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On the critical group of the n-cube

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Abstract

Reiner proposed two conjectures about the structure of the critical group of the n-cube Q_n. In this paper we confirm them. Furthermore we describe its p-primary structure for all odd primes p. The results are generalized to Cartesian product of complete graphs $K_{n_1} \times \cdots \times K_{n_k}$ by Jacobson, Niedermaier and Reiner.

Keywords: n-Cube; Critical group; Sandpile group; Laplacian matrix; Smith normal form; Sylow p-group

1. Introduction

The *n-cube* or *hypercube* Q_n is the simple graph whose vertices are the n-tuples with entries in {0, 1} and whose edges are the pairs of n-tuples that differs in exactly one position.

Let $G = (V, E)$ be a finite graph without self-loops, but, with multiple edges allowed and let $n = |V|$. The $n \times n$ *Laplacian matrix* $L(G)$ for this graph G, is defined by

$$L(G)_{u,v} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{u,v} & \text{otherwise,} \end{cases}$$

where $m_{u,v}$ denotes the multiplicity of the edge $\{u, v\}$ in E. The Laplacian matrix of a graph, which dates back to Kirchhoff's matrix-tree theorem, plays an important role in the study of spanning trees, graph spectra, and the graph isomorphism problem, see [2,10,15].

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When G is connected, the kernel of $L(G)$ is spanned by the all-1 vector $(1, 1, \ldots, 1)^{\mathrm{t}}$ in $\mathbb{R}^n$, where superscript t denotes the transpose. Considering $L(G)$ as a linear map from $\mathbb{Z}^n$ to itself, its cokernel has the form

$$\mathbb{Z}^n/\mathrm{Im}\, L(G) \cong \mathbb{Z} \oplus K(G),$$

where $K(G)$ is defined to be the *critical group* (also called the *Picard group* [1], *Jacobian group* [3] or *sandpile group* [6]). It follows from Kirchhoff's matrix-tree theorem that the order $|K(G)|$ is known to be $\mathscr{K}(G)$, the number of spanning trees in G.

Kirchhoff's Matrix-Tree Theorem (see, e.g. [2, Chapter 6]).

(i)

$$\mathscr{K}(G) = (-1)^{i+j} \det \overline{L(G)}$$

here $\overline{L(G)}$ *is a reduced Laplacian matrix obtained from* $L(G)$ *by deleting row* i *and column* j.

(ii) *For any graph* G *with* n *vertices, index the eigenvalues of the Laplacian* $L(G)$ *in weakly decreasing order*:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n = 0$$

then

$$\mathscr{K}(G) = \frac{\lambda_1\lambda_2 \cdots \lambda_{n-1}}{n}.$$

Our main tools will be the use of Smith Normal Form for integer matrices. Given a square integer matrix A, its *Smith normal form* is the unique *equivalent* diagonal matrix $S(A) = \mathrm{diag}[s_1, s_2, \ldots, s_n]$ whose entries s_i are nonnegative and s_i divides s_{i+1}. The s_i are known as the *invariant factors* of A [16]. Two integral matrices A and B are *equivalent* (denoted as $A \sim B$) if there exist integer matrices P and Q of determinant ± 1 that satisfy $PAQ = B$.

The structure of the critical group is closely related with the Laplacian matrix: if the Smith normal form of $L(G)$ is $\mathrm{diag}[s_1, s_2, \ldots, s_n]$, then $K(G)$ is the torsion subgroup of $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_n}$. Here $\mathbb{Z}_n$ denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ for any nonnegative integer n.

There are very few families of graphs for which the critical group structure has been completely determined, such as complete graphs and complete bipartite graphs [13], cycles [14], wheels [4], generic threshold graphs [5], etc.

Reiner [17] proposed two conjectures on the structure of the critical group of n-cube $K(Q_n)$. His first conjecture is the following theorem, which we prove in Section 2.

Theorem 1.1. *The critical group of* Q_n *has exactly* $2^{n-1} - 1$ *invariant factors, for* $n \geqslant 1$.

We show the structure of the Sylow p-group of the critical group of the n-cube when p is an odd prime.

Theorem 1.2. *For any odd prime number p, the Sylow p-group of the critical group of the n-cube $\mathrm{Syl}_p K(Q_n)$ has the following expression*:

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p\left(\prod_{k=1}^{n} \mathbb{Z}_k^{\binom{n}{k}}\right).$$

Recently Jacobson, Niedermaier and Reiner generalize this result to the Cartesian product of complete graphs $K_{n_1} \times \cdots \times K_{n_k}$.

Jacobson–Niedermaier–Reiner Theorem [12, Theorem 2]. *For every prime p that divides none of $n_1, \ldots, n_k$, the Sylow p-subgroup (or p-primary component) of the critical group $K(K_{n_1} \times \cdots \times K_{n_k})$ has the following description*:

$$\mathrm{Syl}_p K(K_{n_1} \times \cdots \times K_{n_k}) \cong \bigoplus_{\emptyset \neq S \subseteq [k]} \mathrm{Syl}_p(\mathbb{Z}_{N_S})^{\prod_{i\in S}(n_i-1)},$$

where $N_S := \sum_{i\in S} n_i$.

One knows from simple eigenvalue calculations (see [12, Section 2]) that

$$\mathscr{K}(K_{n_1} \times \cdots \times K_{n_k}) = \frac{1}{\prod_{i=1}^{k} n_i} \prod_{\emptyset \neq S \subseteq [k]} N_S^{\prod_{i\in S}(n_i-1)}.$$

Note that this general result on p-primary structures (for p that divides none of $n_1, \ldots, n_k$) is as simple as one could hope.

Our next result is related to the 2-primary structure of $K(Q_n)$.

Let a_n denotes the number of occurrences of $\mathbb{Z}_2$ in the elementary divisor form of the critical group of the n-cube $K(Q_n)$. Reiner's second conjecture is confirmed in Section 4.

Theorem 1.3. *The generating function of a_n is*

$$\sum_{n=0}^{\infty} a_{n+3}x^n = \frac{1}{(1-2x)(1-2x^2)}.$$

This leads to a simple expression for a_n ($a_0 = a_1 = 0$):

$$a_n = 2^{n-2} - 2^{\lfloor (n-2)/2 \rfloor} \quad \text{when } n \geqslant 2.$$

For the background of the *elementary divisor form of a finitely generated abelian group* (see [8, p. 163]).

The full structure of the Sylow-2 subgroup of the critical group of the n-cube is still unknown.

2. Some lemmas and the proof of Theorem 1.1

Let $L_n = L(Q_n)$. Clearly $L_0 = 0$ and since $Q_n = Q_{n-1} \times Q_1$ for $n \geqslant 1$, one can order the vertices of Q_n such that

$$L_n = \begin{pmatrix} L_{n-1}+1 & -1 \\ -1 & L_{n-1}+1 \end{pmatrix}.$$

The size of L_n is 2^n, which is the number of vertices of Q_n.

Throughout this paper let $L_{n,k} := L_n + 2k$, n and k be nonnegative integers, where $2k$ denotes the scalar matrix $2kI$ with I an identity matrix of same size as L_n. In general, if A is a matrix and c a constant, $A + c$ denotes $A + cI$. If A and B are two square matrices, we also make following convention:

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad A^{\oplus k} := \overbrace{A \oplus A \oplus \cdots \oplus A}^{k \text{ times}}.$$

Let $M_n(R)$ denote the set of $n \times n$ matrices with entries in the ring R. If the size of the matrices does not matter or is clear, sometimes we simply write $M_n(R)$ as $M(R)$.

Lemma 2.1

$$\frac{1}{m!}\prod_{i=0}^{m-1} L_{n,k+i} \in M(\mathbb{Z})$$

for any $m \in \mathbb{N}$. In particular, $L_n(L_n+2) \in M(2\mathbb{Z})$.

Proof. Let

$$T_n := \begin{pmatrix} I_{2^{n-1}} & -I_{2^{n-1}} \\ -I_{2^{n-1}} & I_{2^{n-1}} \end{pmatrix}, \quad \tilde{L}_{n-1,k} := L_{n-1,k} \oplus L_{n-1,k}.$$

Then $\tilde{L}_{n-1,k}$ commutes with T_n, and $(\frac{1}{2}T_n)^2 = \frac{1}{2}T_n$. Hence

$$\begin{aligned} \left(\frac{1}{2}L_{n,k}\right)^j &= \left(\frac{1}{2}(\tilde{L}_{n-1,k} + T_n)\right)^j \\ &= \left(\frac{1}{2}\tilde{L}_{n-1,k}\right)^j + \frac{1}{2}T_n \sum_{l=0}^{j-1}\binom{j}{l}\left(\frac{1}{2}\tilde{L}_{n-1,k}\right)^l \\ &= \left(\frac{1}{2}\tilde{L}_{n-1,k}\right)^j + \frac{1}{2}T_n\left(\left(1+\frac{1}{2}\tilde{L}_{n-1,k}\right)^j - \left(\frac{1}{2}\tilde{L}_{n-1,k}\right)^j\right). \end{aligned}$$

Note that

$$\prod_{i=0}^{m-1}(x+i) = \sum_{j=1}^{m} c(m, j)x^j$$

where $c(m, j)$ is the signless Stirling number of the first kind,

$$\begin{aligned}\prod_{i=0}^{m-1} L_{n,k+i} &= \sum_{j=1}^{m} 2^m c(m, j) \left(\frac{1}{2} L_{n,k}\right)^j \\ &= \prod_{i=0}^{m-1} \tilde{L}_{n-1,k+i} + \frac{1}{2} T_n \left(\prod_{i=0}^{m-1} (\tilde{L}_{n-1,k+i} + 2) - \prod_{i=0}^{m-1} \tilde{L}_{n-1,k+i}\right) \\ &= \prod_{i=0}^{m-1} \tilde{L}_{n-1,k+i} + m T_n \prod_{i=1}^{m-1} \tilde{L}_{n-1,k+i}.\end{aligned}$$

Now by induction on $n + m$ (the lemma is trivial when $n = 0$ since $L_0 = 0$),

$$\frac{1}{m!} \prod_{i=0}^{m-1} \tilde{L}_{n-1,k+i}$$

is in $M(\mathbb{Z})$, and so is

$$\frac{1}{m!} \left(m T_n \prod_{i=1}^{m-1} \tilde{L}_{n-1,k+i}\right) = T_n \left(\frac{1}{(m-1)!} \prod_{i=0}^{(m-1)-1} \tilde{L}_{n-1,(k+1)+i}\right).$$

So the lemma is true for all n, m. □

Lemma 2.2

$$L_{n+1,k} \sim I_{2^n} \oplus L_{n,k} L_{n,k+1}.$$

Consequently, we have a group isomorphism

$$\mathbb{Z}^{2^{n+1}} \Big/ \operatorname{Im} L_{n+1,k} \cong \mathbb{Z}^{2^n} \Big/ \operatorname{Im} L_{n,k} L_{n,k+1}.$$

Proof

$$\begin{aligned}L_{n+1,k} &= \begin{pmatrix} 1 + L_{n,k} & -1 \\ -1 & 1 + L_{n,k} \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & -1 \\ L_{n,k}(L_{n,k} + 2) & 1 + L_{n,k} \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 \\ 0 & L_{n,k} L_{n,k+1} \end{pmatrix}. \qquad \square\end{aligned}$$

Corollary 2.3. *The Smith normal form of L_n has exactly 2^{n-1} occurrences of* 1, *for all $n \geqslant 1$.*

Proof. Let $k = 0$ and replace n by $n - 1$ in the above lemma. Note that there is no occurrence of 1 in the Smith normal form of $L_{n-1}(L_{n-1} + 2)$, which is in $M_{2^{n-1}}(2\mathbb{Z})$. Hence it concludes. □

This statement is equivalent to (cf. Fiol [9]) the following theorem, which is Reiner's first conjecture on the number of invariant factors of $K(Q_n)$:

Theorem 1.1. *The critical group of Q_n has exactly $2^{n-1}-1$ invariant factors.*

3. The Sylow p-group of $K(Q_n)$ for $p \neq 2$

In this section we will determine the structure of the Sylow p-group of the critical group $K(Q_n)$ for all odd primes p.

If A is an $n \times n$ integer matrix, let $K(A)$ be the torsion subgroup of the finitely generated abelian group $\mathbb{Z}^n/\mathrm{Im}\,A$. If G is an abelian group, let $\mathrm{Syl}_p G$ denote the *Sylow p-group* of G.

Proposition 3.1

$$\mathrm{Syl}_p K(L_{n+1,k}) \cong \mathrm{Syl}_p K(L_{n,k}) \times \mathrm{Syl}_p K(L_{n,k+1}).$$

Proof. We have two group homomorphisms ψ and ϕ:

$$K(L_{n,k}L_{n,k+1}) \underset{\phi}{\overset{\psi}{\rightleftharpoons}} K(L_{n,k}) \times K(L_{n,k+1});$$

$$\psi\big(w + \mathrm{Im}\,L_{n,k}L_{n,k+1}\big) := \big(w + \mathrm{Im}\,L_{n,k},\ w + \mathrm{Im}\,L_{n,k+1}\big),$$

$$\phi\big(u + \mathrm{Im}\,L_{n,k},\, v + \mathrm{Im}\,L_{n,k+1}\big) := L_{n,k+1}u - L_{n,k}v + \mathrm{Im}\,L_{n,k}L_{n,k+1}.$$

Let $2|_G$ denote the 'multiplication by 2' homomorphism of G for any abelian group G. It is straightforward to check that

$$\psi \circ \phi = 2\|_{K(L_{n,k})\times K(L_{n,k+1})} \quad \text{and} \quad \phi \circ \psi = 2|_{K(L_{n,k}L_{n,k+1})}.$$

Now consider the restriction of ϕ and ψ to Sylow p-subgroups for p an odd prime. Because $2|_G$ is an isomorphism for any finite abelian p-group G, both ψ and ϕ must be isomorphisms, so

$$\mathrm{Syl}_p K(L_{n+1,k}) \cong \mathrm{Syl}_p K(L_{n,k}) \times \mathrm{Syl}_p K(L_{n,k+1}). \qquad \square$$

Theorem 1.2. *For any odd prime number p, the Sylow p-group of the critical group of the n-cube $\mathrm{Syl}_p K(Q_n)$ has the following expression*:

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p \left(\prod_{k=1}^{n} \mathbb{Z}_k^{\binom{n}{k}} \right).$$

Proof. Let us compute the Sylow p-group of $K(Q_n)$ recursively:

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p K(L_{n-1,0}) \times \mathrm{Syl}_p K(L_{n-1,1})$$

$$
\begin{aligned}
&\cong \mathrm{Syl}_p K(L_{n-2,0}) \times \left(\mathrm{Syl}_p K(L_{n-2,1})\right)^2 \times \mathrm{Syl}_p K(L_{n-2,2}) \\
&\cong \cdots \\
&\cong \prod_{k=0}^{j} \left(\mathrm{Syl}_p K(L_{n-j,k})\right)^{\binom{j}{k}} \\
&\cong \prod_{k=0}^{n} \left(\mathrm{Syl}_p K(L_{0,k})\right)^{\binom{n}{k}} \quad (\text{let } j = n).
\end{aligned}
$$

In the final step, $\mathrm{Syl}_p K(L_{0,k})$ by definition is $\mathrm{Syl}_p(\mathbb{Z}^{2^0}/\mathrm{Im}(L_0 + 2k))$, which is just $\mathrm{Syl}_p \mathbb{Z}_{2k}$, thus we obtain

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p \left(\prod_{k=1}^{n} \mathbb{Z}_k^{\binom{n}{k}}\right). \qquad \square$$

4. Proof of Theorem 1.3

In this section we get a partial decomposition of $\mathrm{Syl}_2 K(Q_n)$, which leads to the proof of Reiner's second conjecture on the occurrences of $\mathbb{Z}_2$ in $\mathrm{Syl}_2 K(Q_n)$.

Lemma 4.1. *If m is an even positive integer, then*

$$\prod_{i=0}^{m-1} L_{n+2,k+i} \sim \left(\prod_{i=1}^{m} L_{n,k+i}\right)^{\oplus 2} \oplus \prod_{i=2}^{m-1} \tilde{L}_{n,k+i} \begin{pmatrix} L_{n,k}L_{n,k+1} & m \\ 0 & L_{n,k+m}L_{n,k+m+1} \end{pmatrix}$$

in $M_{2^{n+2}}(\mathbb{Z}_{2^N})$, where N is any positive integer.

Proof. Recall that by Lemma 2.1 we have:

$$
\begin{aligned}
\prod_{i=0}^{m-1} L_{n,k+i} &= \prod_{i=0}^{m-1} \tilde{L}_{n-1,k+i} + mT_n \prod_{i=1}^{m-1} \tilde{L}_{n-1,k+i} \\
&= \prod_{i=1}^{m-1} \tilde{L}_{n-1,k+i} \begin{pmatrix} L_{n-1,k} + m & -m \\ -m & L_{n-1,k} + m \end{pmatrix}.
\end{aligned}
$$

We will try to do some elementary row and column operations [11] to get desired decomposition of the Smith normal form. These operations consist of: adding an integer multiple of one row (or column) to another; negating a row (or column); interchanging two rows (or columns). Here we will use the theory of the Smith normal form for matrices with entries in a principal idea domain (cf. [7]), in this case it is $\mathbb{Z}_{2^N}$.

$$\prod_{i=0}^{m-1} L_{n+2,k+i} \sim \prod_{i=1}^{m-1} \tilde{L}_{n+1,k+i} \begin{pmatrix} L_{n+1,k} & -m \\ 0 & L_{n+1,k+m} \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix},$$

where

$$\Gamma_{11} = \prod_{i=1}^{m-1} \tilde{L}_{n,k+i} \begin{pmatrix} L_{n,k} + m & -m \\ -m & L_{n,k} + m \end{pmatrix},$$

$$\Gamma_{12} = -m \prod_{i=2}^{m-1} \tilde{L}_{n,k+i} \begin{pmatrix} L_{n,k+1} + (m-1) & -(m-1) \\ -(m-1) & L_{n,k+1} + (m-1) \end{pmatrix},$$

$$\Gamma_{22} = \prod_{i=2}^{m} \tilde{L}_{n,k+i} \begin{pmatrix} L_{n,k+1} + m & -m \\ -m & L_{n,k+1} + m \end{pmatrix}.$$

By the same sort of operations, it is easy to see that

$$\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} \sim \left(\prod_{i=2}^{m-1} L_{n,k+i} \right)^{\oplus 4} \Lambda,$$

where

$$\Lambda = \begin{pmatrix} L_{n,k}L_{n,k+1} & -mL_{n,k+1} & -mL_{n,k+1} & m(m-1) \\ 0 & L_{n,k+1}L_{n,k+m} & 0 & -mL_{n,k+m} \\ 0 & 0 & L_{n,k+1}L_{n,k+m} & -mL_{n,k+m} \\ 0 & 0 & 0 & L_{n,k+m}L_{n,k+m+1} \end{pmatrix}.$$

Since m is even as assumption, there exists an integer r such that $(m-1)r \equiv 1 (\bmod 2^N)$. Similarly, there exists an integer s making $(1-2mr)s \equiv 1 (\bmod 2^N)$, or, equivalently, $1 + 2mrs \equiv s (\bmod 2^N)$. Let P, Q be block matrices in $SL_{2^{n+2}}(\mathbb{Z}_{2^N})$ as

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ rsL_{n,k+m} & 1 + rsm & rsm & 0 \\ 2r^2sL_{n,k+m} & rs & rs & 0 \\ 0 & -L_{n,k+m+1} & -L_{n,k+m+1} & (1-2mr)(m-1) \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -rsL_{n,k} & 1 & 0 & 0 \\ -rsL_{n,k} & -1 & m-1 & 0 \\ -2r^2sL_{n,k}L_{n,k+1} & 0 & L_{n,k+1} & r \end{pmatrix}.$$

Then we can check that in $M(\mathbb{Z}_{2^N})$

$$P\Lambda Q = \begin{pmatrix} L_{n,k}L_{n,k+1} & 0 & 0 & m \\ 0 & L_{n,k+1}L_{n,k+m} & 0 & 0 \\ 0 & 0 & L_{n,k+1}L_{n,k+m} & 0 \\ 0 & 0 & 0 & L_{n,k+m}L_{n,k+m+1} \end{pmatrix}.$$

Thinking of the "coefficient" $(\prod_{i=2}^{m-1} L_{n,k+i})^{\oplus 4}$, we get the desired decomposition of the Smith normal form in the lemma. □

In the special case that $m = 2$, we already know that $\frac{1}{2}L_{n,k}L_{n,k+1}$ is still an integer matrix, so we may continue elementary row/column operations:

$$\begin{aligned}
&\begin{pmatrix} L_{n,k}L_{n,k+1} & 2 \\ 0 & L_{n,k+2}L_{n,k+3} \end{pmatrix} \\
&\quad\sim \begin{pmatrix} 0 & 2 \\ -\frac{1}{2}L_{n,k}L_{n,k+1}L_{n,k+2}L_{n,k+3} & L_{n,k+2}L_{n,k+3} \end{pmatrix} \\
&\quad\sim 2I_{2^n} \oplus \frac{1}{2}\prod_{i=0}^{3} L_{n,k+i}.
\end{aligned}$$

So we have (*only* in $M(\mathbb{Z}_{2^N})$):

$$L_{n+2,k}L_{n+2,k+1} \sim 2I_{2^n} \oplus (L_{n,k+1}L_{n,k+2})^{\oplus 2} \oplus \frac{1}{2}\prod_{i=0}^{3} L_{n,k+i}$$

Knowing that $K(L_{n+1,k}) \cong K(L_{n,k}L_{n,k+1})$ (Lemma 2.2), we can rewrite this as

Corollary 4.2

$$\mathrm{Syl}_2 K(L_{n,k}) \cong \mathbb{Z}_2^{2^{n-3}} \times \left(\mathrm{Syl}_2 K(L_{n-2,k+1})\right)^2 \times \mathrm{Syl}_2 K\left(\frac{1}{2}\prod_{i=0}^{3} L_{n-3,k+i}\right),$$

for any $n \geqslant 3$.

Let $a(n,k)$ be the number of the occurrences of $\mathbb{Z}_2$ in $\mathrm{Syl}_2 K(L_{n,k})$. In particular Reiner's second conjecture concerns of $a_n := a(n,0)$ for $\mathrm{Syl}_2 K(Q_n)$. Note that $\mathbb{Z}_2$ does not show up in $\mathrm{Syl}_2 K(\frac{1}{2}\prod_{i=0}^{3} L_{n,k+i})$ for any n and k, since the matrix $\prod_{i=0}^{3} L_{n,k+i}$ is in $M(24\mathbb{Z})$ by Lemma 2.1.

Now we can confirm Reiner's second conjecture:

Theorem 1.3. *The generating function of a_n is*

$$\sum_{n=0}^{\infty} a_{n+3}x^n = \frac{1}{(1-2x)(1-2x^2)}.$$

Proof. By Corollary 4.2, $a(n,k)$ has recurrence

$$a(n,k) = 2^{n-3} + 2a(n-2,k+1), \quad \forall n \geqslant 3.$$

We evaluate directly the initial conditions:

$$a(0,k) = a(1,k) = a(2,k) = 0, \quad \forall k \geqslant 0.$$

These initial conditions imply that $a(n,k) = a_n$.

Let $f(x)=\sum_{n=0}^{\infty} a_{n+3}x^n$, then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 2a_{n+1}x^n \\ &= \sum_{n=0}^{\infty} (2x)^n + 2\sum_{n=2}^{\infty} a_{n+1}x^n \\ &= \sum_{n=0}^{\infty} (2x)^n + 2x^2\sum_{n=0}^{\infty} a_{n+3}x^n \\ &= \frac{1}{1-2x} + 2x^2 f(x). \end{aligned}$$

Hence we conclude that $f(x)=1/[(1-2x)(1-2x^2)]$. $\square$

Reiner shares with the author the data computed by the Smith normal form program at web site `http://linbox.pc.cis.udel.edu:8080/gap/SmithForm.html`

n	$\mathrm{Syl}_2 K(Q_n)$
2	$\mathbb{Z}_4$
3	$\mathbb{Z}_2\,\mathbb{Z}_8^2$
4	$\mathbb{Z}_2^2\,\mathbb{Z}_8^4\,\mathbb{Z}_{32}$
5	$\mathbb{Z}_2^6\,\mathbb{Z}_8^4\,\mathbb{Z}_{16}\,\mathbb{Z}_{64}^4$
6	$\mathbb{Z}_2^{12}\,\mathbb{Z}_4^4\,\mathbb{Z}_8\,\mathbb{Z}_{32}^4\,\mathbb{Z}_{64}^{10}$
7	$\mathbb{Z}_2^{28}\,\mathbb{Z}_4\,\mathbb{Z}_{16}^8\,\mathbb{Z}_{32}^6\,\mathbb{Z}_{64}^{14}\,\mathbb{Z}_{128}^6$
8	$\mathbb{Z}_2^{56}\,\mathbb{Z}_4^2\,\mathbb{Z}_{16}^{16}\,\mathbb{Z}_{32}^{12}\,\mathbb{Z}_{64}^{28}\,\mathbb{Z}_{128}^{12}\,\mathbb{Z}_{1024}$
9	$\mathbb{Z}_2^{120}\,\mathbb{Z}_4^{10}\,\mathbb{Z}_{16}^{16}\,\mathbb{Z}_{32}^{26}\,\mathbb{Z}_{64}^{48}\,\mathbb{Z}_{128}^{26}\,\mathbb{Z}_{512}\,\mathbb{Z}_{2048}^8$
10	$\mathbb{Z}_2^{240}\,\mathbb{Z}_4^{36}\,\mathbb{Z}_8^{26}\,\mathbb{Z}_{32}^{16}\,\mathbb{Z}_{64}^{148}\,\mathbb{Z}_{256}\,\mathbb{Z}_{1024}^{26}\,\mathbb{Z}_{2048}^{18}$
11	$\mathbb{Z}_2^{496}\,\mathbb{Z}_4^{66}\,\mathbb{Z}_8^{32}\,\mathbb{Z}_{16}^{100}\,\mathbb{Z}_{64}^{164}\,\mathbb{Z}_{128}\,\mathbb{Z}_{512}^{100}\,\mathbb{Z}_{2048}^{64}$

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