Combinatorics of Gelfand-Tsetlin Polytopes

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Overview

GT Polytopes

Definition (GT Polytope)

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the Gelfand-Tsetlin Polytope GT_λ is the set of points $\vec{x} = (x_{i,j})_{1 \le j \le i \le n} \in \mathbb{R}^{n(n+1)/2}$ with $x_{i,i} = \lambda_i$ satisfying the following inequalities:

1. $x_{i-1,j} \le x_{i,j} \le x_{i+1,j}$,
2. $x_{i,j-1} \le x_{i,j} \le x_{i,j+1}$.

GT Polytopes

$$
\begin{array}{ccccccccccc}
\lambda_1 \\
\rotatebox{90}{$\leq$} \\
x_{2,1} & \leq & \lambda_2 \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
x_{3,1} & \leq & x_{3,2} & \leq & \lambda_3 \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
x_{4,1} & \leq & x_{4,2} & \leq & x_{4,3} & \leq & \lambda_4 \\
\vdots & & \vdots & & \vdots & & \ddots \\
x_{n,1} & \leq & x_{n,2} & \leq & \cdots & \leq & x_{n,n-1} & \leq & \lambda_n
\end{array}
$$

Figure: Inequality constraints of GT polytopes.

Main Results

Theorem (Diameter)

$diam(GT_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

Theorem ($m = 2$ Automorphism Group)

Suppose $\lambda = (1^{a_1}, 2^{a_2})$ *and* $a_1, a_2 \geq 2$. *Then*

$$Aut(GT_\lambda) = D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\delta_{a_1, a_2 \neq 2}}.$$

Theorem ($m \geq 3$ Automorphism Group)

Suppose $\lambda = 1^{a_1} \dots m^{a_m}$ *and* $m \geq 3$. *Let* $t = 1$ *if* λ *is reverse symmetric and let* $t = 0$ *otherwise. Let* j *be the number of pairs* $a_k, a_{k+1} \geq 2$. *Then*

$$Aut(GT_\lambda) \cong \mathbb{Z}_2^t \ltimes_\varphi (S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{j+1})$$

Ladder Diagrams

Definition (Ladder Diagrams)

For $\lambda = (1^{a_1}, \ldots, m^{a_m})$, the grid Γ_λ is an induced subgraph of Q constructed as follows. Let the **origin** be the vertex $(0, 0)$. Set $s_j := \sum_{i=1}^{j} a_i$, and define **terminal vertices** $t_j = (s_j, n - s_j)$ for $0 \leq j \leq m$. Γ_λ consists of all vertices and edges appearing on any North-East path between the origin and a terminal vertex.
A **ladder diagram** is a subgraph of Γ_λ such that

1. the origin is connected to every terminal vertex by some North-East path.
2. every edge in the graph is on a North-East path from the origin to some terminal vertex.

Face Posets

Theorem (ACK)

Let $\mathcal{F}(\Gamma_\lambda)$ denote the poset of ladder diagrams induced by λ ordered by inclusion. Then $\mathcal{F}(GT_\lambda) \cong \mathcal{F}(\Gamma_\lambda)$.

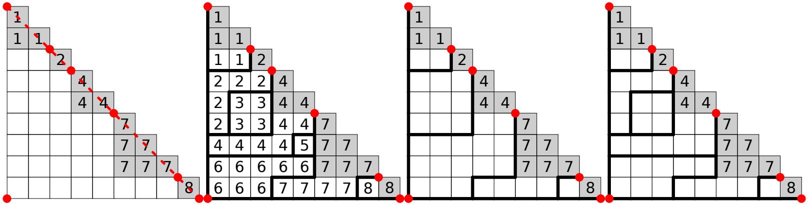


Figure: Let $\lambda = (1^2, 2^1, 4^2, 7^3, 8^1)$. From left to right: Γ_λ with origin and terminal vertices in red and a dashed line indicating the main diagonal, ladder diagram for a point in GT_λ, ladder diagram for a 0-dimensional face (vertex), and ladder diagram for a 2-dimensional face.

Diameter Theorem

By the previous Theorem, it suffices to consider $\lambda = (1^{a_1}, \ldots, m^{a_m})$.

Our proofs will use ladder diagrams to model faces of GT_λ.

Theorem (Diameter)

$diam(GT_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

Diameter Upper Bound

Lemma

Any two vertices v and w of GT_λ are separated by at most $2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$ edges.

As ladder diagrams, a vertex is a set of $m - 1$ noncrossing paths.

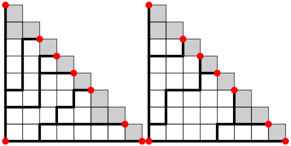
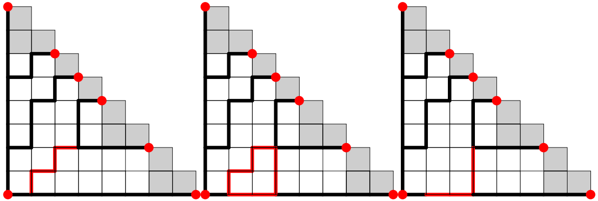


Figure: Vertices v and w.

For each terminal vertex t_i, there is a path $v_i \in v$ and a path $w_i \in w$. We want to change each v_i to w_i by traveling along edges.

Diameter Lower Bound: Phase 1

Traveling along an edge corresponds to moving a **subpath** of the diagram. We call this a **move**.



Formally, two vertices are adjacent iff the union of two vertices is (the ladder diagram of) an edge.

Diameter Lower Bound: Phase 1

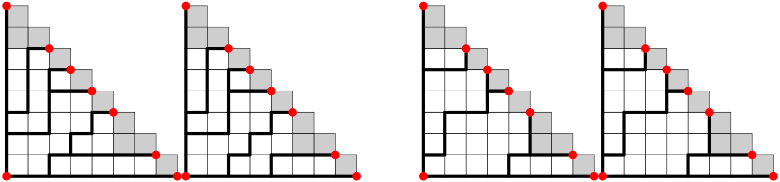


Figure: Phase 1 of the algorithm. $v \to v'$, $w \to w'(= w)$.

Diameter Lower Bound: Phase 2

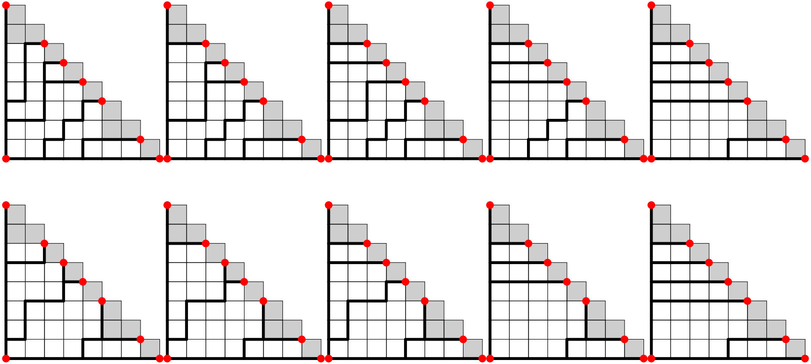


Figure: Phase 2 of the algorithm. First line: $v' \to u$. Second line: $w' \to u$.

Diameter Lower Bound

Lemma

There exist two vertices separated by $\geq 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$ *edges.*

We construct the vertices z_h and z_v that have this separation.

Definition (Zigzag lattice path)

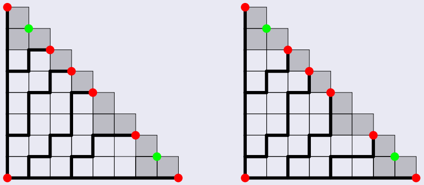


Figure: Vertices z_h and z_v of GT_λ.

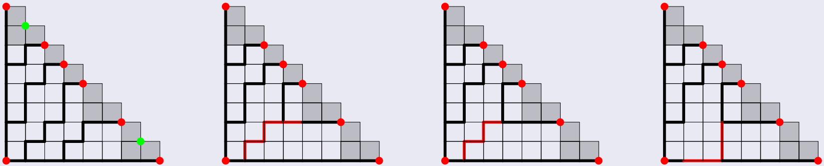
Diameter Lower Bound

Lemma

There exist two vertices separated by $\geq 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$ *edges.*

Proof outline.

One would like to argue that each path of z_h requires two **moves** to be changed into the corresponding path of z_v. But a single move can alter multiple paths. To do this, paths must be merged together first.



We create sets to account for the merges that occur before altering ≥ 2 paths simultaneously.

Diameter Lower Bound

Proof outline cont.

For any sequence of ℓ edges (moves) between z_h and z_v, we can associate sets $X_1, \ldots, X_\ell$ where X_i is the set of indices of paths altered by the ith move.

Claim: $X_1, \ldots, X_\ell$ satisfies the following conditions:

1. Any index (except possibly 1 and $m-1$) appears in at least two sets.
2. The last set one index appears cannot be the last set another index appears in.
3. If $X_k = \{i, i+1, \ldots, j\}$, then at least $j-i$ of $i, i+1, \ldots, j$ appear in sets before X_k.
4. If $X_k = \{i, i+1, \ldots, j\}$ and is the last set an index appears in, then each of $i, i+1, \ldots, j$ appears in sets before X_k.

Diameter Lower Bound

Proof outline cont.

1. Any index (except possibly 1 and $m-1$) appears in at least two sets.
2. The last set one index appears cannot be the last set another index appears in.
3. If $X_k = \{i, i+1, \ldots, j\}$, then at least $j-i$ of $i, i+1, \ldots, j$ appear in sets before X_k.
4. If $X_k = \{i, i+1, \ldots, j\}$ and is the last set an index appears in, then each of $i, i+1, \ldots, j$ appears in sets before X_k.

Claim: Any sequence of sets satisfying these conditions has length $\geq 2m-2-\delta_{1,a_1}-\delta_{1,a_m}$.

Idea: Starting at the end of the sequence $X_1, \ldots, X_\ell$, we replace any tuples by singletons. After each replacement, the sequence still satisfy these conditions. At the end, we are left with $\geq 2m-2-\delta_{1,a_1}-\delta_{1,a_m}$ singletons. □

Proof of Diameter

Theorem (Diameter)

$diam(GT_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

Proof.

Combine the upper and lower bounds in the previous two lemmas. □

Definition (The Corner Symmetry)

For any λ, there is a $\mathbb{Z}_2$ automorphism μ on $\mathcal{F}(\Gamma_\lambda)$ given by swapping two pairs of edges $((0,0),(1,0))$ with $((0,0),(0,1))$ and $((1,0),(1,1))$ with $((0,1),(1,1))$ in any positive path leaving $(0,0)$

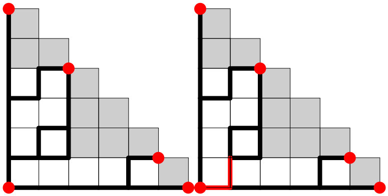


Figure: Action of μ

Generators

Definition (The k-Corner Symmetry)

Denote the k^{th} terminal vertex by $(n-i, i)$, and suppose that $a_k, a_{k+1} \geq 2$. There is a $\mathbb{Z}_2$ automorphism μ_k on $\mathcal{F}(\Gamma_\lambda)$ given by swapping two pairs of edges, $((n-i,i)(n-i,i-1))$ with $((n-i,i)(i-1,i)$ and $((n-i,i-1),(n-i-1,i-1))$ with $((n-i-1,i),(n-i-1,i-1))$ in any positive path going to $(n-i,i)$.

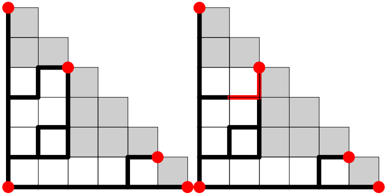


Figure: Action of μ_k

Generators

Definition (Symmetric Group Symmetry)

Suppose that $a_1 = 1$. Then there is a S_{a_2} automorphism group acting on $\mathcal{F}(\Gamma_\lambda)$ in the following way. Take the first column of possible horizontal edges, and label the top a_2 edges 1 though a_2. S_{a_2} then acts by if $\sigma(i) = j$, the edges corresponding to i are mapped to edges corresponding to j.

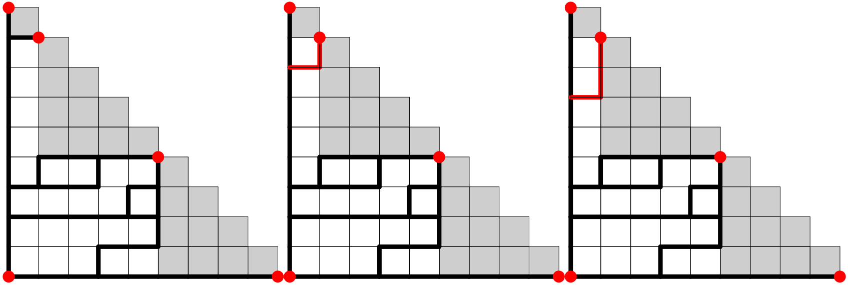


Figure: Action of (123)

Definition (The Flip Symmetry)

Suppose that $\lambda = (1^{a_1}, 2^{a_2}, \dots, m^{a_m}) = (1^{a_m}, 2^{a_{m-1}}, \dots, m^{a_1}) =: \lambda'$. There is a $\mathbb{Z}_2$ automorphism ρ on $\mathcal{F}(\Gamma_\lambda)$ given by reflecting a subgraph over the line $y = x$.

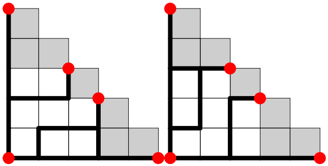


Figure: Action of ρ.

Generators

Definition (The $m = 2$ Rotation Symmetry)

Suppose that $m = 2$. Note that any ladder diagram only has 3 terminal vertices, two on the the x or y axis and one not on the axes, call it v. There is a $\mathbb{Z}_2$ automorphism τ on $\mathcal{F}(\Gamma_\lambda)$ taking paths from $(0, 0)$ to v and rotating them 180° so that they are paths from v to $(0, 0)$.

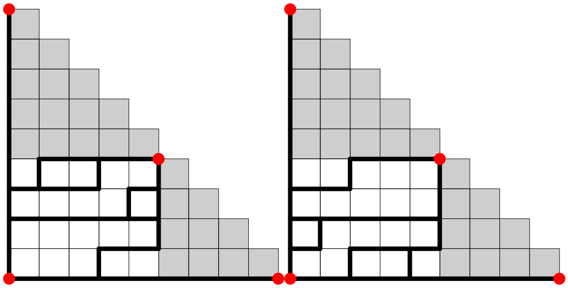


Figure: Action of τ

Generators

Definition (The $m = 2$ Vertex Symmetry)

When $m = 2$, there are two special vertices that are connected to every vertex. This symmetry α maps these two vertices to each other.

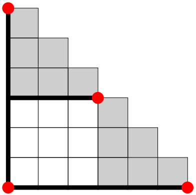


Figure: Vertices acted on by α

Classifying Automorphism Groups

Theorem ($m = 2$ Automorphism Group)

Suppose $\lambda = (1^{a_1}, 2^{a_2})$ *and* $a_1, a_2 \geq 2$.
If $a_1 = a_2 = 2$, *then*

$$Aut(GT_\lambda) \cong D_4 \times \mathbb{Z}_2.$$

Otherwise,

$$Aut(GT_\lambda) \cong D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\delta_{a_1,a_2}}.$$

Theorem ($m \geq 3$ Automorphism Group)

Suppose $\lambda = 1^{a_1} \dots m^{a_m}$ *and* $m \geq 3$. *Let* $t = 1$ *if* $\lambda = \lambda'$ *and let* $t = 0$ *otherwise. Let* j *be the number of pairs* $a_k, a_{k+1} \geq 2$. *Then*

$$Aut(GT_\lambda) \cong \mathbb{Z}_2^t \ltimes_\varphi (S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{j+1})$$

Representing Facets

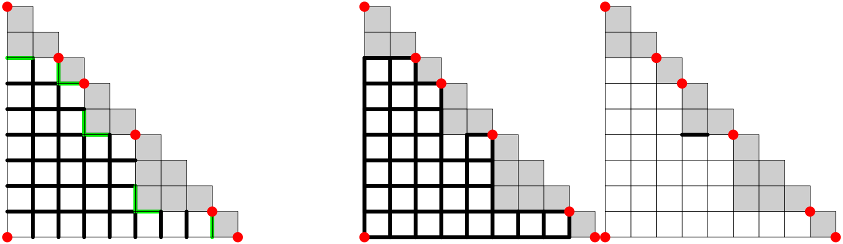


Figure: Left: interior edges of Γ_λ. Right: representing a facet.

Facets of GT_λ are in bijection with interior edges of Γ_λ.
We will denote a facet by its corresponding interior edge.

Dependent Facets

Two facets are called **dependent** if their intersection is a $d-3$ dimensional face. This occurs iff they are arranged in one of two ways.

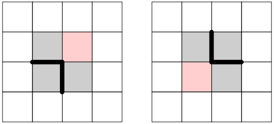
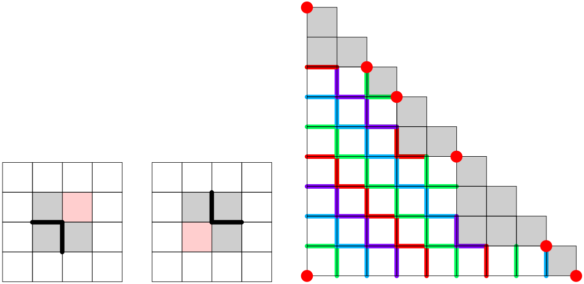


Figure: The gray boxes indicate entries $x_{i,j}$ that are equal on each facet. The red box indicates the entry forced to be equal to the other three.

Facet Chains

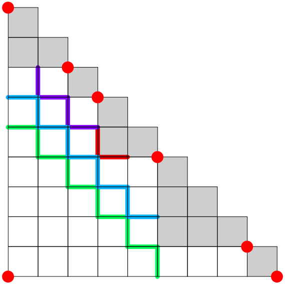
We can form maximal chains of dependent facets. These chains partition the interior edges of Γ_λ.
There is always a unique longest chain.



Adjacent Chains

Chains C_1, C_2 are **adjacent** if the intersection of two facets of C_1 equals the intersection of two facets of C_2.

This occurs iff one chain sits directly to the North-East of the other chain.



Proof of Automorphism Group

Theorem ($m \geq 3$ Automorphism Group)

Suppose $\lambda = 1^{a_1} \dots m^{a_m}$ and $m \geq 3$. Let $t = 1$ if $\lambda = \lambda'$ and let $t = 0$ otherwise. Let j be the number of pairs $a_k, a_{k+1} \geq 2$. Then

$$Aut(GT_\lambda) \cong \mathbb{Z}_2^t \ltimes_\varphi (S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{j+1})$$

Idea of proof:

We know $\mathbb{Z}_2^t \ltimes_\varphi (S_{a_2}^{\delta_{1,a_1}} \times S_{a_{m-1}}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{j+1}) \subseteq \mathrm{Aut}(\mathrm{GT}_\lambda)$.

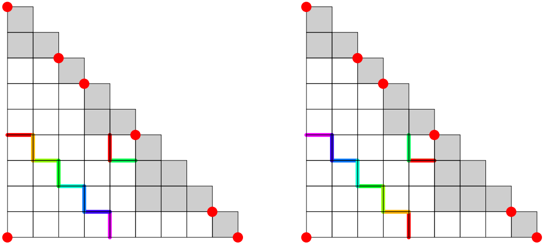
Fact: Any $\phi \in \mathrm{Aut}(\mathrm{GT}_\lambda)$ is determined by where it sends the facets of GT_λ.

We upperbound the size of $\mathrm{Aut}(\mathrm{GT}_\lambda)$ by looking at the action of any $\phi \in \mathrm{Aut}(\mathrm{GT}_\lambda)$ on facets and applying the Orbit-Stabilizer theorem. This suffices to show equality.

Proof of Automorphism Group

Any $\phi \in \mathrm{Aut}(\mathrm{GT}_\lambda)$ must preserve many of the properties we've described. Useful facts:

- ϕ preserves dependency of facets. If $\phi(C_1) = C_2$, then C_1 is mapped to C_2 or the **flip** of C_2.



- ϕ preserves the lengths of chains.
- ϕ preserves adjacency of chains.

Proof of Automorphism Group

Useful facts:

- If $\phi(C_1) = C_2$, then C_1 is mapped to C_2 or the **flip** of C_2.
- ϕ preserves the lengths of chains.
- ϕ preserves adjacency of chains.

Proof outline.

First fix the facets in chains of length ≤ 2 and the facets in C_{long}. This is sufficient to fix the image of *every facet*.

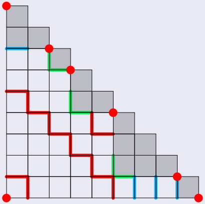


Figure: Flipping short red chains accounts for $\mu, \mu_1, \ldots, \mu_{m-1}$. Permuting blue chains accounts for $\sigma \in S_{a_2}, S_{a_{m-1}}$.

Proof of Automorphism Group

We show this determines the image of *every facet*.

Proof outline.

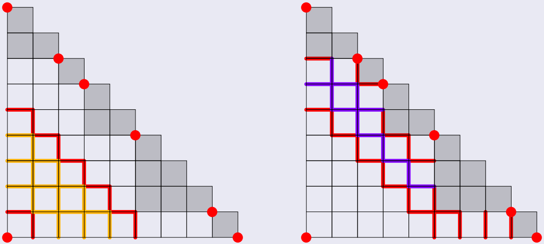


Figure: Arguing towards C_{long}.

□

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