Binary matroids & sandpile groups

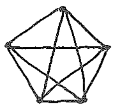
6/12/18

1. Counting trees (ref: Loebl §3.3 or Stanley Ch.9)
2. Sandpile graph
3. Cayley graphs of $\mathbb{F}_2^r$ ← two element field (ref: Stanley Ch.2)
4. REU problem #2
5. Ring theory

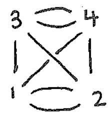
1. Counting trees

$G = (V, E)$ an undirected graph, with no self-loops ()
parallel edges () are fine

EX|| K_n = complete graph on n vertices

K_5 

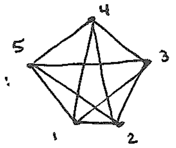
EX||

$G =$ 

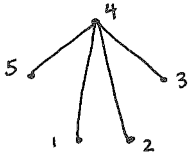
Def: A spanning tree in G is a subset $T \subseteq E$ with no cycles that connects all of V
$\Upsilon(G)$ = # of spanning trees in G

Theorem: (Cayley 1889, Borchardt 1860)
$$\Upsilon(K_n) = n^{n-2}$$

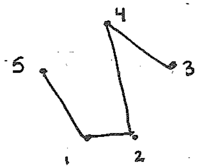
EX|| $\Upsilon(K_5) = 5^{5-2} = 5^3 = 125$

K_5: 

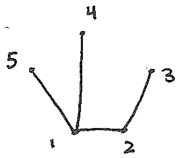
has spanning trees of form



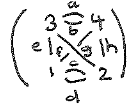
(5 total)



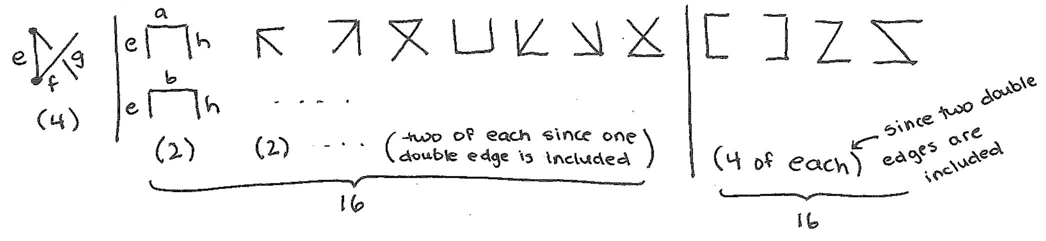
$\left(\frac{5!}{2} = 60 \text{ total}\right)$



$(5 \cdot 4 \cdot 3 = 60 \text{ total})$

EX|| Υ () $= 36$

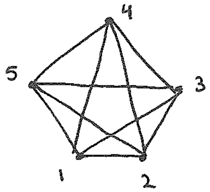
has spanning trees of type

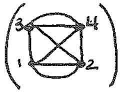


It's easier to find $\Upsilon(G)$ two ways using

Def: $L(G)$ = graph Laplacian matrix $\in \mathbb{Z}^{n \times n}$ $(n = \#V)$
$$L(G)_{ij} = \begin{cases} \deg_G(i) & \text{if } i=j \\ -(\#\text{edges}_{i-j}) & \text{if } i \neq j \end{cases}$$

EX|| $L(K_5) = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 4 & -1 & -1 & -1 & -1 \\ 2 & -1 & 4 & -1 & -1 & -1 \\ 3 & -1 & -1 & 4 & -1 & -1 \\ 4 & -1 & -1 & -1 & 4 & -1 \\ 5 & -1 & -1 & -1 & -1 & 4 \end{array}$



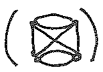
EX|| L () $= \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 4 & -2 & -1 & -1 \\ 2 & -2 & 4 & -1 & -1 \\ 3 & -1 & -1 & 4 & -2 \\ 4 & -1 & -1 & -2 & 4 \end{array}$

$L(G)$ has $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ in its kernel / null space, so it's always singular.

Theorem:

(a) (Kirchoff 1847, Matrix Tree Theorem)
$\Upsilon(G) = \left(\det \overline{L(G)}^{i,i}\right) \ \forall\, i \in [n]$
↑ "reduced Laplacian", $\overline{L(G)}^{ii} = L(G)$ with row i, column i removed

(b) (eigenvalue version)
If $L(G)$ has eigenvalues $(0<)\ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $\Upsilon(G) = \frac{\lambda_2 \cdots \lambda_n}{n}$ $\left(= 0 \text{ if } G \text{ disconnected}\right)$

EX|| Υ () $= \det\left(\overline{L(G)}^{4,4}\right) = \det \begin{bmatrix} 4 & -2 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix} = 36$
↑ can be computed quickly via Gaussian elimination in $\leq cn^3$ steps

EX!! For  , $L(G)$ has eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 4, 6, 6)$

Try in Sage / CoCalc –

```
L = matrix([[4,-2,-1,-1],
            [-2,4,-1,-1],
            [-1,-1,4,-2],
            [-1,-1,-2,4]])
L.eigenvalues()
```

<u>Note</u>: $\det(tI - L(G)) = \prod_{i=1}^{n}(t-\lambda_i)$ helps pass between parts (a) and (b) of the theorem

There are three proofs of the theorem - we'll use the one that appears in Loebl, Stanley.

<u>REU Exercise #4</u>:

(a) Show $L(K_n) = n\underset{\text{identity matrix}}{I_n} - J_n$ $\leftarrow J_n = \begin{bmatrix} 1 & 1 & \cdots \\ 1 & 1 & \\ \vdots & & \ddots \end{bmatrix}$ (all ones)

(b) Show J_n has eigenvalues $(n, \underbrace{0, \ldots, 0}_{n-1 \text{ times}})$

(c) Deduce $\tau(K_n) = n^{n-2}$

2. Sandpile group

For connected G, $\det(\overline{L(G)}^{i,i}) \neq 0$ shows $\mathbb{R}^n \xrightarrow{L(G)} \mathbb{R}^n$ has $\operatorname{rank}(L(G)) = n-1$

$\ker(L(G)) = \mathbb{R}^1$

col-space $= \operatorname{im}(L(G)) = \mathbb{R}^{n-1}$

$\operatorname{coker}(L(G)) = \mathbb{R}^n / \operatorname{im}(L(G))$

$\cong \mathbb{R}^1$

But $L(G) \in \mathbb{Z}^{n\times n}$, so what about $\mathbb{Z}^n \xrightarrow{L(G)} \mathbb{Z}^n$?

It's not too hard to see

$$\ker(L(G)) = \mathbb{Z}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \cong \mathbb{Z}^1$$

$$\operatorname{im}(L(G)) \cong \mathbb{Z}^{n-1} \subset \mathbb{Z}^n$$

$$\operatorname{coker}(L(G)) \cong \mathbb{Z}^1 \oplus \underbrace{K(G)}$$

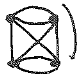
this is <u>defined</u> as the <u>sandpile group of G</u>

$$\text{so, } K(G) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}/\underset{\uparrow \text{ where } d_i | d_{i+1}}{d_i}\mathbb{Z} \cong \bigoplus_{\text{primes } p}\bigoplus_{e\geq 1}(\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)}$$

One can compute $K(G)$ and $\operatorname{coker}(L(G))$ via a change of basis in $\mathbb{Z}^n \xrightarrow{L(G)} \mathbb{Z}^n$ that puts $L(G)$ into <u>Smith normal form</u>,

$$\underset{\text{row ops over } \mathbb{Z}}{P} \cdot L(G) \cdot \underset{\text{col ops over } \mathbb{Z}}{Q} = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & 0 \end{bmatrix}$$

where $d_i | d_{i+1}$, $P, Q \in GL_n(\mathbb{Z})$
det $\in \{-1, 1\}$

EX!! $L($$)$ $= \begin{bmatrix} 4 & -2 & -1 & -1 \\ -2 & 4 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix}$ $\underset{\rightsquigarrow}{\text{row \& col ops}}$ $\begin{bmatrix} 4 & -2 & -1 & 0 \\ -2 & 4 & -1 & 0 \\ -1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\overline{L(G)}^{4,4}$

$\rightsquigarrow$ smith form algorithm (or SAGE!)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $d_1 = 1$
$d_2 = 3$
$d_3 = 12$

quick aside–

<u>Note</u>: $\operatorname{coker}(\overline{L(G)}^{i,i}) = \bigoplus_{i=1}^{n-1} \mathbb{Z}/d_i\mathbb{Z} = K(G)$

which implies $|K(G)| = \det(\overline{L(G)}^{i,i}) = \tau(G)$

In our example,

$$\operatorname{coker}(L(G)) \cong \operatorname{coker}(P\,L(G)\,Q) = \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/0\mathbb{Z}$$
$$= \mathbb{Z}^1 \oplus \underbrace{\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}}_{K(G)}$$

$$K(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$$
$$\cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z} \quad \text{(by Chinese remainder theorem)}$$

$\Rightarrow |K(G)| = 36$
$(= 3 \cdot 12)$

Recall, $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ if $\gcd(m,n) = 1$

EX!! $P\,L(G)\,Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ entries are mod 2

$\operatorname{rank}_{\mathbb{F}_2} L(G) = 2$

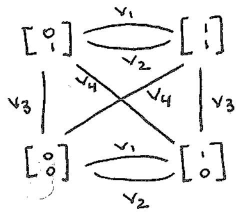
$\operatorname{rank}_{\mathbb{F}_3} L(G) = 1$ since $P\,L(G)\,Q = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ mod 3

3. Cayley graphs of $\mathbb{F}_2^r$

<u>Def</u>: Given a group Γ and generating set $M = \{v_1, \ldots, v_n\}$ which are <u>involutions</u> (so $v_i^2 = 1$, $v_i = v_i^{-1}$), the <u>Cayley graph</u> $G = G(\Gamma, M)$ has vertices Γ and edges $E = \{ g \overset{v_i}{—} g v_i \}_{\substack{i=1,2,\ldots,n \\ g\in\Gamma}}$

EX!! $\Gamma = \mathbb{F}_2^2$, M = columns of $\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left[\begin{matrix}1\\0\end{matrix}\right. & \begin{matrix}1\\0\end{matrix} & \begin{matrix}0\\1\end{matrix} & \left.\begin{matrix}1\\1\end{matrix}\right] \end{array}$

$G = G(\mathbb{F}_2^2, M)$ is



It turns out there's an eigenbasis for $\mathbb{R}^{2^r} \xrightarrow{L(G)} \mathbb{R}^{2^r}$ simultaneously for all choices of M. (c.f. Stanley Ch.2) ($G := G(\mathbb{F}_2^r, M)$)

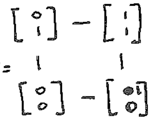
Def: For $u, v \in \mathbb{F}_2^r$, $u \cdot v = u_1v_1 + u_2v_2 + \dots + u_nv_n \in \mathbb{F}_2$.

For $u \in \mathbb{F}_2^r$, define $f_u \in \mathbb{R}^{2^r}$, ← with standard basis

as $$f_u := \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} e_v$$

i.e., $(f_u)_v = (-1)^{u \cdot v} \in \{+1,-1\} \in \mathbb{Z} < \mathbb{R}$

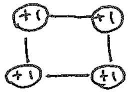
Ex!! $\mathbb{F}_2^r = \Gamma =$



has

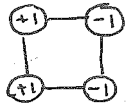
L()

$f_{\begin{bmatrix}0\\0\end{bmatrix}}$:



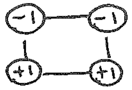
0 (= 4 − 1 − 1 − 1 − 1)

$f_{\begin{bmatrix}1\\0\end{bmatrix}}$:



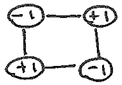
6 (= 4 + 1 + 1 − 1 + 1)

$f_{\begin{bmatrix}0\\1\end{bmatrix}}$:



4 (= 4 − 1 − 1 + 1 + 1)

$f_{\begin{bmatrix}1\\1\end{bmatrix}}$:



6 (= 4 + 1 + 1 + 1 − 1)

(eigenvectors) (eigenvalue)

REU Exercise #5:

(a) Prove that $\{f_u\}_{u \in \mathbb{F}_2^r}$ are an orthogonal 2^r basis for $\mathbb{R}^{2^r}$, and that the standard basis $\{e_u\}_{u \in \mathbb{F}_2^r}$ has $e_u = \frac{1}{2^n} \sum_{r \in \mathbb{F}_2^r} (-1)^{u \cdot v} f_v$

(b) Show that for any $M = \{v_1, \dots, v_n\}$ generating $\Gamma = \mathbb{F}_2^r$, the Cayley graph $G = (\Gamma, M)$ has every f_u as an eigenvector with eigenvalue $\lambda_{u,M} = n - \sum_{i=1}^{n} (-1)^{u \cdot v_i}$

(c) Show that if we define the ring $R = \mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{a}{2^m} : a \in \mathbb{Z}, m \geq 0 \right\}$, then
$$\underset{\mathbb{Z}}{\overset{\cup}{R}} = \mathbb{Z}\underset{\mathbb{Q}}{\overset{\cap}{\left[\tfrac{1}{2}\right]}}$$
"localization at 2"

there exists an R-basis for R^{2^r} in which $R^{2^r} \xrightarrow{L(G)} R^{2^r}$ acts diagonally, with $\underbrace{\text{eigenvalues}}_{\text{i.e, diagonal entries}}$ $\{\lambda_{u,M}\}_{u \in \mathbb{F}_2^r}$

(d) Explain why (c) shows that
$$K(G) \cong \bigoplus_{\substack{\text{primes}\\ p}} \underbrace{\left(\bigoplus_{e \geq 1} (\mathbb{Z}/p^e\mathbb{Z})^{m(p^e)} \right)}_{p\text{-Sylow subgroup of } K(G),\ \mathrm{Syl}_p K(G)}$$

then ∀ odd primes ($p \neq 2$),
$$\mathrm{Syl}_p K(G) = \mathrm{Syl}_p \Big(\bigoplus_{u \in \mathbb{F}_2^r - \{0\}} \mathbb{Z}/\lambda_{u,M}\mathbb{Z} \Big)$$

Ex!! $M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$K(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} = (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}$$

$$\mathrm{Syl}_3 K(G) = (\mathbb{Z}/3\mathbb{Z})^2 \cong \mathrm{Syl}\Big(\underbrace{\underset{(u=\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])}{\mathbb{Z}/6\mathbb{Z}}}_{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}} \oplus \underset{(u=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right])}{\mathbb{Z}/4\mathbb{Z}} \oplus \underbrace{\underset{(u=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right])}{\mathbb{Z}/6\mathbb{Z}}}_{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}} \Big)$$

$\mathrm{Syl}_3 K(G)$

$\boxed{\text{REU Problem \#2}}$

Describe $\mathrm{Syl}_2 K(G)$ for $G = G(\overset{\mathbb{F}_2^r}{\overset{\shortparallel}{\Gamma}}, \overset{\{v_1,\dots,v_n\}}{\overset{\shortparallel}{M}})$, in terms of the matrix
$$M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in \mathbb{F}_2^{r \times n}$$

e.g.,
- How many generators does it need?
- Is there a bound on e in $\bigoplus_{e \geq 1} (\mathbb{Z}/2^e\mathbb{Z})^{m(2^e)}$?
- How does it depend on the matroid of M?
  (i.e., features that are invariant under $M \mapsto PM$, $P \in GL_r(\mathbb{F}_2)$)

5. Ring theory (applicable here!)

Ex!! $G = G(\mathbb{F}_2^2, M)$

$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

we can model $\mathbb{Z}^{2^2}$ as $\mathbb{Z}[x_1, x_2] \big/ (x_1^2 - 1, x_2^2 - 1)$ with $\mathbb{Z}$-basis $\underset{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}{1}, \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{x_1}, \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{x_2}, \overset{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{x_1x_2}$

L(G) is multiplication by $4 - (2x_1 + x_2 + x_1x_2)$

Hence, $$\mathrm{coker}(G) = \mathbb{Z} \oplus K(G) = \mathbb{Z}[x_1, x_2] \Big/ \underbrace{\left(x_1^2 - 1, x_2^2 - 1, 4 - (2x_1 + x_2 + x_1x_2)\right)}_{I}$$

$$\underset{(\text{as } \mathbb{Z}\text{-modules})}{\overset{\uparrow}{\cong}} \mathbb{Z}[x_1, x_2] \Big/ \underbrace{\left(\begin{matrix} x_1^2 - 1,\ x_1x_2 + 2x_1 + x_2 - 4,\ x_2^2 - 1, \\ 3x_1 + 6x_2 - 9,\ 12x_2 - 12 \end{matrix} \right)}_{\text{Gröbner basis for } I}$$

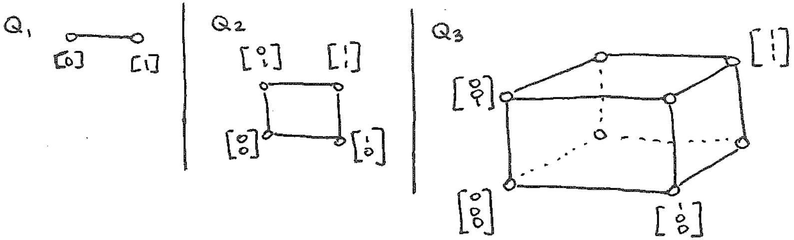
$$\cong \mathbb{Z} \cdot 1 \oplus \underbrace{(\mathbb{Z}/3\mathbb{Z})x_1 \oplus (\mathbb{Z}/12\mathbb{Z})x_2}_{K(G)}$$

REU Exercise #6:

Show $G = G(\mathbb{F}_2^r, M)$, $M = \left.\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\right\}r$ has
$$\mathrm{coker}(L(G)) \cong \mathbb{Z}[x_1, \dots, x_r] \Big/ \left(\begin{matrix} x_1^2 - 1, \dots, x_r^2 - 1, \\ n - \sum_{i=1}^{n} x_1^{(v_i)_1} \cdots x_r^{(v_i)_r} \end{matrix} \right)$$

For the special case $M = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$, then $G = G(\mathbb{F}_2^r, M)$ = r-dimensional cube graph Q_r

<u>EX!!</u>



a lot of work has been done, but $Syl_2(Q_r)$ is not <u>fully known</u>.

<u>References</u>:

- paper of H. Bui for partial results & data
- see REU report of Anzis & Prasad for ring approach
- paper of Chandler-Sin-Xiang for coker A(G)

$A\Big($  $\Big) = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

(which turns out to be totally predictable from eigenvalues)