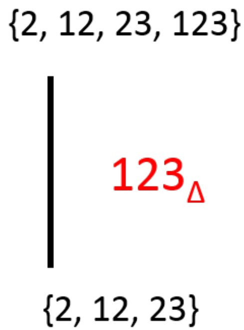

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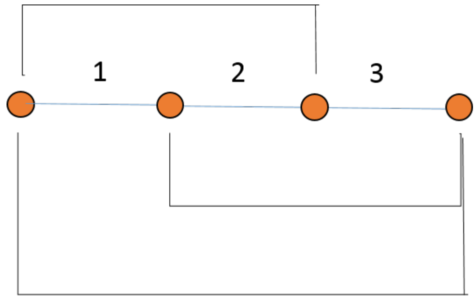

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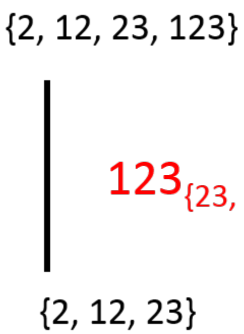

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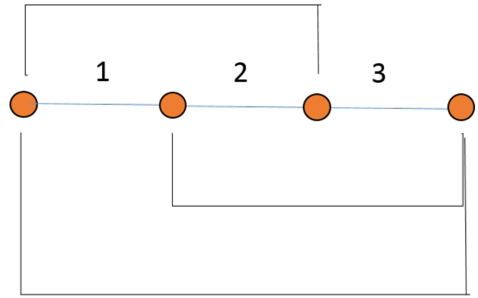

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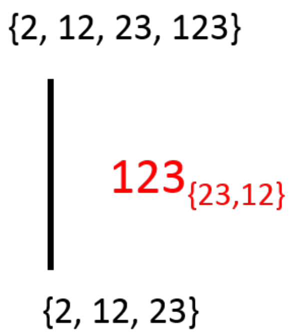
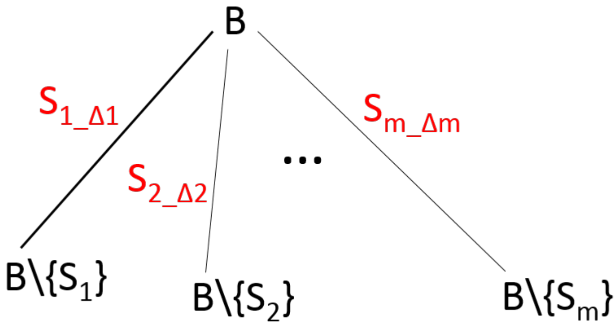


$\Psi(Bic(T))$ has a maximum element.

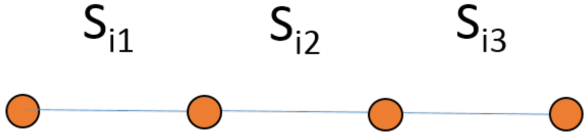
If we can show that for all $C, D \in Bic(T)$, there exists some $B \in Bic(T)$ such that $\psi(C) \cap \psi(D) = \psi(B)$, then we can conclude that $\Psi(Bic(T))$ is a lattice.



Elements of $\psi(B)$ are those of the form S_Δ where S is a composition of some of $S_1, S_2, \ldots, S_m$ and Δ is a set of splits of S with certain stipulations.

Vertices within S which are endpoints of some S_i correspond to **faultline** breaks.

Other vertices correspond to **non-faultline** breaks.



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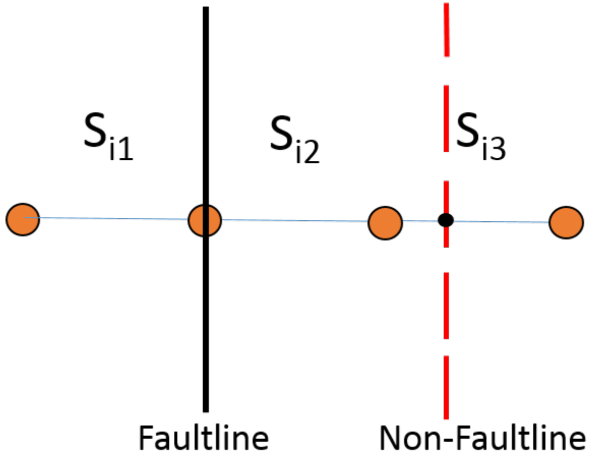
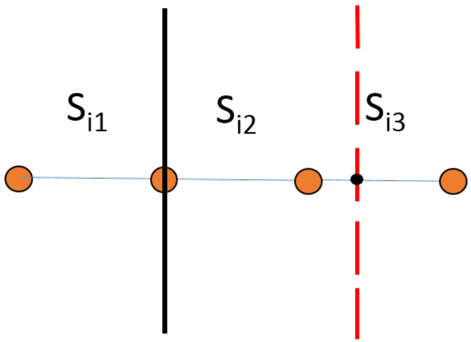


What splits of S are in Δ?



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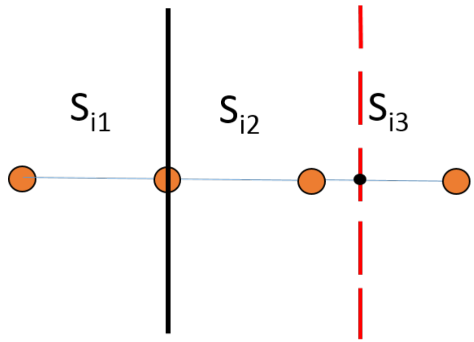

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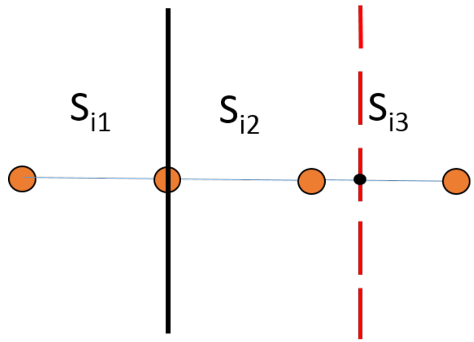
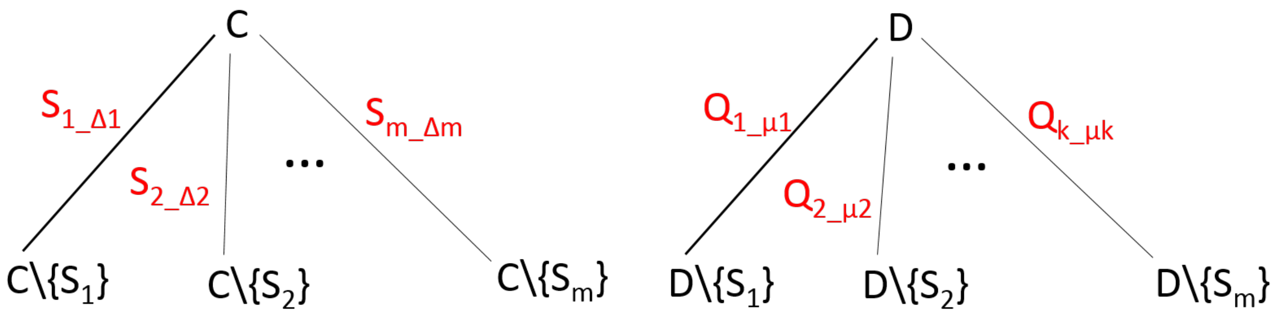
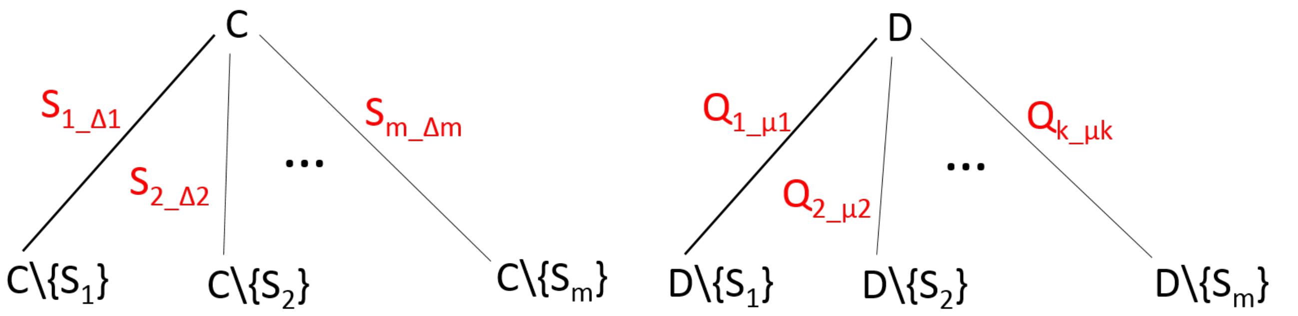


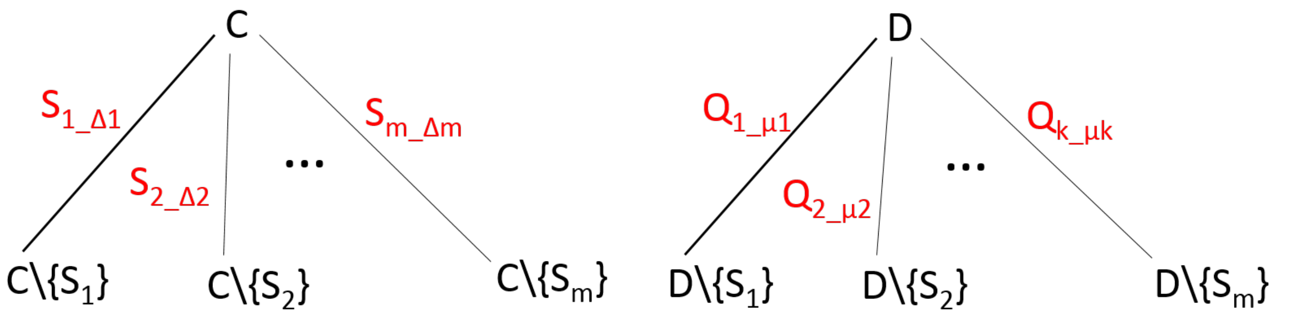
For non-faultline breaks, these are predetermined by $\Delta_1, \Delta_2, \ldots, \Delta_m$. For each faultline break, there is an independent choice.



Labels in $\psi(C) \cap \psi(D)$ are of the form S_Δ where:



Labels in $\psi(C) \cap \psi(D)$ are of the form S_Δ where:
S must simultaneously be a composition of S_i's and Q_i's.



Labels in $\psi(C) \cap \psi(D)$ are of the form S_Δ where:
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Furthermore, the splits determined by the corresponding Δ_i's and μ_i's must be compatible.

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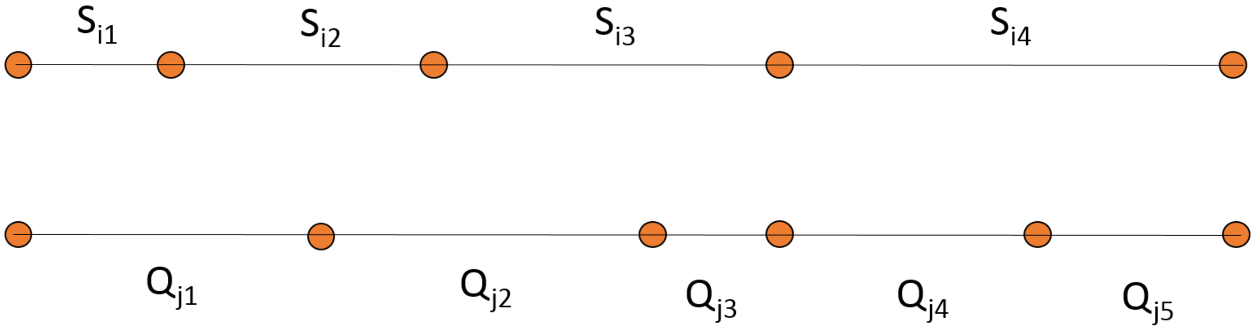
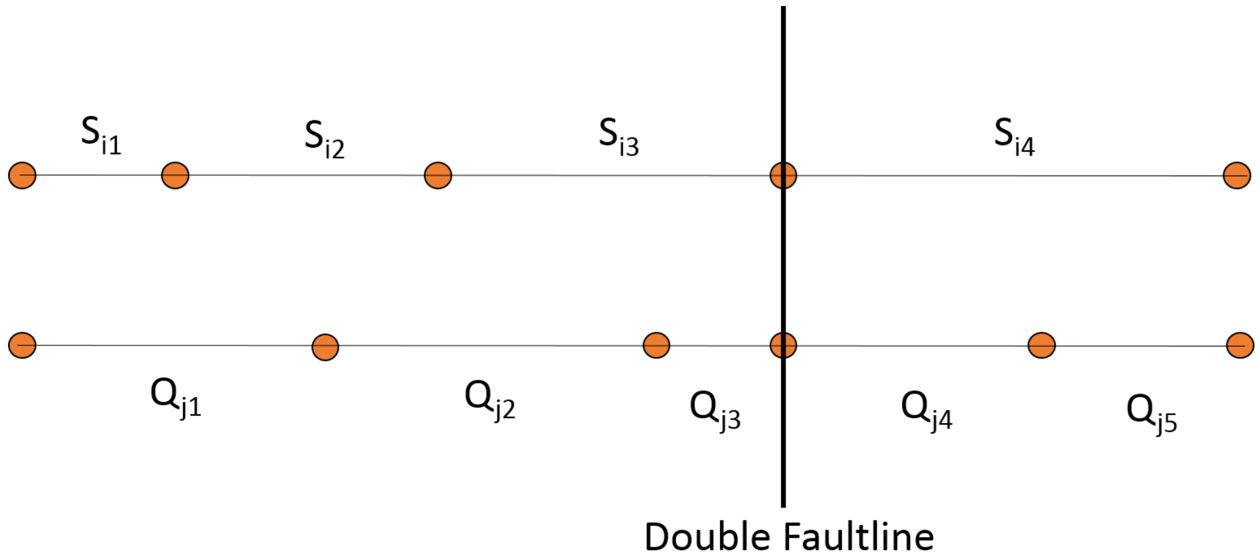


The only breaks for which there is a choice of what split of S to include in Δ are when the break is a faultline for S viewed as a composition of S_i's **and** S viewed as a composition of Q_i's.



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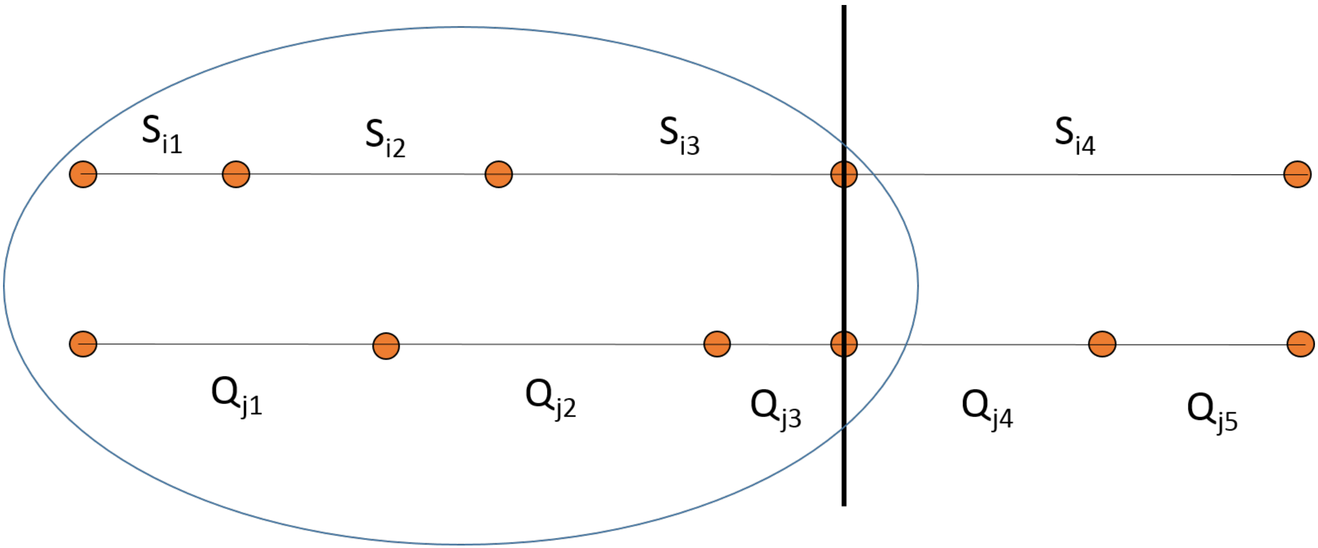

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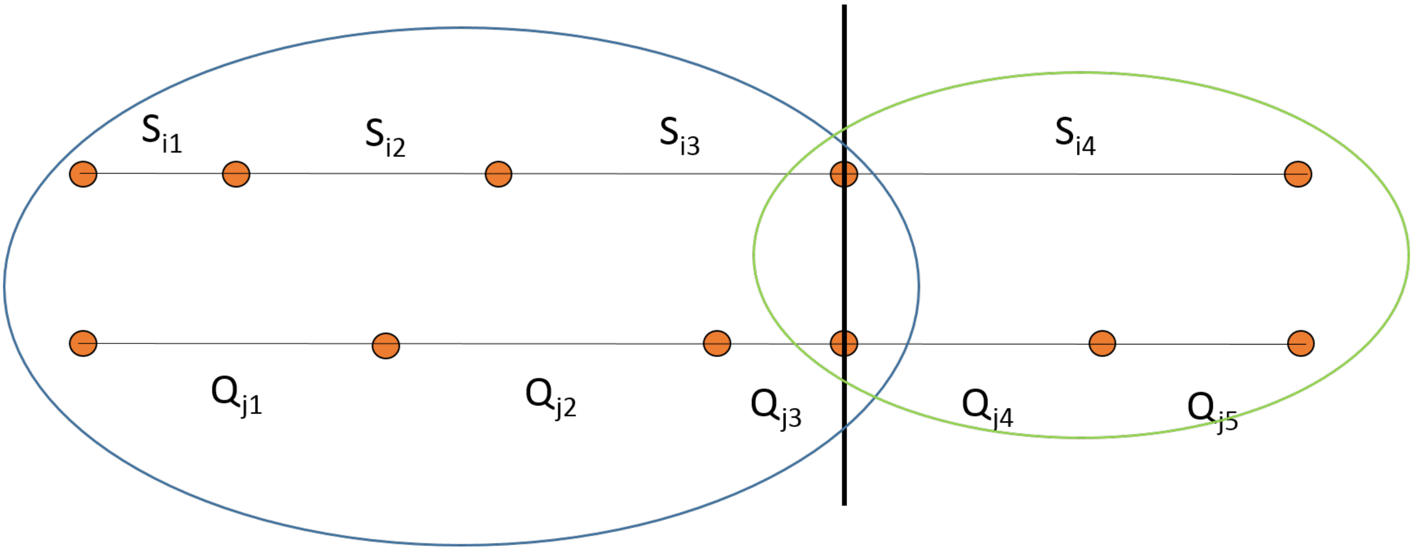

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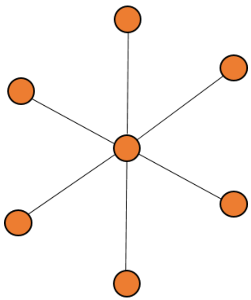
1. S is a composition of the segment parts of pseudominimal labels.
2. The only choices for which splits of S to include in Δ occur at breaks where two such pseudominimal lables are joined together.

Pseudominimal elements of $\psi(C) \cap \psi(D)$ generate $\psi(C) \cap \psi(D)$ the same way $\psi(B)$ is generated by the labels on its covering relations.

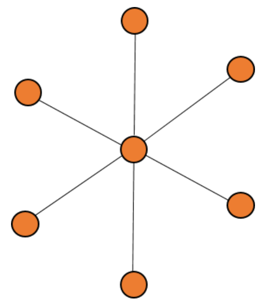
We can conceivably take
$B = \vee\{$ pseudominimal elements of $\psi(C) \cap \psi(D)\}$ to obtain
$\psi(B) = \psi(C) \cap \psi(D)$.

For the star graph S_k:



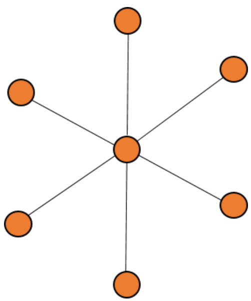
$$|NCP(S_k)| =$$

For the star graph S_k:



$$|NCP(S_k)| = 2|NCP(S_{k-1})| + |NCP(S_{k-2})|$$

For the star graph S_k:



$|NCP(S_k)| = 2|NCP(S_{k-1})| + |NCP(S_{k-2})|$
with $|NCP(S_3)| = 14, |NCP(S_4)| = 34$

Straight trees like

are analagous to classical non-crossing partitions.

Thank you to project mentor Al Garver, project TA Craig Corsi, Thomas McConville, Vic Reiner, the University of Minnesota, and the National Science Foundation.

References

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