REU 2017 Day 8 P. Pylyavskyy
Correlations among patterns in permutations

w a permutation of $1,2,\ldots,n$

EXAMPLE

$w=(5,2,1,3,6,4)\in S_6$

Permutation w contains permutation u as a pattern if restricted to some subset of letters, w becomes ordered the same way as u

EXAMPLE $(5,2,1,3,6,4)$ contains $(3,1,2)$, but does not contain $(1,2,3,4)$.

REU EXERCISE 17

Prove the number of w in S_n that

- avoid $(3,2,1)$ is Catalan # $C_n = \frac{1}{n+1}\binom{2n}{n}$
- avoid $(3,1,2)$ is also C_n
- avoid both $(3,1,2)$ and $(3,2,1)$ is 2^{n-1}.

QUESTION: Pick $v \in S_n$ for n large, uniformly at random. For a given pair of (small) permutations w and u, are the events of avoiding w and u positively or negatively correlated?

EXAMPLE $w=(3,2,1)$
vs. $u=(3,1,2)$

$$P\big(\text{avoid}(3,2,1) \text{ (in } S_n)\big) \ \overset{<}{\underset{>}{}} \ ?? \ \ P\big(\text{avoid}(3,2,1) \,\big|\, \text{avoid}(3,1,2)\big)$$

$$\frac{\left(\dfrac{(2n)!}{n!\,(n+1)!}\right)}{n!} \qquad\qquad \frac{2^{n-1}}{\left(\dfrac{2n!}{n!\,(n+1)!}\right)}$$

$n=3$ $\quad \frac{5}{6} > \frac{4}{5}$ is misleading!

REU EXERCISE 18: Show that for n large, the left side is smaller, so they are positively correlated.

EXAMPLE: $(3,2,1)$ and $(1,2,3)$ are negatively correlated.

Eventually, $P(\text{avoiding both}) = 0$.

REU EXERCISE 19: Show that $(1,2,\ldots,n)$ and $(n,\ldots,3,2,1)$ are negatively correlated.

EXAMPLE: $(4,3,2,1)$, $(3,2,1)$ are positively correlated.

In fact, Joel Lewis explained (see his attached PDF note on website) why this always occurs when w, u are not as in EXERCISE 19.

On the other hand, this question still makes sense . . .

REU PROBLEM 8

Fix 3 permutations w, u, v and consider only permutations in S_n (for n large) that avoid v.

How can one decide whether the events

$$A = \{ t \in S_n \text{ avoids } w \mid t \text{ avoids } v \}$$

$$B = \{ t \in S_n \text{ avoids } u \mid t \text{ avoids } v \}$$

are positively or negatively correlated?