REU 2018 Day 4 Ben Brubaker

Special functions in combinatorial representation theory and ice

<u>Extra reference</u>: Barcelo-Ram
arXiv: 9707221

<u>Schur functions</u>

- symmetric functions
  (invariant under permuting variables)

3 definitions today,
2 as combinatorial generating functions

First definition:

Given a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 0)$$

then the Schur function is

$$s_\lambda(z_1, \dots, z_n) = \sum_{P \in B(\lambda)} \underline{z}^{wt(P)}$$

$wt(P)$ is an n-tuple

$$= (wt(P)_1, \dots, wt(P)_n)$$

$$\underline{z}^{wt(P)} = z_1^{wt(P)_1} \cdots z_n^{wt(P)_n}$$

$B(\lambda) :=$ set of triangular arrays with top row λ

$$\begin{array}{ccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_n \\
 & a_{22} & & a_{23} & \cdots & a_{2n} & \\
 & & a_{33} & & a_{3n} & & \\
 & & & \ddots & & & \\
 & & & a_{nn} & & &
\end{array}$$

with inequalities
$$\begin{array}{ccc}
a_{i,j} & & a_{i,j+1} \\
 & \rotatebox{-45}{$\geq$} \quad \rotatebox{45}{$\leq$} & \\
 & a_{i+1,j+1} &
\end{array}$$

throughout.

They're called Gelfand-Tsetlin patterns.

EXAMPLE

For $\lambda = (3,1,0)$, then

$$\begin{array}{ccccc} 3 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \in B(\lambda)$$

but

$$\begin{array}{ccccc} 3 & & 1 & & 0 \\ & \cancel{4} & & 1 & \\ & & 1 & & \end{array} \notin B(\lambda)$$

Given $P \in B(\lambda)$, then

$R_i :=$ i^{th} row sum, and

$$wt(P) := (R_1 - R_2, R_2 - R_3, R_3 - R_4, \ldots, R_n)$$

EXAMPLES $\quad \lambda = (3,1)$

$$
\begin{array}{llll}
B(\lambda) & \begin{array}{cc} 3 & 1 \\ \multicolumn{2}{c}{3} \end{array} & \begin{array}{cc} 3 & 1 \\ \multicolumn{2}{c}{2} \end{array} & \begin{array}{cc} 3 & 1 \\ \multicolumn{2}{c}{1} \end{array} \\[2ex]
wt(P) & \begin{array}{c}(4-3,3) \\ \parallel \\ (1,3)\end{array} & (2,2) & (3,1)
\end{array}
$$

$$\Rightarrow \quad S_{(3,1)}(z_1,z_2) = z_1^1 z_2^3 + z_1^2 z_2^2 + z_1^3 z_2^1$$

(invariant under swapping z_1, z_2)

Why do we care?

- $s_\lambda(\underline{z})$ are amazing symmetric functions, giving a nice basis for the ring of symmetric functions; there is a natural inner product on the space of symmetric functions in which s_λ are orthonormal.
- they arise in representation theory of $GL_n(\mathbb{C})$ (= general linear group of all $n \times n$ invertible matrices over $\mathbb{C}$)

A representation of $G = GL_n(\mathbb{C})$ is a homomorphism $\rho : G \to \mathrm{Aut}(V)$

$$g \mapsto (\rho(g) : v \mapsto v')$$

$$\underbrace{\rho(g * g')}_{\text{mult. in } G} = \rho(g) \underbrace{\circ}_{\text{composition in } \mathrm{Aut}(V)} \rho(g')$$

Or more simply, we have an action of G on a vector space V (over $\mathbb{C}$).

Want to study irreducible, polynomial representations of $GL_n(\mathbb{C})$

no proper G-stable subspaces

$\boxed{\text{can write } \rho(g) \text{ entries as polynomials in entries } a_{ij} \text{ of } g = (a_{ij})}$

These rep'ns are indexed by partitions λ with n parts!

Where do $s_\lambda(\underline{z})$ arise?

We study characters of rep'ns

$$\operatorname{tr}(\rho(g)) =: \chi_\rho(g)$$

- Most g are diagonalizable with eigenvalues $z_1, \ldots, z_n$, and

then $$\operatorname{tr}(\rho_\lambda(g)) = \operatorname{tr}\left(\rho_\lambda \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix}\right)$$

$$= s_\lambda(z_1, \ldots, z_n)$$

BIG THM!

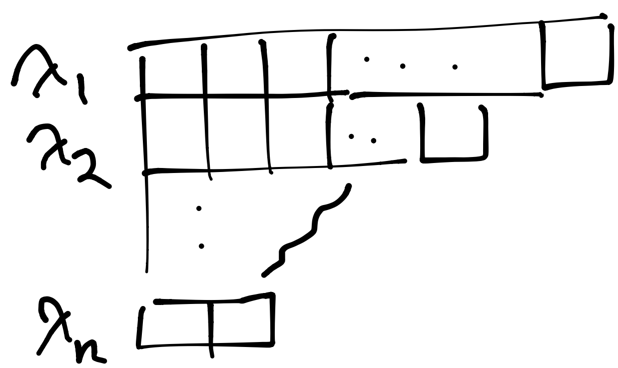
Better yet, given two repns V_λ and V_μ,

$$V_\lambda \oplus V_\mu \rightsquigarrow s_\lambda + s_\mu$$

$$V_\lambda \otimes V_\mu \rightsquigarrow s_\lambda s_\mu$$

<u>Second</u> <u>description</u> of s_λ

$\lambda = (\lambda_1, \dots, \lambda_n) \rightsquigarrow$ Young diagram



Y_λ

e.g. $\lambda = (4,2,1)$ has $Y_\lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}$

$B(\lambda)$ = fillings of Y_λ with alphabet $\{1, 2, \ldots, n\}$

- weakly increasing along rows
- strictly increasing down columns

called semistandard Young tableaux of shape λ (SSYT)

e.g. $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ is OK, but

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \cancel{1} \\ \cline{1-1}\end{array}$ is bad.

CLAIM: $SSYT(\lambda) \xleftrightarrow[\text{bijection}]{} GT(\lambda)$

Gelfand-Tsetlin patterns of shape λ

$$GT \longmapsto SSYT$$

$$\begin{array}{ccccc} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & & \\ 3 & & & \end{array}$$

i^{th} row from bottom in GT pattern $\longmapsto$ shape of the tableaux restricted to entries $1, 2, \dots, i$

REU Exercise 9

(0) Check this is a bijection

Q: How do we know $s_\lambda(\underline{z})$ is a symmetric function?

(1) Show GT-patterns with 2 parts in λ give symmetric functions in 2 variables

(2) What about 3 parts?

Given T a SSYT, then

$wt(T) = (\#n\text{'s in } T, \#(n-1)\text{'s in } T, \ldots, \#1\text{'s in } T)$

e.g. $T = \begin{matrix} 1\,1\,2\,3 \\ 2\,3 \\ 3 \end{matrix} \rightsquigarrow \underline{z}^{wt(T)} = z_1^3 z_2^2 z_3^2$

Our last definition of s_λ

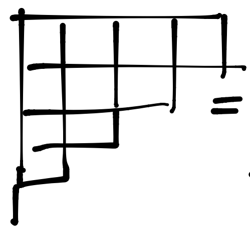
$$s_\lambda(\underline{z}) \overset{(*)}{\underset{\text{CLAIM}}{=}} \sum_{w \in S_n} \operatorname{sgn}(w)\, \underline{z}^{w(\lambda+\rho)} \Big/ \sum_{w \in S_n} \underline{z}^{w(\rho)}$$

symmetric group on n letters

$\underline{z} = (z_1, \dots, z_n)$

$\lambda = (\lambda_1, \dots, \lambda_n)$

$\rho := (n-1, n-2, \dots, 0)$

e.g. $n=5$  $= \rho$

EXAMPLE $\lambda = (3,1)$ $\rho = (1,0)$, $n=2$

$S_2 = \{1, (12)\}$

sgn $+1$, -1

$$s_{(3,1)}(z_1,z_2) = \sum_{w \in S_2} \operatorname{sgn}(w)\, \underline{z}^{w(4,1)} \Big/ \sum_{w \in S_2} \operatorname{sgn}(w)\, \underline{z}^{(1,0)}$$

$$= \left(z_1^4 z_2^1 - z_1^1 z_2^4\right) \Big/ \left(z_1^1 z_2^0 - z_1^0 z_2^1\right)$$

$$= \underbrace{z_1^3 z_2}_{\begin{smallmatrix}3&1\\1&\end{smallmatrix}} + \underbrace{z_1^2 z_2^2}_{\begin{smallmatrix}3&1\\2&\end{smallmatrix}} + \underbrace{z_1 z_2^3}_{\begin{smallmatrix}3&1\\3&\end{smallmatrix}}$$

One approach to Exercise 9.1 is to verify claim (*) that the GT description of s_λ is equal to the symmetric group description.

Note that the denominator

$$\sum_{w \in S_n} \operatorname{sgn}(w)\, \underline{z}^{w(\rho)} = \prod_{1 \le i < j \le n} (z_i - z_j)$$

Tokuyama (1987)

Gave a generating function for

$$\prod_{1 \le i < j \le n} (z_i + t z_j)\, s_\lambda(\underline{z}).$$

$\left(\begin{matrix} t=-1 \\ \leadsto \end{matrix}\right.$ numerator before$\left.\right)$

Let $SGT(\lambda+\rho) :=$ set of strict Gelfand-Tsetlin patterns with top row $\lambda+\rho$.

e.g. $\lambda=(3,1,0)$, $\rho=(2,1,0)$

$\lambda+\rho=(5,2,0)$

$$\begin{array}{ccccc} 5 & & 2 & & 0 \\ & 3 & & 1 & \\ & & 2 & & \end{array}$$

is a strict Gelfand-Tsetlin pattern

rows are strictly decreasing

$$\begin{array}{ccccc} 5 & & 2 & & 0 \\ & 2 & & 2 & \\ & & 2 & & \end{array}$$

is the only nonstrict GT pattern with top row $(5,2,0)$

<u>Tokuyama's formula</u>

$$\sum_{T\in SGT(\lambda+\rho)} (1+t)^{S(T)}\, t^{L(T)}\, z^{wt(T)} = \prod_{1\le i<j\le n}(z_i+tz_j)\, s_\lambda(z_1,\dots,z_n)$$

where

$S(T)$ = # of "special" entries, where given $\begin{matrix} a & & b \\ & c & \end{matrix}$ in the GT-pattern, c is special if $\begin{matrix} a & & b \\ & {>}\,c\,{>} & \end{matrix}$

$L(T)$ = # of left-leaning entries

given $\begin{matrix} a & & b \\ & c & \end{matrix}$, $\begin{matrix} a & & b \\ & {=}\,c & \end{matrix}$ is left-leaning at c.

EXAMPLE: $\lambda=(3,1,0)$ $\rho=(2,1,0)$

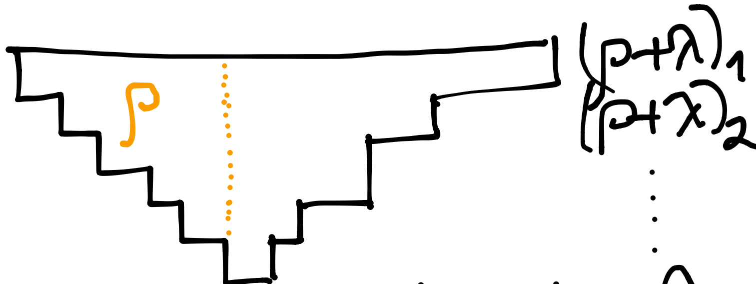
$SGT(\lambda+\rho)$ $\quad \begin{array}{ccc} 4 & & 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} 4 & & 1 \\ & 2 & \end{array} \quad \begin{array}{ccc} 4 & & 1 \\ & 3 & \end{array} \quad \begin{array}{ccc} 4 & & 1 \\ & 4 & \end{array}$

weighted sum $\quad z_1^4 z_2 \quad (1+t) z_1^3 z_2^2 \quad (1+t) z_1^2 z_2^3 \quad t z_1 z_2^4$

EXERCISE (10.1) Show that at $t=0,-1$ one gets the original GT description and the numerator of the S_n description.

(10.2) Give a tableaux version of Tokuyama's formula (in particular, find tableaux in bijection with strict GT patterns)

Hint for (10.2): Given $\lambda+\rho$, form the tableaux of shape

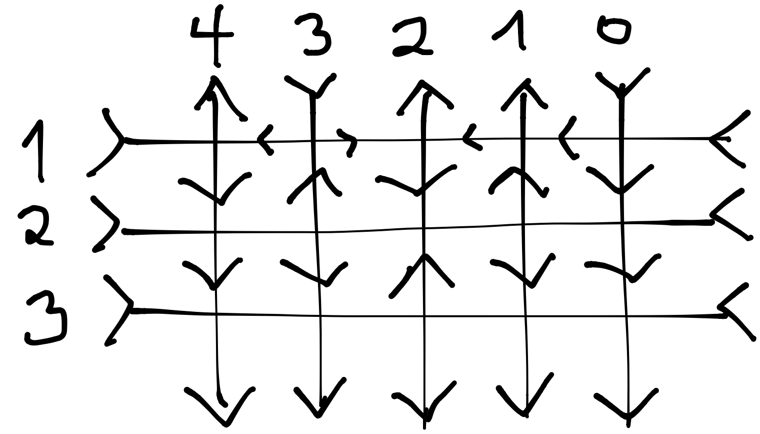


and come up with rules for fillings – diagonals will play a role.

ICE

Our final bijection:
Given $T \in SGT(\lambda+\rho)$,
produce a state of ice $\in \underline{ICE}(\lambda+\rho)$

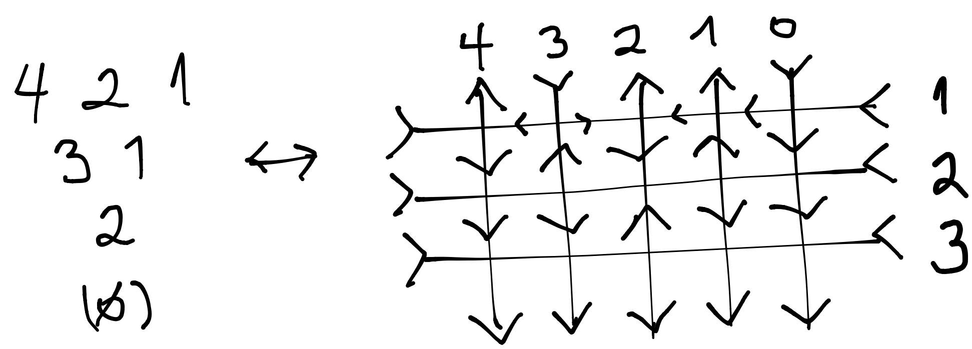
$$\begin{array}{ccccc} 4 & & 2 & & 1 \\ & 3 & & 1 & \\ & & 2 & & \\ & & (\emptyset) & & \end{array} \quad \longleftrightarrow$$



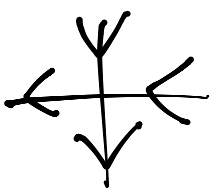
make rectangle with

- n rows and λ_1+n columns,
- boundary arrows as shown (down on bottom, up on the entries of $\rho+\lambda$), inward on sides.
- 2 arrows in, 2 out at each internal vertex

The vertical up/down arrows already determine the horizontal left/right arrows. So in the bijection, make the up arrows in the i^{th} row of the ice state be the entries of the i^{th} row of SGT.

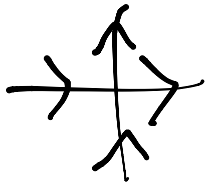


(Local)
Weights on the ice state give rise to Tokuyama's generating function.



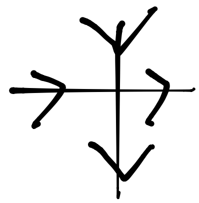
NE

z_i



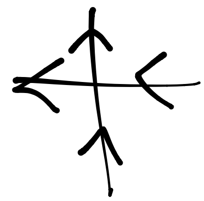
SW

t



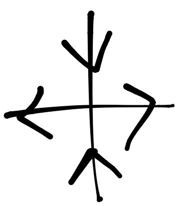
NW

1



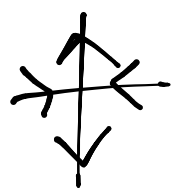
SE

z_i



NS

$z_i(t+1)$



EW

1

i = <u>row</u> index always

total weight of one state of ice (one filling)
$wt(I)$
is the product of weights at all vertices.

CLAIM:

$$\sum_{I \in ICE(\lambda+\rho)} wt(I) = \text{Tokuyama's formula}$$

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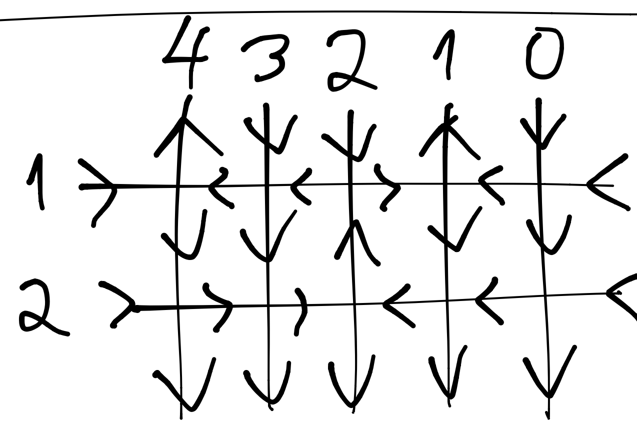
4 1

2

$(t+1)z_1^3 z_2^2$



row1: $\overset{EW_1}{1} \cdot \overset{NE_1}{z_1} \cdot \overset{NS_1}{z_1(t+1)} \cdot \overset{EW_1}{1} \cdot \overset{NE_1}{z_1} = (t+1)z_1^3$

row2: (check it gives z_2^2)

REU Problem 4:

There is an ice model for $\underbrace{\text{generalizations}}_{\text{Tokuyama's formula}}$ of $\underbrace{\text{characters of } GL_n(\mathbb{C})}_{\text{Schur polynomials}}$

Find ice models for the group $\underbrace{SO(2n+1, \mathbb{C})}_{\text{type B}}$

Odd orthogonal tableaux

Alphabet:

$1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n} < 0$

think $0 = \text{infinity}$

Given a partition λ, fill Y_λ with alphabet according to these rules:

- weakly increasing along rows
- weakly increasing in columns
- no entry i or $\bar{i}$ appears below row i
- no two non-zero entries in column are equal
- at most one 0 in any row.

The resulting generating function on tableaux gives the character of the polynomial representation of $SO(2n+1, \mathbb{C})$ corresponding to λ.

How one might approach this ...

STEP 1: Give a shifted version of the tableaux when of the form $\lambda+\rho$.

STEP 2: Give a bijection with an ice model & GT-patterns.

STEP 3: Attach weights to vertices in ice model which respect bijection (like Tokuyama).