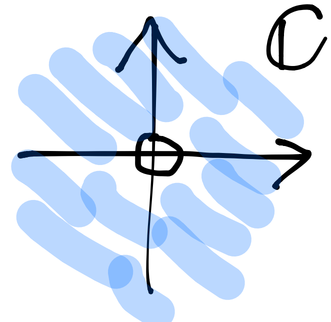
REU 2018 Day 5
Christine Berkesch
Virtual Resolutions

We'll work over a field k, and think $k=\mathbb{C}$ so that polynomials have roots.

$$\mathbb{C}^{\times} = \mathbb{C}^{*} = \mathbb{C} \setminus \{0\}$$



a (1-dimensional) algebraic torus

Given $\bar{x} = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$, consider

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^{n+1} &\longrightarrow \mathbb{C}^{n+1} \\ (t, \bar{x}) &\longmapsto (tx_0, tx_1, \dots, tx_n) \\ &\qquad =: t\cdot\bar{x} \end{aligned}$$

For $\bar{x} \in \mathbb{C}^{n+1} \setminus \{\bar{0}\}$, write

$$\underset{\substack{\shortparallel \\ \mathbb{C}^\times\text{-orbit of } x}}{[\bar{x}]} = \{t\bar{x} \mid t \in \mathbb{C}^\times\}$$

$\mathbb{C}^\times$-orbit of x $\longleftrightarrow$ a line thru $\bar{0}$ in $\mathbb{C}^{n+1}$

Projective space $\mathbb{P}^n = \{[\bar{x}] \mid \bar{x} \in \mathbb{C}^{n+1} \setminus \{\bar{0}\}\}$

Note

$$\begin{aligned} \mathbb{C}^{n+1} \setminus \{\bar{0}\} &\longrightarrow \mathbb{P}^n \\ \bar{x} &\longmapsto [\bar{x}] \end{aligned}$$

and $[\bar{x}] = [\bar{y}] \iff \exists t \in \mathbb{C}^\times$ with $t\bar{x} = \bar{y}$

For $\alpha \in \mathbb{N}^{n+1}$, let $\bar{x}^{\alpha} = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

and $|\alpha| = \sum_{i=0}^{n} \alpha_i$

If $f(\bar{x}) = \sum_{\alpha} a_\alpha \bar{x}^{\alpha} \in \mathbb{C}[x_0, x_1, \dots, x_n]$

polynomials

a sum over finitely many α's

and $t \in \mathbb{C}^{\times}$, then

$$f(t \cdot \bar{x}^{\alpha}) = \sum_{\alpha} a_\alpha (t \cdot \bar{x}^{\alpha}) = \sum_{\alpha} a_\alpha t^{|\alpha|} \bar{x}^{\alpha}$$

DEF'N: Say $f(\bar{x})$ is homogeneous (of degree d) if $\exists\, d$ with $|\alpha| = d$ $\forall \alpha$ above with $a_\alpha \neq 0$.

<u>Note</u>: $f(\bar{y})=0 \ \forall \bar{y}\in[\bar{x}]$
follows from $f(\bar{x})=0$
$\iff f(\underline{x})$ is homogeneous

Given $X \subseteq \mathbb{P}^n$, set

$$I(X) := \langle f(\bar{x}) \in S \mid f(\bar{c})=0 \ \forall \bar{c}\in X \rangle$$

$$S = \mathbb{C}[x_0,\dots,x_n]$$

(<u>later</u>: $I(X)$ is always generated by homogeneous polynomials)

EXAMPLES:

① $\mathbb{P}^2 \supset X = \left\{ \begin{array}{c} x_0 \; x_1 \; x_2 \\ [1:0:0] \end{array}, [0:1:0], [0:0:1] \right\}$

$$[1:0:0] := \mathbb{C}^\times(1,0,0) = \{(t,0,0) : t \in \mathbb{C}^\times\}$$

Here,

and

$$I(X) = \langle x_1, x_2 \rangle \cap \langle x_0, x_2 \rangle \cap \langle x_0, x_1 \rangle$$

$$= \langle x_0x_1, x_1x_2, x_0x_2 \rangle$$

---

② $\mathbb{P}^2 \supset Y = \{ [1:0:0], [1:1:0], [2:0:1] \}$

$$I(Y) = \langle x_1, x_2 \rangle \cap \langle x_0 - x_1, x_2 \rangle \cap \langle x_0 - 2x_2, x_1 \rangle$$

$$= \langle x_1x_2, x_0x_2 - 2x_2^2, x_0x_2 - x_1^2 \rangle$$

How'd we compute those ideal intersections?

Crash course in algebraic geometry

DEFN: $I \subseteq S = \mathbb{C}[x_0, x_1, \ldots, x_n]$ is an *ideal* if

(1) $0 \in I$ (or $I \neq \emptyset$)

(2) $a, b \in I \Rightarrow a + b \in I$

(3) $a \in I,\ f \in S \Rightarrow af \in I$

Claim: $I(X)$ above is always an ideal

Given $f_1, \dots, f_r \in S$, the ideal generated by $f_1, \dots, f_r$ is

$$(*) \quad \langle f_1, \dots, f_r \rangle := \left\{ \sum_{i=1}^{r} h_i f_i : h_i \in S \right\}$$

---

e.g.

$$\langle x_0 x_1, x_0 x_2, x_1 x_2 \rangle = \{ a(\bar{x}) x_0 x_1 + b(\bar{x}) x_0 x_2 + c(\bar{x}) x_1 x_2 \}$$

Hilbert's Basis Theorem

S is a Noetherian ring, meaning every ideal $I \subset S$ is finitely generated, that is, of the form $I = \langle f_1, \dots, f_r \rangle$.

DEF'N: An ideal $I \subseteq S$ is homogeneous if it can be generated by homogeneous polynomials.

CLAIM: $X \subset \mathbb{P}^n \Rightarrow$

$I(X)$ homogeneous

DEF'N: If $I = \langle f_1, \ldots, f_r \rangle$ is a homogeneous ideal in S, then

$$V(I) := \{ p \in \mathbb{P}^n \mid f_1(p) = \ldots = f_r(p) = 0 \}$$
$$= \{ p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in I \}$$

is a projective algebraic variety.

EXAMPLE:

$$V(\langle x_0x_1, x_0x_2, x_1x_2\rangle) = \{[1:0:0], [0:1:0], [0:0:1]\} \subset \mathbb{P}^2$$

THE GAME:

Geometric properties of $V(I)$

$\longleftrightarrow$ Algebraic properties of the ring $\mathbb{C}[x_0, \ldots, x_n]/I = S/I$.

e.g. irreducible varieties

$\longleftrightarrow$ domains S/I (i.e. I a prime ideal)

Recall: $X \subset \mathbb{P}^n$

$$\Rightarrow I(X) = \{f \in S \mid f(p) = 0 \ \forall p \in X\}.$$

THEOREM: For k infinite, the maps

$$\left\{\begin{matrix}\text{projective} \\ \text{varieties}\end{matrix}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{matrix}\text{homogeneous} \\ \text{ideals}\end{matrix}\right\}$$

are inclusion-reversing.

Furthermore, for any proj. variety V,

$$V(I(V)) = V.$$

In other words,

$$X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$$

$$I \subseteq J \Rightarrow V(I) \supseteq V(J).$$

EXAMPLE: $V(x_0) = V(x_0^2)$,

so $I(V(I)) \supsetneq I$

(e.g. take $I = \langle x_0^2 \rangle$)

How to fix this?

DEF'N: If I is an ideal of S, then the radical of I is

$$\sqrt{I} := \{ f \in S : \exists n \in \mathbb{Z}_{>0} \text{ with } f^n \in S \}$$

EXAMPLES:

- $\sqrt{\langle x_0^2 \rangle} = \langle x_0 \rangle$
- $\sqrt{\langle x^2, y^3 \rangle} = \langle x, y \rangle$ need binomial theorem

THEOREM

(Projective strong Nullstellensatz)

k algebraically closed, I a homogeneous ideal in $S = k[x_0, \ldots, x_n]$.

If $V(I) =: V$ is a nonempty projective variety in $\mathbb{P}^n$, then

$$I(V(I)) = \sqrt{I}.$$

Projective Ideal-Variety Correspondence

If we restrict the earlier correspondence, we get

$$\left\{\begin{array}{c}\text{nonempty}\\ \text{projective}\\ \text{varieties}\end{array}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{array}{l}\text{radical,}\\ \text{homogeneous}\\ \text{ideals}\\ I \subsetneq \langle x_0,\dots,x_n\rangle\end{array}\right\}$$

as inclusion-reversing mutually inverse bijections.

Note: Primary decomposition of ideals explains how to write varieties down as unions of irreducible varieties.

REU Exercise 11 (consider each a separate exercise!)

a) Prove that (*) is an ideal.

b) If $I \subset S$ is a (homogeneous) ideal, show that $\sqrt{I}$ is (homogeneous) ideal.

c) Let $f, g \in \mathbb{C}[x, y]$ be distinct nonconstant polynomials. Let $I = \langle f^2, g^3 \rangle$. Is it true that $\sqrt{I} = \langle f, g \rangle$. Explain.

d) Let I, J be homog. ideals in S. Show $V(I \cap J) = V(I) \cup V(J)$.

Hilbert functions

$S_d := \{\text{homog. polynomials of degree } d\}$

$\subset \mathbb{C}[x_0, \dots, x_n] = S$

a $\mathbb{C}$-vector space

$$S = \mathbb{C}[\bar{x}] = \bigoplus_{d=0}^{\infty} S_d$$

$\dim_{\mathbb{C}} S_d = \#\{\text{monomials of degree } d$

$\text{in } \mathbb{C}[x_0, \dots, x_n]\}$

$$= \binom{n+d}{n} = \frac{(n+d)!}{n!\,d!}$$

Why? $m=2$, $d=5$

$x_0^3 x_2^2 \rightsquigarrow (0,0,0,2,2)$

$\{0+1, 0+2, 0+3, 2+4, 2+5\}$

$= \{1,2,3,6,7\} \subset \{1,2,\dots,n+d\}$

Let's consider the function

$$HF_{S/I} : \mathbb{Z} \longrightarrow \mathbb{N}$$
$$d \longmapsto \dim_{\mathbb{C}} (S/I)_d$$

Hilbert function

if I is homogeneous

EXAMPLE: $S \supset I(X) = \langle x_0 x_1, x_1 x_2, x_0 x_2 \rangle$

d	monomials of degree d in S	$\dim_{\mathbb{C}} (S/I)_d$
0	1	1
1	x_0, x_1, x_2	3
2	$x_0^2, x_1^2, x_2^2, \cancel{x_0 x_1}, \cancel{x_0 x_2}, \cancel{x_1 x_2}$	3
$\vdots$		$\vdots$
d	x_0^d, x_1^d, x_2^d	3

EXAMPLE: $S \supset I(C) = \langle x_0x_2 - x_1^2 \rangle$

d	monomials	$\dim_{\mathbb{C}}(S/I)_d$
0	1	1
1	x_0, x_1, x_2	3
2	$x_0^2, x_1^2, x_2^2, x_0x_1, x_1x_2, \underset{= x_1^2}{x_0x_2}$	5
3	(check $\longrightarrow$)	7
$\vdots$		
d		$2d+1$

C is a curve (1-dimensional in $\mathbb{P}^2$)
and $HF_{S/I}(d)$ had degree 1 as
a polynomial in d.

Hilbert's polynomial theorem

Given homogeneous $I \subset S$,
$\exists$ a polynomial $P(z) \in \mathbb{Q}[z]$
such that, for d sufficiently large,
$HF_{S/I}(d) := \dim_{\mathbb{C}}(S/I)_d = P(d)$.

(The polynomial $P(z)$ is called the Hilbert polynomial of I.)

We'll deduce this from free resolutions of S/I

EXAMPLE: $I = I(X) = \langle x_0x_1, x_0x_2, x_1x_2 \rangle$

$$0 \leftarrow S/I \leftarrow S \xleftarrow{\begin{bmatrix} x_0x_1 & x_0x_2 & x_1x_2 \end{bmatrix}} S^3(-2) \xleftarrow{\begin{bmatrix} x_2 & x_2 \\ -x_1 & 0 \\ 0 & -x_0 \end{bmatrix}} S^2(-3) \leftarrow 0$$

$$\begin{matrix} ax_0x_1 \\ +bx_0x_2 \\ +cx_1x_2 \end{matrix} \longmapsfrom \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

this is a free S-module with 3 S-basis elements, all in degree 2.

free S-module of rank 2 with basis elements in degree 3

$$\Rightarrow \dim_{\mathbb{C}} (S/I)_d = \dim_{\mathbb{C}} S_d - 3 \dim_{\mathbb{C}} S(-2)_d + 2 \dim_{\mathbb{C}} S(-3)_d$$

$$= \dim_{\mathbb{C}} S_d - 3 \dim_{\mathbb{C}} S_{d-2} + 2 \dim_{\mathbb{C}} S_{d-3}$$

$$= 3 \text{ for } d \gg 0$$

If $m+d-a \geq 0$, then

$$\dim_{\mathbb{C}} S_{d-a} = \binom{m+d-a}{m}$$

$$= \frac{(m+d-a)(m+d-a-1)\cdots(d-a+1)}{m!}$$

Hilbert's Syzygy Theorem $X \subset \mathbb{P}^m$,

$I(X) \subset S = \mathbb{C}[x_0, \ldots, x_m]$ always

a free resolution

$$0 \leftarrow S/I \leftarrow \underset{\overset{\parallel}{F_0}}{S} \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_m \leftarrow 0$$

of length at most m.

REU Exercise 12

a) $I = \langle x_0, x_1 \rangle \cap \langle x_2, x_3 \rangle$

(2 skew lines in $\mathbb{P}^3$)

Prove $I = \langle x_0x_2, x_1x_3, x_0x_3, x_1x_2 \rangle$

Compute the

Hilbert function

polynomial

free resolution

(find one of length 3; show this is the minimal length)

b) Show $V(J) = V(I)$ for

$J = \langle x_0x_2 - x_1x_3, x_0x_3, x_1x_2 \rangle$,

but $J \subsetneq I(V(I))$

Compute the

Hilbert function,

polynomial

free resolution

<u>Hint</u>: Show

$J = I \cap \langle x_3^2, x_0x_3, x_2^2, x_1x_2, x_0x_2 - x_1x_3, x_1^2, x_0^2 \rangle$

c) $R = k[x]/\langle x^3 \rangle$

Compute a free resolution of $R/\langle x^2 \rangle$ as an R-module, not S-module

In particular, show it is infinite.

Virtual resolutions

$$\mathbb{P}^{\bar{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$$

$$(\mathbb{C}^\times)^r \times \mathbb{C}^{\overbrace{(n_1+1)+\cdots+(n_r+1)}^{N:=}} \longrightarrow \mathbb{C}^N$$

$$((t_1,\ldots,t_r),(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_r)) \longmapsto (t_1\bar{x}_1,\ldots,t_r\bar{x}_r)$$

Set $\deg(\bar{x}_i) := i^{th}$ standard basis vector in $\mathbb{Z}^r$

$S = \mathbb{C}[\bar{x}_1,\ldots,\bar{x}_r]$ is a $\mathbb{Z}^r$-graded ring

EXAMPLE: $\mathbb{P}^{1,2} := \mathbb{P}^1 \times \mathbb{P}^2$

$$(\mathbb{C}^\times)^2 \times \mathbb{C}^5 \longrightarrow \mathbb{C}^5$$

$$((t_1,t_2),(x_0,x_1,y_0,y_1,y_2)) \mapsto (t_1x_0, t_1x_1, t_2y_0, t_2y_1, t_2y_2)$$

$$\deg(x_0) = \deg(x_1) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\deg(y_0) = \deg(y_1) = \deg(y_2) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\deg(x_0^2x_1y_0^5) = \begin{pmatrix}3\\5\end{pmatrix}$$

Instead of throwing away $(0,0,\dots,0)$ in $\mathbb{C}^{n+1}$ for $\mathbb{P}^n$,

here we throw away $(\{\bar{0}\}\times\mathbb{C}^3)\cup(\mathbb{C}^2\times\{\bar{0}\})$

$$\updownarrow$$

$$B = \langle x_0,x_1\rangle \cap \langle y_0,y_1,y_2\rangle$$

Geometrically, $V(B) = \cancel{\emptyset} \subset \mathbb{P}^1 \times \mathbb{P}^2$!

<u>DEFN</u>: A <u>virtual resolution</u> of S/I is a sequence of $F_i = \bigoplus_\alpha S(-\alpha)^{\beta_{i,\alpha}}$

such that $0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \leftarrow F_t \leftarrow 0$

has $\operatorname{ann}\left(\frac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}\right) \supseteq B^\ell$ for $\ell \gg 0$

and $\operatorname{ann}\left(F_0 / \operatorname{im} \partial_1\right) = I$, up to components of B.

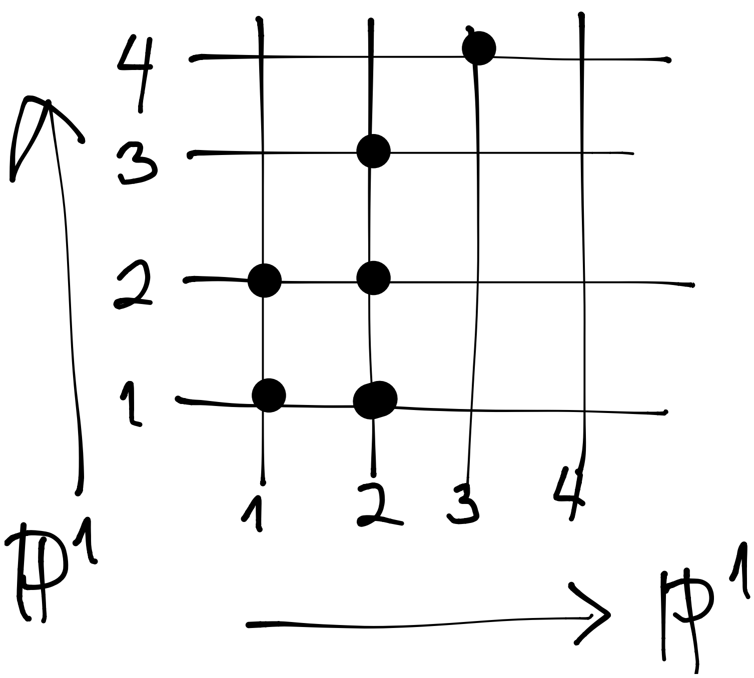
see examples in Macaulay2 demo to eventually appear

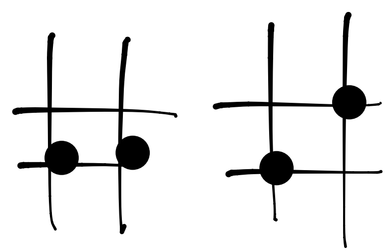
Virtual Hilbert Syzygy Theorem

[B–Erman–Smith 2017]

$\forall\, Y \subset \mathbb{P}^{\bar n}$, $I(Y)$ has a virtual resolution of length $\leq |\bar n| = n_1 + n_2 + \cdots + n_r$.

Points in $\mathbb{P}^1 \times \mathbb{P}^1$





REU Exercise 13

a) What are all possible configurations of 3 points in $\mathbb{P}^1 \times \mathbb{P}^1$?

b) Write out their defining ideals.

c) Compute their corresponding
Hilbert functions HF
polynomials HP
free resolutions

d) Compute virtual resolutions in each case (get length 2).

e) Do same for 4 points

f) Write Macaulay2 code to compute $I(Y)$

REU Problem 5

For configurations of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (later $\mathbb{P}^a \times \mathbb{P}^b$),

a) What powers of components of B give a "short" virtual resolution.

b) What is the minimal number of generators needed to generate an ideal of points virtually?

c) When does the ideal of points have a virtual resolution that is a Koszul complex

d) more to do!

. . . switched to Macaulay 2 .
demo of virtual
resolutions.

Starting with free resolution, can intersect with powers of components of B, and sometimes this gives virtual resolutions of $I(Y)$, smaller than the original.