(1)

2017 REU Day 1

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Symmetric functions

$x_1, \ldots, x_m$ are variables (sometimes we allow ∞ many: $x_1, x_2, \ldots$)

Symmetric functions are polynomials fixed by permuting variables

EXAMPLE $x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2 + 2x_1x_2x_3$

THM: They form a ring (under usual + and $\times$), called Λ

DEF'N: A partition λ of n is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ with $\lambda_i \in \mathbb{Z}_{>0}$. Then $n = |\lambda|$ and we write "$\lambda \vdash n$"

EXAMPLE $(4,2,1) = \lambda$ has $\lambda \vdash 7 = |\lambda|$

Define the monomial symmetric function

$$m_\lambda := \sum x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

over all ordered subsets of the variables

EXAMPLE: $m_{(1,1)} = x_1x_2 + x_1x_3 + \ldots$

$m_{(2)} = x_1^2 + x_2^2 + \ldots$

$m_{(2,1)} + 2m_{(1,1,1)}$ is the symm. fn. in the earlier example

THM: Λ in ∞ many variables has $\{m_\lambda\}_{\lambda \text{ all partitions}}$ as a basis

$\Lambda(x_1, \ldots, x_m)$ in variables $x_1, \ldots, x_m$ has $\{m_\lambda\}_{\lambda \text{ with } \leq m \text{ parts}}$ as a basis.

Define the elementary symmetric functions

$$e_k := m_{\underbrace{(1,1,\ldots,1)}_{k \text{ portions}}} \quad \text{for } k = 1, 2, \ldots$$

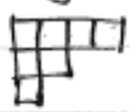
THM: The $\{e_k\}_{k \geq 1}$ generate Λ as an algebra

EXAMPLE: $m_{(2,1)} = e_1e_2 - 3e_3$

REU EXERCISE 1 : Learn about h_k, the <u>complete</u> <u>homogeneous</u> <u>symmetric functions</u>.

Schur functions

Partitions λ $\longleftrightarrow$ Young diagrams

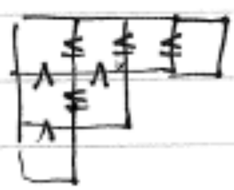
e.g. $\lambda = (4,2,1) \longmapsto$  $\longleftarrow$ λ_i boxes in row i (from the top)

$= (\lambda_1, \lambda_2, \lambda_3)$

A semistandard filling (Young tableau) of λ is a filling T of the boxes of its Young diagram, weakly increasing in rows, strictly increasing in columns, with elements of $\mathbb{Z}_{>0}$.

EXAMPLE

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = T$$



$$\mathrm{weight}(T) := \prod_{i \geq 1} x_i^{\#i\text{'s in } T}$$

e.g. T above has $\mathrm{weight}(T) = x_1^2 x_2^2 x_3 x_4^2$

DEF'N: The Schur function $\displaystyle s_\lambda := \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda}} \mathrm{weight}(T)$

semistandard Young tableaux

EXAMPLE: $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \ldots \left(= m_{(2,1)} + 2m_{(1,1,1)}\right)$

$T = \begin{smallmatrix}1&1\\2\end{smallmatrix} \quad \begin{smallmatrix}1&2\\2\end{smallmatrix} \quad \begin{smallmatrix}1&2\\3\end{smallmatrix} \; \begin{smallmatrix}1&3\\2\end{smallmatrix}$

EXAMPLE: $s_{\underbrace{(1,1,\ldots,1)}_{k}} = e_k = m_{(1,1,\ldots,1)}$

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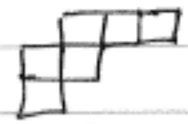
REU EXERCISE 2: Prove that
(a) the $\{s_\lambda\}$ are symmetric functions, and
(b) they form a linear basis for the ring Λ of symmetric fns.

A *skew shape* or *skew Young diagram* λ/μ for $\mu \subseteq \lambda$ is this:

EXAMPLE: $(4,2,1)/(1) =$



The *skew Schur function* $$s_{\lambda/\mu} := \sum_{\substack{\text{SS skew tableaux } T \\ \text{of shape } \lambda/\mu}} \text{weight}(T)$$

(In doing REU EXERCISE 2(a), actually prove that the $s_{\lambda/\mu}$ are also symmetric functions)

There are ~~too~~ many skew shapes λ/μ for the $s_{\lambda/\mu}$ to be linearly independent, or a basis for Λ.

One can expand them uniquely as ...

THM: If $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu$ then $c^\lambda_{\mu,\nu} \in \mathbb{Z}_{\geq 0}$

$c^\lambda_{\mu,\nu}$:= Littlewood-Richardson coefficient

(It has many interpretations & meanings, e.g. in representation theory of symmetric groups, general linear groups, and in cohomology of Grassmannians ...)

EXAMPLE: $s_{(2,1)/(1)} = s_{\square\square} = (s_\square)^2 = s_{\substack{\square\\\square}} + s_{\square\square}$
$= s_{(1,1)} + s_{(2)}$

EXAMPLE: $S_{(2,2)/(1)} = S_{(2,1)}$

i.e. 

References on equalities among skew Schur functions: (in ∞ many variables)

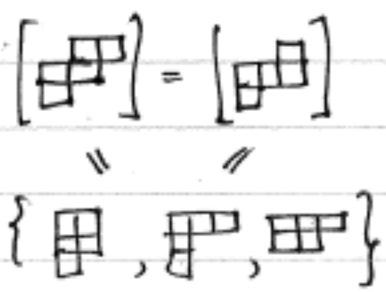
- Reiner, Shaw, and van Willigenburg
- McNamara and van Willigenburg

Two variations

(a) Define the (Schur function) support of $s_{\lambda/\mu}$ as

$$[s_{\lambda/\mu}] := \{\nu : c^{\lambda}_{\mu,\nu} > 0\}.$$

REU PROBLEM 1(a)
When is $[S_{\lambda/\mu}] = [S_{\rho/\delta}]$?

EXAMPLE: 

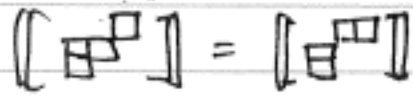
(b) Since the $\{m_\lambda\}$ are a basis for Λ, one can also uniquely express

$$s_{\lambda/\mu} = \sum_{\nu} K^{\lambda}_{\mu,\nu}\, m_\nu$$

EXAMPLE: $S_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$

Define the (monomial) support $[[S_{\lambda/\mu}]] := \{\nu : K^{\lambda}_{\mu,\nu} > 0\}$

REU PROBLEM 1(b) When is $[\![s_{\lambda/\mu}]\!] = [\![s_{\rho/\delta}]\!]$?

EXAMPLE: 

(This turns out to be the same question as when the tropicalizations of $s_{\lambda/\mu}$ and $s_{\rho/\delta}$ are the same)

People have asked "When is $c^{\lambda}_{\mu,\nu} > 0$?"

A: The Horn-Klyachko inequalities on (μ,ν,λ) characterize it.

Reference: Dobrovolska - Pylyavskyy

People have also asked "When is $K^{\lambda}_{\mu,\nu} > 0$?"

(equivalent to characterizing the Newton polytope of $s_{\lambda/\mu}$)

A: See the recent preprint by Monical-Tokcan-Yong.

REU EXERCISE 8:

Determine which skew shapes with 4 cells have equal support.
Do the same for 5 cells.
Can you do it for 6 cells?