GEOMETRICAL REALIZATIONS OF SHADOW GEOMETRIES

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Contents

Introduction

This paper deals with geometries and geometrical objects which are usually symbolized by diagrams of the following kind:

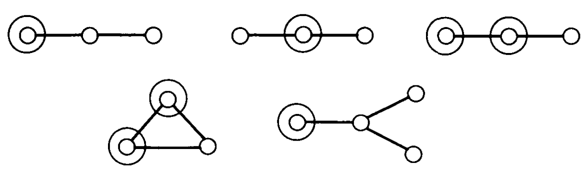

FIG. 1

Such symbols occur in the literature in two different contexts; on the one hand, they represent *shadow geometries of incidence geometries* in the sense of Tits, on the other hand, they are used in connection with *Wythoff's construction* for regular polytopes [**7**, **8**, **9**, **10**, **11**].

An example of a shadow geometry is the 'space' which is obtained from a three-dimensional projective space in the following way: the 'points' are the lines of the original space, there is one 'line' for each flag $\{\text{point } p \subset \text{plane } z\}$ of the original space, consisting of all 'points' l such that $p \subset l \subset z$, and there are two classes of 'planes', a 'plane' consisting of all 'points' containing a given point or contained in a given plane of the original space. The associated diagram is of course the second example given above. This example of a shadow geometry generalizes to a projective n-space and any distinguished set $I_0 \subseteq \{0, \ldots, n-1\}$, of dimensions. The 'points' are the flags (totally ordered sets of subspaces) P of the distinguished dimension type, and each subspace by definition comes from a

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flag F and consists of all P such that $F \cup P$ is again a flag in the original projective space. (Not all types of flags F are actually needed to produce all subspaces.) In the case of a general incidence geometry, the definition is the same, just replacing $\{0, \ldots, n-1\}$ by the set of types of that geometry.

An example of a 'semiregular' polytope obtained by Wythoff's construction is the truncated cube (Fig. 2).

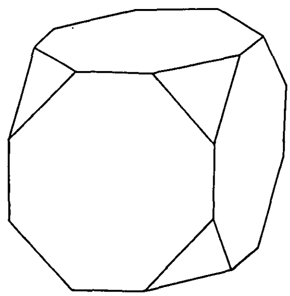


FIG. 2

The vertices of this figure are by definition all the transforms of some given point under the symmetry group of the cube. The group being fixed, different choices of the starting vertex lead to different polytopes. Combinatorially, there are seven different choices, corresponding to the subsets of the set of 'types' of reflecting hyperplanes. (Once a fundamental chamber of the group is fixed, this set of types is identified with a set of generating reflections.) The seven polytopes are the cube, the octahedron, the cubeoctahedron, the truncated cube, the truncated octahedron, the rhombicubeoctahedron [**8**, p. 17], and the truncated cubeoctahedron [**8**, p. 18].

The idea of Wythoff's construction is also used in the case of planar tilings, and more generally of tilings of Euclidean n-space. An example is shown in Fig. 3.

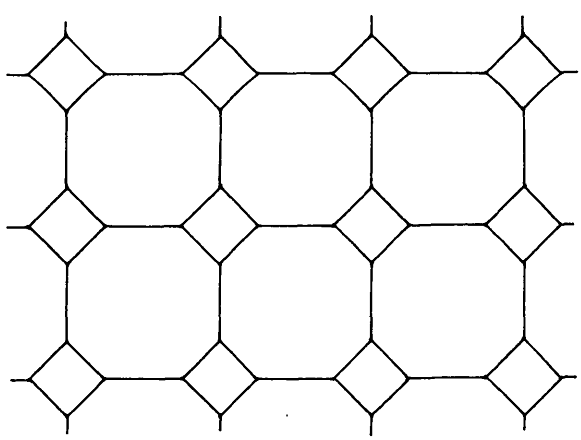


FIG. 3

It has certainly been observed before that the combinatorial properties of the tiling in Fig. 3 and of the truncated tube, that is, their vertices, edges and faces and their inclusion relation, are described by shadow geometries. The incidence geometry to start from is, of course, the set of vertices, edges, and faces of the regular square tiling, respectively the cube. However, it seems that the relation between shadow geometries and Wythoff's construction had never been investigated carefully and exhaustively until the recent paper **[28]** of G. Maxwell which was written independently of this paper.

In the present paper, the shadow geometry associated to an incidence geometry (more generally, to a chamber system **[42]**), and a distinguished set of types, is an abstract poset which in important cases is isomorphic to the poset of subspaces of the classical shadow geometry **[41**, Appendix I; **5**; **32]**. One of my main results (Theorem 1.5) states that the standard geometrical realization (in the sense of combinatorial topology) of such a poset is (more or less canonically) homeomorphic to the standard geometrical realization of the original incidence geometry (that is, of its flag complex). On the other hand, it is known **[16]** that any pure poset $\mathcal{S}$ can be considered as a tiling or tessellation of its standard geometrical realization $\|\mathcal{S}\|$ in the following way. For each $s \in \mathcal{S}$ one considers the geometrical realization $\|s_*\|$ of $s_* := \{t \in \mathcal{S} | \ t \leqslant s\}$, viewed as a subset of $\|\mathcal{S}\|$. These 'cells' $\|s_*\|$ are the tiles of the associated tiling, the dimension of $\|s_*\|$ being the combinatorial dimension of s in the pure poset $\mathcal{S}$. Combining my above-mentioned result with this general construction, we see that every shadow geometry of a given incidence geometry Δ (technically, Δ is a 'numbered complex') can be considered as a tiling of its standard realization $\|\Delta\|$. More generally, in the case of a chamber system $\mathcal{C}$, one has not only the standard realization, but also various geometrical realizations $E(\mathcal{C}, M)$ depending on the choice of a 'typical chamber' M. An appropriate subgeometry, depending on M, of a shadow geometry $\mathcal{S}$ of $\mathcal{C}$, can be represented by a family of 'cells' in $E(\mathcal{C}, M)$ (which is canonical in a certain sense if one allows the orderings to be reversed, that is, small elements in $\mathcal{S}$ correspond to big cells). Now a generalization (Theorem 3.1) of the above-mentioned Theorem 1.5 says that under certain conditions on $\mathcal{C}$ and M, these families of 'cells' are tilings of $E(\mathcal{C}, M)$.

In the case of a spherical or Euclidean reflection group, one can apply this result to the Coxeter–Tits complex or chamber system of this group and thereby obtain—at least combinatorially—the semi-regular polytopes (considered as tilings of the sphere) and the Euclidean tilings which are known from Wythoff's construction. Thus, the relationship between shadow geometries and Wythoff's construction is clarified and is not restricted to examples any more. Indeed, although formally and technically this paper stays inside the framework of incidence geometries, I would rather view it as a contribution to discrete geometry. Even in the classical cases, at least for the combinatorial side of Wythoff's construction, a new foundation is given by defining the faces of the new objects in all dimensions and describing them in a unique fashion, whereas the original idea applied to the vertices alone. The very first place in the literature where a general treatment with full proofs of Wythoff's construction has been given is, to my best knowledge, the paper **[28]** by G. Maxwell already mentioned above. In that work, a description of the face lattice of a Wythoff polytope or tessellation is given which easily shows that it is equivalent to a set of shadows as described by Tits in **[41]**. Thus the face lattice is indeed a shadow geometry in our sense. Strictly speaking, it is this result of Maxwell's which gives the full justification to my claim that the present paper can be viewed as a generalization of Wythoff's construction.

This paper is organized into five sections as follows. In § 1, after some preparation concerning the basic graph of an incidence geometry, the shadow geometries of incidence geometries are introduced and the main result about the standard geometrical realization, that is, the interpretation as tessellations, is formulated. In the easy but important Proposition 1.7, some basic properties of the 'separation relation' in a finite graph are collected which are fundamental for all further developments of this paper. In § 2, I present the framework which allows us to treat simultaneously the shadow geometries of incidence geometries and the shadow geometries of reflection groups. To this end, I introduce shadow geometries and geometrical realizations of chamber systems, that is, spaces which are constructed by pasting copies of a 'fundamental cell' along its panels ('faces' of codimension 1). In § 3, the main theorem of § 1 about shadow geometries and tessellations is extended to the shadow geometries of chamber systems. In § 4 we derive additional properties of these tessellations in the case of reflection groups: the tiles are convex, all tiles are faces of the maximal tiles and intersections of maximal tiles. Furthermore, it is shown that the geometrical realizations of the shadow geometries of a reflection group are essentially the only tessellations on which the group acts transitively on the maximal tiles. In § 5, I derive the Delaney symbols (or generalized Schläfli symbols) [**16, 17**] of the shadow geometries of reflection groups. This is a fairly straightforward combination of the ideas of [**16, 17**] with previous results of the present paper. Section 5 can be regarded as complementary to the explicit listings of numerical parameters of semiregular polytopes and tessellations as given in [**10, 11**].

Sections 1, 2, the first part of § 3, and § 5 are completely elementary and can in principle be read without any particular prerequisites. In the second half of § 3 and particularly in § 4, the theory of reflection groups on, for example, Euclidean or hyperbolic spaces [**25**; **2**, chapitre V; **44**] and the related theory of abstract Coxeter groups [**2**, chapitre IV; cf. **41**] is presupposed.

Acknowledgement. I am indebted to Andreas Dress for helpful suggestions and comments.

General definitions and notation

A *poset* is a set X together with a partial ordering, that is, a transitive and antisymmetric relation $\leqslant$. It is called *pure* if any two maximal flags (totally ordered subsets) have the same finite cardinality $n+1$ (or 'length' n). This number n is the *dimension* of X. If X is pure and $x \in X$, and $x_0 < x_1 < \ldots < x_d = x < \ldots < x_n$ a maximal flag, then d, the *dimension of x in X* is independent of the choice of the flag.

A *simplicial complex* Δ with *vertex set* X is a set of finite subsets of X such that $A \subseteq B \in \Delta$ implies $A \in \Delta$, and $\{x\} \in \Delta$ for all $x \in X$. Such a Δ is a poset with respect to the inclusion relation, and any poset isomorphic to such a Δ is also called a simplicial complex. Therefore, we will often not distinguish between a vertex x and the one-element set $\{x\}$. If I is any finite set, then the power set $P(I)$ is a simplicial complex. If Δ is a simplicial complex, then the (*standard*) *geometrical realization* of Δ is obtained by replacing each 'abstract' simplex A (element of Δ) by a 'geometrical' simplex $\|A\|$ in some vector space, and respecting inclusion. Formally:

$$\|\Delta\| := \{\sum r_x x \mid r_x \in \mathbb{R}, r_x \geq 0, \sum r_x = 1, \{x \mid r_x > 0\} \in \Delta\},$$
$$\|A\| := \{\sum r_a a \mid r_a \geq 0, \sum r_a = 1\} \subseteq \|\Delta\|.$$

If $(X, \leq)$ is a pure poset, then the set of flags in X is a simplicial complex, denoted by $\Delta(X, \leq)$. The geometrical realization $\|X, \leq\|$ is, by definition, the geometrical realization of the flag complex.

We shall distinguish between the two notions of a '*Coxeter group*' and a '*reflection group*'. A Coxeter group is a pair (W, S), where W is a group, S a finite generating set, and the well-known conditions of [**2**, chapitre IV] hold. A reflection group is a transformation group W on some space E subject to certain conditions which have to be specified and which imply that W, together with a generating set coming from a 'fundamental chamber', is a Coxeter group. An important class consists of the *properly discontinuous reflection groups*. Here, E is a differentiable manifold, the 'reflections' in W are involutionary diffeomorphisms such that the complement of the fixed point set consists of two connected components and W operates properly discontinuously [**25**, Chapter III, § 3]. Another important class consists of the *linear reflection groups*. Here, E is a subset of a vector space, the reflections are linear mappings with fixed point set a hyperplane. The most important examples of spherical, Euclidean, or hyperbolic reflection groups are properly discontinuous and can also be regarded as linear reflection groups.

1. *Shadow geometries of incidence geometries*

At the beginning of this first section, an introduction to the Tits incidence geometries [**38, 3**], more precisely to the slightly more general 'numbered complexes' [**41**; **2**, chapitre IV, § 1, Exercices 15 à 24] is given. This short presentation of well-known material had to be included because Proposition 1.1 (the 'main theorem about the basic graph') cannot be found in the literature in the form given here.

After these generalities, the shadow geometries of a numbered complex are introduced. There is one such geometry for each non-empty subset I_0 of the type set I of the given complex. We begin with a certain condition which in [**41**, Appendix I] in the case of buildings occurs as a criterion for the inclusion of two 'shadows'. My starting point was the observation that, without any particular assumption on the complex, this condition defines a partial ordering on certain 'reduced' simplices of the complex. These posets are the shadow geometries of the present paper.

If one examines the construction of shadow geometries with respect to some basic combinatorial properties of complexes and posets, one observes that certain properties of a given complex are inherited by all its shadow geometries. It is known that the properties in question, 'purity', 'strong connectedness', 'thinness', 'pseudomanifold' are topological invariants. Therefore one looks for a topological 'explanation' of the observation mentioned. Indeed, one main result of the present paper states that the geometrical realization of each shadow geometry $\mathcal{S}$ of a given complex Δ is homeomorphic to the geometrical realization of Δ. More precisely, the geometrical simplicial complex $\|\mathcal{S}\|$ can be considered as a *subdivision* of $\|\Delta\|$. This result is formulated in 1.5; in 1.6 a special case independent of incidence geometries is emphazised; the theorem will be proved only in § 3 below.

Proposition 1.7 collects some basic facts about the 'separation relation' in a finite graph which are not difficult but essential for all further developments of the present paper. This proposition deals with certain 'reduced' and certain 'closed' vertex sets of the graph and with their relations to each other. In the applications, this graph is the basic graph of a numbered complex, and the reduced sets have already occurred in the definition of the shadow geometries. The utility of the closed sets has been pointed out to me by A. Dress. He also observed that the relation between reduced and closed sets as described in Proposition 1.7 is already sufficient to construct a subdivision of a single simplex. This result is included as the final Theorem 1.8 of this first section.

A *numbered complex* is a simplicial complex Δ together with a partitioning $X = \bigcup_{i \in I} X_i$ of its vertex set X such that each maximal simplex contains precisely one vertex of each X_i. Here, I is some finite index set which is called the *type set* of Δ. Every simplex $A \in \Delta$ is of the form $A = \{x_j | \; j \in J\}$ for some $J \subseteq I$ and $x_j \in X_j$; the set J is called the *type of* A and denoted by $J = \operatorname{type} A$. An important example is the *flag complex* (complex of totally ordered subsets) of a pure poset $(X, \leqslant)$, where $I = \{0, 1, \ldots, \dim X\}$ and $X_i = \{x \in X | \; \dim x = i\}$. More generally, X_i can be the set of 'objects of type i' in an incidence geometry in the sense of Tits. Such a geometry by definition consists of a set X, partitioned as above and an 'incidence relation' on X such that each maximal set of pairwise incident elements contains precisely one object in each X_i. The incidence geometries can be identified with a subclass of the numbered complexes, a complex 'being a geometry' if and only if each set of vertices such that any two of them form a simplex is a simplex itself.

Following **[41]**, we shall use the terminology of incidence geometries also for the numbered complexes. For instance, two vertices x, y are *incident* if $\{x, y\}$ is a simplex, and two simplices are *incident* if their union is also a simplex. At some places, for instance in the title of this section, I shall even use the term '(incidence) geometry' for a member of the more general class of numbered complexes.

If Δ is numbered with type set I and $A \in \Delta$, the *star* $\operatorname{St} A = \{B \in \Delta | \; B \supseteq A\}$ is numbered with type set $\operatorname{cotype} A := I \backslash \operatorname{type} A$. The maximal simplices are called *chambers*, the simplices of codimension 1 are called *panels*. Two chambers C, C' are called *adjacent* if $C \cap C'$ is a panel, more precisely *i-adjacent* if $\operatorname{cotype}(C \cap C') = i$. A numbered complex Δ is *connected* if any two chambers can be connected by a gallery, that is, a sequence of chambers which are successively adjacent or equal. It is called *strongly connected* if, furthermore, the stars of all simplices are connected.

DEFINITION. Let Δ be a numbered complex with type set I. The *basic graph* of Δ has the vertex set I, and $\{i, j\} \subseteq I$ is not an edge if for each $A \in \Delta$ of cotype $\{i, j\}$, the star $\operatorname{St} A$ is a 'generalized digon', that is, each vertex of type i in $\operatorname{St} A$ is incident with each vertex of type j in $\operatorname{St} A$.

If Δ is the flag complex of a pure poset X of dimension n, then $i, j \in \{0, \ldots, n\}$ are not connected if $|i - j| \geqslant 2$. The basic graph of each star $\operatorname{St} A$, with $A \in \Delta$, is a subgraph of the basic graph of Δ. That is, if $i, j \in \operatorname{cotype} A$ are connected with respect to $\operatorname{St} A$, then they are also connected with respect to Δ.

If some graph with vertex set I is given and $J, K \subseteq I$, we say that J, K are *separated* from each other if there exists no path in the graph starting inside J and ending inside K. Equivalently, the graph is the disjoint union of two subgraphs having vertex sets J', K' such that $J \subseteq J'$ and $K \subseteq K'$. If L is a third vertex set, then J and K are *separated by* L if $J \backslash L$ and $K \backslash L$ are separated from each other in the subgraph induced on $I \backslash L$. This means that there exist $\tilde{J} \supseteq J$ and $\tilde{K} \supseteq K$ such that $I \backslash \tilde{J}$ and $I \backslash \tilde{K}$ are not connected and $I \backslash L = I \backslash \tilde{J} \dot{\cup} I \backslash \tilde{K}$.

The following 'Main Theorem on the basic graph' is due to Tits for the case of buildings, and has been proved for general incidence geometries by Buekenhout in [**3**].

1.1. Proposition. *Let Δ be a strongly connected numbered complex and let A_1, A_2, A_3 be simplices of Δ; set $J_t := \operatorname{type} A_t$. If $A_1 \cup A_2$ and $A_2 \cup A_3$ are simplices, and J_1 and J_3 are separated by J_2 in the basic graph, then $A_1 \cup A_2 \cup A_3$ is a simplex.*

Proof. By strong connectedness, it is sufficient to prove the claim inside $\operatorname{St} A_2$. Therefore consider two simplices A, B such that, for $J := \operatorname{type} A$, $K := \operatorname{type} B$, the whole type set J equals $J \dot{\cup} K$, and J, K are separated from each other. We have to show that $A \cup B$ is a simplex. Set $A_0 = A$ and choose B_0 and D such that $A_0 \dot{\cup} B_0$ and $D \dot{\cup} B$ are chambers. Connect $A_0 \cup B_0$ and $D \cup B$ by some gallery, and write this gallery in the form

$$(*) \qquad (A_0 \cup B_0, A_1 \cup B_1, \ldots, A_m \cup B_m = D \cup B),$$

where $\operatorname{type} A_t = J$, $\operatorname{type} B_t = K$ for all t. Consider three consecutive chambers,

$$(**) \qquad A_{t-1} \cup B_{t-1}, \quad A_t \cup B_t, \quad A_{t+1} \cup B_{t+1},$$

and assume that in the first step a vertex b_1 such that $\operatorname{type} b_1 \in J$ and in the second step a vertex b_2 such that $\operatorname{type} b_2 \in K$ is exchanged:

$$A' \cup a_1 \cup B' \cup b_1, \quad A' \cup a_2 \cup B' \cup b_1, \quad A' \cup a_2 \cup B' \cup b_2.$$

Considering $\operatorname{St}(A' \cup B')$ whose basic graph consists of two isolated vertices, we see that also $A' \cup a_1 \cup B' \cup b_2$ is a simplex. Therefore we can modify $(**)$ such that the change takes place at first in K and then in J, and not conversely. Therefore the gallery $(*)$ can be modified in such a way that $B_0 \neq B_1 \neq \ldots \neq B_r = B_{r+1} = \ldots = B_m$ for some $r < m$. So $A \cup B = A_0 \cup B_m = A_r \cup B_r$ is a simplex, as desired.

After this short introduction to our general framework we now come to the various 'shadow geometries', derived from a given geometry. We first have to repeat the following observation from [**41**, Appendix I]; cf. [**4**].

1.2. Remark and Definition. *Fix a subset I_0 of the vertex set I of some finite graph. Consider, for subsets $J, K \subseteq I$, the relation*

$$J \preccurlyeq K \quad \Leftrightarrow \quad I_0 \text{ and } K \text{ are separated by } J.$$

For each K, there exists an (obviously unique) smallest subset $K_{\text{red}} \subseteq K$ such that $K_{\text{red}} \preccurlyeq K$, that is, $J \preccurlyeq K$ and $J \subseteq K$ imply $K_{\text{red}} \subseteq J$. Vertex sets of the form K_{red} are called reduced.

Proof. By the finiteness of I, it is sufficient to show that $J \preccurlyeq K$, $J' \preccurlyeq K$, $J \subseteq K$, $J' \subseteq K$ imply $J \cap J' \preccurlyeq K$. If $k \in K$ and $i_0, i_1, \ldots, i_m = k$ is a path in the graph, $i_0 \in I_0$, then the first i_t lying in $J \cup J'$ actually must lie in $J \cap J'$. This is true because $J \preccurlyeq J' \preccurlyeq J$.

In Proposition 1.7 below, we shall collect various properties of the function $K \mapsto K_{\text{red}}$.

1.3. Proposition and Definition. *Let Δ be a strongly connected numbered complex with type set I, and let $I_0 \subseteq I$.*

(a) *On* $\mathcal{S}(\Delta, I_0) := \{A \in \Delta \mid \text{type } A = (\text{type } A)_{\text{red}}\}$ *the relation*

$$A \preccurlyeq B \quad :\Leftrightarrow \quad A \cup B \in \Delta, \ \text{type}\, A \preccurlyeq \text{type}\, B$$

is a partial ordering, the so-called shadow geometry *of Δ with respect to I_0. Here,* 'red' *and* '$\preccurlyeq$' *are defined as in* 1.2, *with respect to the basic graph of Δ.*

(b) *If $A_1 < A_2 < \ldots < A_d$ is a flag in $\mathcal{S}(\Delta, I_0)$, then $A_1 \cup A_2 \cup \ldots \cup A_d \in \Delta$.*

(c) *Assume furthermore that I_0 meets every connected component of the basic graph. Then all maximal flags of the shadow geometry have cardinality $n = |I|$, and $A_1 \cup A_2 \cup \ldots \cup A_n$ is a chamber for each maximal flag $A_1 < A_2 < \ldots < A_n$. That is, the chamber set of the shadow geometry can be canonically identified with the product of the chamber set of Δ by the set of maximal flags of reduced type sets (with respect to the separation relation $\preccurlyeq$).*

Proof. Trivially, $A \preccurlyeq A$ for each $A \in \mathcal{S} := \mathcal{S}(\Delta, I_0)$. If $A, B \in \mathcal{S}$ and $A \preccurlyeq B \preccurlyeq A$, then, in particular, $\text{type}\, A \preccurlyeq \text{type}\, B \preccurlyeq \text{type}\, A$, and $\text{type}\, A$ and $\text{type}\, B$ have to coincide because they are reduced. Then also A and B have to coincide, because they are incident.

For the proof of (b), which contains the transitivity of '$\preccurlyeq$' as a special case, we use induction on d, and we may suppose that $d \geqslant 3$. Let $A_1, \ldots, A_d \in \mathcal{S}$ be given such that $A_1 < A_2 < \ldots < A_d$. By assumption, $A_1 \cup A_2 \in \Delta$, and by the induction hypothesis, $A_2 \cup (A_3 \cup \ldots \cup A_d) \in \Delta$. Furthermore,

$$\text{type}\, A_1 \preccurlyeq \text{type}\, A_2 \preccurlyeq \text{type}(A_3 \cup \ldots \cup A_d)$$

from which it easily follows that $\text{type}\, A_1$ and $\text{type}(A_3 \cup \ldots \cup A_d)$ are separated by $\text{type}\, A_2$. Now Proposition 1.1 implies that

$$A_1 \cup A_2 \cup (A_3 \cup \ldots \cup A_d) \in \Delta,$$

as desired.

For technical reasons, the proof of (c) will only be given after Proposition 1.7.

I want to point out the following special case of Proposition 1.3. If the basic graph of Δ is a 'chain', that is, $I = \{0, 1, \ldots, n\}$, i is connected with j if and only if $|i - j| = 1$, then the vertex set of Δ becomes a poset by setting

$$x \preccurlyeq y \quad \Leftrightarrow \quad \{x, y\} \in \Delta, \ \text{type}\, x \preccurlyeq \text{type}\, y,$$

and Δ is the flag complex of this poset. This means that in the case of chain basic graphs, firstly all strongly connected numbered complexes actually 'are' incidence geometries, and secondly all incidence geometries are posets. These two substatements or special cases thereof have been considered before by various authors.

The following remark relates the definition of shadow geometries given above to [**41**, Appendix I], which was the starting point of my consideration.

1.4. REMARK. The 'classical' shadow geometry of Δ with respect to I_0 consists of

$$S_{I_0} := \{A \in \Delta \mid \text{type } A = I_0\}$$

as the *set of points* and the *shadows*

$$S_{I_0}(B) := \{A \in S_{I_0} \mid A \preccurlyeq B\},$$

where B runs through Δ, as the *subspaces*.

By Proposition 1.3 and its proof, we have

$$S_{I_0}(B) = S_{I_0}(B_{\text{red}}),$$

where B_{red} is defined by $B_{\text{red}} \subseteq B$ and type $B_{\text{red}} = (\text{type } B)_{\text{red}}$, and

$$B_1 \preccurlyeq B_2 \quad \Rightarrow \quad S_{I_0}(B_1) \subseteq S_{i_0}(B_2).$$

We say that the shadow geometry is *faithful* if the converse of the last implication holds for any two B_1, B_2 of reduced type. In this case, the shadow geometry in the sense of the above definition is isomorphic to the set of subspaces of the 'classical' shadow geometry, partially ordered by inclusion. A basic theorem by Tits [**41**, Appendix I] states that all shadow geometries of buildings are faithful. A new proof of this result will be given in § 4 below.

Before stating my main result about shadow geometries of numbered complexes, I recall that the (standard) geometrical realization of a poset is by definition the (standard) geometrical realization of its flag complex.

1.5. THEOREM. *If Δ is a strongly connected numbered complex and I_0 a subset of its type set I meeting each connected component of the basic graph of Δ, then the geometrical realization of the shadow geometry of Δ with respect to I_0 is canonically isomorphic to the geometrical realization of Δ itself.*

More precisely: the mapping which associates each element A of the shadow geometry to the barycentre of the geometrical simplex $\|A\| \subseteq \|\Delta\|$, extends to a simplicial isomorphism of $\|\mathscr{S}(\Delta, I_0)\|$ onto $\|\Delta\|$.

In view of Proposition 1.3(b) and (c), the theorem means that, starting from the standard realization of Δ, one can realize the shadow geometries $\mathscr{S}$ of Δ in such a way that each maximal geometrical simplex of Δ is the union of such simplices of $\mathscr{S}$. That is, $\|\mathscr{S}\|$ can be considered as a *subdivision* of $\|\Delta\|$.

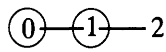
Theorem 1.5 will be a corollary of a more general result on shadow geometries of chamber systems which will be proved in § 3 below.

1.6. SPECIAL CASE. *For each non-empty subset $J \subseteq I$ denote by p_J the barycentre of the standard simplex $\|J\|$ (considered as a face of $\|I\|$). For all flags $J_1 < J_2 < \ldots < J_d$ of reduced type sets, the point sets $p_{J_1}, \ldots, p_{J_d} \subseteq \|I\|$ are affinely independent, and $\|I\|$ is the disjoint union of the open simplices spanned by these point sets.*

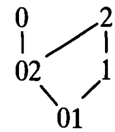
We illustrate the Special Case of Theorem 1.5 by the following two examples, the first of which is directly related to the examples given in the Introduction.

EXAMPLE 1.

Graph and I_0:



poset of reduced types:



subdivision of $\|I\|$:

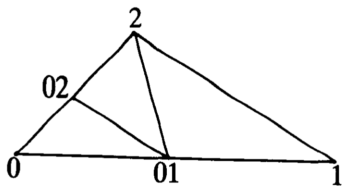
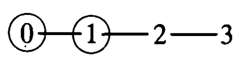


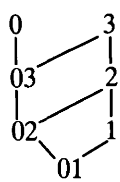
FIG. 4

EXAMPLE 2.

Graph and I_0:



poset of reduced types:



subdivision of $\|I\|$:

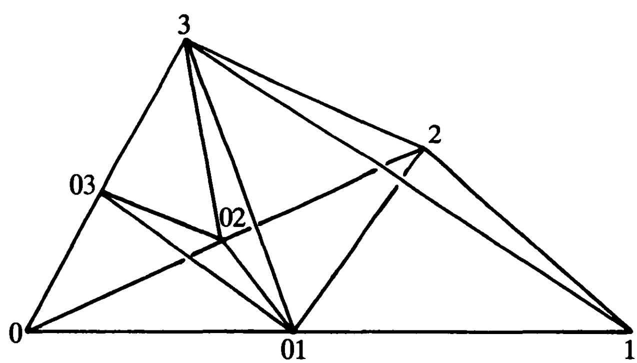


FIG. 5

The figures in Examples 1 and 2 can also be considered as an illustration for the general case of Theorem 1.5: using the numbering of any complex Δ over I, identify each maximal simplex $\|C\|$, for $C \in \mathscr{C}_\Delta$, with $\|I\|$, and subdivide each $\|C\|$ in the same way, as indicated. Obviously, this construction can be carried out for any I-numbered complex and any simplicial subdivision of $\|I\|$. The particular feature of the present construction is the fact that the new complex is always the geometrical realization of a poset, viz. $\mathcal{S}(\Delta, I_0)$. The geometric relevance of this property will be the subject of § 4.

The following proposition collects some basic properties and technicalities about the separation relation $\preccurlyeq$. It will be used without comment throughout the remainder of this paper.

1.7. Proposition. *For a finite graph with vertex set I and some distinguished subset $I_0 \subseteq I$, the separation relation '$\preccurlyeq$' and the corresponding equivalence relation '$J \preccurlyeq K \preccurlyeq J$' on $P(I)$ have the following properties.*

(a) *The equivalence class of each $J \subseteq I$ contains a smallest set J_{red}.*

(b) *The equivalence class of each $J \subseteq I$ contains a largest set $\bar{J}$.*

(c) *The mapping $J \mapsto \bar{J}$ is a monotonous closure operation on $P(I)$, that is, $J \subseteq \bar{J} = \bar{\bar{J}}$, $J \subseteq K \Rightarrow \bar{J} \subseteq \bar{K}$.*

(d) *The mappings $J \mapsto J_{\mathrm{red}}$ and $J \mapsto \bar{J}$ induce bijections, inverse to each other, between the* closed *sets $J = \bar{J}$ and the* reduced *sets $J = J_{\mathrm{red}}$.*

(e) *$J \preccurlyeq K \Leftrightarrow \bar{J} \supseteq \bar{K}$, for any two subsets $J, K \subseteq I$.*

(f) *If $J = \bar{J}$ is closed and $j \in J$, then $J \backslash j$ is closed if and only if $j \in J_{\mathrm{red}}$.*

(g) *J_{red} consists of the* boundary *of J in the graph and of $J \cap I_0$:*

$$J_{\mathrm{red}} = \partial \bar{J} \cup (J \cap I_0),$$

where $\partial \bar{J} := \{ j \in \bar{J} \mid$ there exists an edge $\{j, k\}$ such that $k \notin \bar{J}\}$.

(h) *If J and K are reduced and $J \preccurlyeq K$, then $\bar{K} \backslash K \subseteq \bar{J} \backslash J$.*

Proof. (a) This is identical with 1.2.

(b) This is obviously true if we set $\bar{J} = \{ i \in I \mid J \preccurlyeq i\}$.

(c), (d), (e) These are trivial.

(f) Here $J \backslash j$ is closed if and only if $j \notin \overline{J \backslash j}$, that is, if j and I_0 are not separated by $J \backslash j$.

(g) An element $j \in J$ is in J_{red} if and only if there exists a path $i_0, i_1, \ldots, i_m = j$ such that $i_0 \in I_0$ and $i_t \notin \bar{J}$ for $t < m$.

(h) Let $\bar{K} \subseteq \bar{J}$ (cf. (e)). Applying (g) twice, we conclude that $\bar{K} \cap J \subseteq K$, that is, $\bar{K} \backslash K \subseteq \bar{J} \backslash J$.

Although the statements 1.7(b) to (e) are obvious once $\bar{J}$ is defined, it seems that they have not been used in a systematic way in previous work on shadow geometries. They demonstrate some properties of the poset of reduced sets which are not completely obvious from the definition. For instance, it is clear by (c) that our closed sets form a lattice with respect to inclusion. Furthermore, it is evident by (f) that they form a pure poset of dimension $|I| - 1$, provided that $J_{\mathrm{red}} \neq \varnothing$ for all $J \neq \varnothing$, that is, provided that I_0 meets each connected component of the graph. More generally, any two closed sets J, K with $K \supset J$ can be joined by a flag of closed sets such that the cardinality decreases only by one at each step. To see this, it is sufficient to find a $k \in K \backslash J$ such that $K \backslash k$ is closed, that is, a $k \in K_{\mathrm{red}} \backslash J$, by 1.7(f). Such a k actually exists, for otherwise $K_{\mathrm{red}} \subseteq J$, which implies that $K = \overline{K_{\mathrm{red}}} \subseteq \bar{J} = J$. Some of these properties have been discussed in **[4]**. In a completely different context, the relation between separation in a graph and lattice theory is also discussed in **[24]**.

Proof of 1.3(c). By 1.3(b) it is sufficient to show that each maximal flag of reduced type sets has the cardinality n and the whole of I as its union. By 1.7(d) and (e), and taking into account that $\bar{\varnothing} = \varnothing$, by assumption, it is sufficient to show the following: if $J_1 \supset J_2 \supset \ldots \supset J_d$ is a maximal flag of closed sets, then $J_t \setminus J_{t+1}$ consists of a single element j_t, and this j_t is in $(J_t)_{\text{red}}$. We have already observed that this follows from 1.7(f).

Finally, I wish to formulate a generalization of the special case 1.6 of Theorem 1.5 which will not be used in the sequel. It is due to A. Dress who also pointed out the usefulness of the closed type sets for the study of the reduced type sets.

1.8. THEOREM. *Let I be a finite set, and let $J \mapsto \bar{J}$ be a monotonous closure operator on $P(I)$ such that, for each $J \subseteq I$, there exists a smallest J_{red} such that $\overline{J_{\text{red}}} = \bar{J}$. Define, for such 'reduced' sets $J = J_{\text{red}}$ a partial ordering by $J \leqslant K \Leftrightarrow \bar{J} \supseteq \bar{K}$. If in addition $\bar{\varnothing} = \varnothing$, or, equivalently, $\{i\}_{\text{red}} = \{i\}$ for all $i \in I$, then all the point sets $p_{J_1}, \ldots, p_{J_d}$, where $J_1 < J_2 < \ldots < J_d$ runs through all flags of non-empty reduced subsets of I, are affinely independent, and the standard simplex $\|I\|$ is the disjoint union of the open simplices spanned by these $p_{J_1}, \ldots, p_{J_d}$.*

The proof uses induction on $|I|$ and the following two facts.

(1) For $\varnothing \neq J \subseteq I$, the simplex $\|I\|$ is a cone with vertex p_J and basis the union of all panels (faces of codimension 1) not containing p_J (cf. § 3 below).

(2) The restriction of one of the closure operators described in 1.8 to one of its closed sets $\bar{J} = J$ again has the same properties, and the K_{red} belonging to this new operator are the old ones.

2. *Shadow geometries of chamber systems and their geometrical realizations*

In the first section of this paper, we started from certain abstract simplicial complexes whose simplices could be thought of as the 'flags' of a more or less unspecified 'geometry'. We defined the shadow geometries of these complexes as certain abstract posets. Then an interpretation of these shadow geometries was given which refers to the geometrical realizations of the complexes and posets and not to the abstract objects alone.

In the present (intermediate) section, I introduce geometrical realizations $E(\mathscr{C}, M)$ of chamber systems which are constructed from an (abstract, combinatorial) chamber system $\mathscr{C}$ together with a 'space with panels' M. This M serves as a typical chamber of the realization. Such spaces in particular occur in connection with reflection groups, for instance, discrete isometry groups of a spherical, Euclidean or hyperbolic space which are generated by reflections in hyperplanes. Already in the hyperbolic case, the chambers can be much more complicated than simplices, and in general, they can be prescribed more or less arbitrarily (cf. [**14**, **37**]).

Independently of the introduction of the spaces $E(\mathscr{C}, M)$, I then define the shadow geometries of a chamber system $\mathscr{C}$ as certain abstract posets, again. Finally, in Proposition 2.3(b) a certain subposet, depending on a given M, of such a shadow geometry $\mathscr{S}$ is realized by a family of subsets of $E = E(\mathscr{C}, M)$. That is, the partial ordering between the members of $\mathscr{S}$ is given by the inclusion of the corresponding subsets of E. This collection of subsets of E is called the *geometrical realization in E* of the shadow geometry $\mathscr{S}$. These realizations must not be confused with the geometrical realizations of chamber systems (i.e. with E itself). They have also to be distinguished from the standard geometrical realizations of the abstract posets $\mathcal{S}$. The relation between the realization in E and the standard realization of some $\mathcal{S}$ is the subject of the following section, § 3.

A *chamber system* is a set $\mathscr{C}$ together with a family $\sim_i$, with $i \in I$, of equivalence relations on $\mathscr{C}$, indexed by some finite 'type set' I. One may, in particular, think of the set of chambers of a numbered complex or of a reflection group, where two chambers are i-equivalent if they have the panel of cotype i in common. Also for general chamber systems, we adopt the term *i-adjacent* for two chambers which are i-equivalent but distinct.

I recall that the general theory of reflection groups is presupposed for this paper, and I only repeat the definition of the chambers and their faces. With a reflection group W on some space E, there is associated a family $\mathscr{H}$ of *reflecting hyperplanes* of W. These are closed subspaces whose complement consists of two connected components, and to each $H \in \mathscr{H}$ there belongs an involution $s_H \in W$ with fixed point set H. Now consider the following equivalence relation on the points of E: two points are equivalent if, for each reflecting hyperplane H, they both lie in H or both in the same open halfspace determined by H. The topological closures of these equivalence classes are the 'faces', the maximal faces are called chambers, and the faces of codimension 1 are called panels. Each chamber C is a fundamental domain for W; that is, for fixed C, the mapping $w \mapsto wC$ from W into the set of chambers $\mathscr{C}$ is bijective, and from $p, q \in C$, $w \in W$ and $wp = q$ it follows that $p = q$. The type set $I = I_W$ can be defined independently of the choice of a chamber as the set of W-orbits of panels. Each chamber contains precisely one panel of each type.

In addition to a chamber system $\mathscr{C}$ over I we now give ourselves a *space with panels* over I. This is an object $(M, M^i, i \in I)$, where M is some set and the M^i are subsets, the 'panels', of M. For $J \subseteq I$, we set

$$M^J := \bigcap_{j \in J} M^j,$$

and we shall occasionally call these sets the *faces* of M. *A priori* no assumptions are made on the intersections of the M^i. Later, M will be a topological space and the M^i closed subspaces, and the most important case is that M is a compact convex polytope and the panels are the faces of codimension 1.

The *geometrical realization* of $\mathscr{C}$ with typical chamber M, denoted by $E(\mathscr{C}, M) = E(\mathscr{C}, \sim_i, M, M^i, i \in I)$ is the quotient of $\mathscr{C} \times M$ by the coarsest equivalence relation which, for any two i-adjacent chambers C and D, identifies the subsets $C \times M^i$ and $D \times M^i$ of $\mathscr{C} \times M$ in the obvious way. The projection $\mathscr{C} \times M \to E(\mathscr{C}, M)$ will be written as $(C, p) \mapsto [C, p]$. The equivalence relation introduced on $\mathscr{C} \times M$ can be described more explicitly as follows. A *gallery* (in the chamber system $\mathscr{C}$) is a sequence $(C_0, C_1, \dots, C_m; i_1, \dots, i_m)$ such that $C_t \in \mathscr{C}$, $i_t \in I$ and $C_{t-1} \sim_{i_t} C_t$ for all t. Two chambers C and D are *J-equivalent*, $C \sim_J D$, for some subset $J \subseteq I$, if there exists a gallery as above such that $C_0 = C$, $C_m = D$, and $i_t \in J$ for all t. For a point $p \in M$ set

$$J(p) := \{j \in I \mid p \in M^j\}.$$

Using these notations, we find that the following holds for $C, D \in \mathscr{C}$, $p, q \in M$:

$$[C, p] = [D, q] \quad \Leftrightarrow \quad p = q,\ C \sim_{J(p)} D.$$

In particular $J(x) := J(p)$ is well defined for all $x = [C, p] \in E := E(\mathscr{C}, M)$, and every $C \times M$ is mapped injectively into E. Often, the image $[C, M]$ of some $C \times M$ in E will be denoted by C, again, and called a *chamber* in E. The subsets

$$C^i := [C, M^i] \subseteq E$$

are the *panels* of $C \subseteq E$, the subsets

$$C^J := \bigcap_{j \in J} C^j, \quad \text{for } J \subseteq I,$$

are the *faces* of C. We have

$$C^J = \{x \in C \mid J(x) \supseteq J\} = [C, M^J].$$

Sometimes it is more convenient to use the faces

$$C_K := C^{I \setminus K};$$

therefore we also introduce the notation

$$I(x) := I \setminus J(x), \quad \text{for } x \in E,$$

and call $I(x)$ the *type* of x. Accordingly, C_K is said to be of type K. We have

$$C_K = \{x \in C \mid I(x) \subseteq K\}.$$

Notice that C^J only depends on the J-equivalence class of C. Type sets J such that $M^J \neq \varnothing$ and thus $C^J \neq \varnothing$ for all C are called *spherical*, their complements *cospherical*.

If $\mathscr{C}$ is the chamber system of a numbered complex Δ, $M = \|I\| \subseteq \mathbb{R}^I$ is the standard simplex, and the $M^i := \|I \setminus i\|$ are its panels, then there is an obvious surjective mapping from $E(\mathscr{C}, M)$ onto the geometrical realization $\|\Delta\|$, which maps each chamber $[C, M]$ of $E(\mathscr{C}, M)$ onto the geometrical simplex spanned by C. This mapping is bijective if and only if any two chambers C, D having the face of type $I \setminus J$ in common are J-equivalent, that is, if Δ is strongly connected.

If $\mathscr{C}$ is the chamber system of a reflection group W on some space E' and $M \subseteq E'$ a chamber of W, the M^i its panels, then $E(\mathscr{C}, M)$ can be identified with E' by observing that the mapping $\mathscr{C} \times M \to E'$, $(wM, p) \mapsto wp$ factors bijectively over $E(\mathscr{C}, M)$. This comes from the fact that the chambers containing some C^J form one orbit under the stabilizer of C^J and therefore are precisely the chambers J-equivalent to C. This reconstruction of E' is classical; see **[25]**.

We have just recorded that for the chamber systems of strongly connected complexes or of reflection groups, the isomorphism between $E(\mathscr{C}, M)$ and $\|\Delta\|$, respectively E', relies on the fact that in Δ, respectively E', all chambers containing a fixed face of type $I \setminus J$ are actually J-equivalent to each other. Our next remark gives a condition under which this last property holds in spaces of the form $E(\mathscr{C}, M)$. Before formulating this precisely, we notice that, in general, a subset $\mathscr{A} \subseteq \mathscr{C}$ may be a J-equivalence class for more than one type set J. Therefore, we shall always consider a J-class as a pair $(\mathscr{A}, J)$ where $J \subseteq I$, and $\mathscr{A} \subseteq \mathscr{C}$ is an equivalence class with respect to the relation $\sim_J$.

2.1. REMARK. Let M be *non-degenerate* in the sense that each non-empty M^J contains an *interior point* p, that is, a point p such that $J(p) = J$. Then, for every chamber system $\mathscr{C}$, every chamber $C \in \mathscr{C}$ and every spherical type set J, the

J-class of C is determined by C^J:

$$C^J = D^K,\ C^J \neq \varnothing \neq D^K,\ J, K \subseteq I,\ C, D \in \mathscr{C} \ \Rightarrow\ J = K,\ C \sim_J D.$$

The product of two spaces with panels over I_1 and I_2, respectively, is a space with panels over $I_1 \times I_2$ in a natural way. It is non-degenerate if and only if both factors are non-degenerate.

Before introducing the shadow geometries of a chamber system $\mathscr{C}$ and their realizations, we finally have to define the *basic graph* of $\mathscr{C}$. It has the vertex set I, and i and j are not connected if the corresponding adjacency relations commute, that is, if for any gallery of the form $(C, D, E\,; i, j)$, $C \neq D \neq E$, there exists a chamber $D' \neq C, E$ such that $(C, D', E\,; j, i)$ is a gallery as well. If $\mathscr{C}$ comes from a reflection group, this is equivalent to the commutativity of the corresponding generators s_i and s_j. If $\mathscr{C}$ is the chamber system of a strongly connected numbered complex, then the basic graph of $\mathscr{C}$ coincides with the basic graph of the complex. The analogue of the 'main theorem about the basic graph', Proposition 1.1, in the case of chamber systems is a triviality.

2.2. Proposition. *Let $(\mathscr{C}, \sim_i, i \in I)$ be a chamber system, $J \subseteq I$ and $(\mathscr{A}, J)$ a J-class, suppose that $J = J_1 \dot{\cup} J_2$, where J_1 and J_2 are not connected in the basic graph. Then every J_1-class contained in $\mathscr{A}$ has a non-empty intersection with every J_2-class contained in $\mathscr{A}$.*

We can now define the *shadow geometry* of a chamber system $(\mathscr{C}, \sim_i, i \in I)$ with respect to a type set $I_0 \subseteq I$, which we shall call the I_0-shadow geometry of $\mathscr{C}$, for short:

$$\begin{aligned}\mathscr{S}(\mathscr{C}, I_0) &:= \mathscr{S}(\mathscr{C}, \sim_i, i \in I, I_0)\\ &:= \{(\mathscr{A}, I\backslash J) \mid J \subseteq I \text{ is } I_0\text{-reduced in the basic graph},\\ &\qquad \mathscr{A} \text{ an } (I\backslash J)\text{-class}\}.\end{aligned}$$

Furthermore, we set

$$(\mathscr{A}, I\backslash J) \preccurlyeq (\mathscr{B}, I\backslash K) \quad :\Leftrightarrow \quad J \preccurlyeq K \text{ and } \mathscr{A} \cap \mathscr{B} \neq \varnothing,$$

where the '$\preccurlyeq$' on the right is the separation relation with respect to I_0, as before.

If $\mathscr{C}$ comes from a strongly connected numbered complex Δ, then the $(I\backslash J)$-classes correspond bijectively to the simplices of type J by

$$\Delta \ni A \leftrightarrow \mathscr{C}(A) := \{C \in \mathscr{C} \mid C \supseteq A\} \subseteq \mathscr{C},$$

and $A \cup B$ exists if and only if $\mathscr{C}(A) \cap \mathscr{C}(B) \neq \varnothing$, by definition. Therefore, the present definition of $\mathscr{S}(\mathscr{C}, I_0)$ is a translation of the definition given in § 1 for complexes.

Following the proof of 1.2, one can easily derive from 2.2 that $\preccurlyeq$ is a partial ordering on $\mathscr{S}(\mathscr{C}, I_0)$. This is also a corollary of the following proposition, taking for M the standard simplex $\|I\|$.

2.3. Proposition. *Let a chamber system $\mathscr{C}$ and a non-degenerate space with panels M with common type set I be given. Fix a set of types $I_0 \subseteq I$. For each class*

$(\mathscr{A}, I\backslash J)$ in $\mathscr{C}$ consider the subset

$$F(\mathscr{A}, I\backslash J) := \bigcup_{C \in \mathscr{A}} C_{\bar{J}}$$

of the geometrical realization $E(\mathscr{C}, M)$ (for $\bar{J}$, cf. 1.7).

(a) *If J is reduced with respect to I_0 and $\mathscr{A}'$ is a $(\bar{J}\backslash J)$-class contained in $\mathscr{A}$, then*

$$F(\mathscr{A}, I\backslash J) = \bigcup_{C \in \mathscr{A}'} C_{\bar{J}}.$$

(b) *If, furthermore, $(\mathscr{B}, I\backslash K)$ is an $(I\backslash K)$-class, J, K are reduced with respect to I_0 and $\bar{J}$, $\bar{K}$ are cospherical, then the following is true:*

$$F(\mathscr{A}, I\backslash J) \supseteq F(\mathscr{B}, I\backslash K) \quad \Leftrightarrow \quad (\mathscr{A}, I\backslash J) \leqslant (\mathscr{B}, I\backslash K).$$

Proof. (a) Let $C \in \mathscr{A}$ be given. Choose some $C' \in \mathscr{A}'$; then $C \sim_{I\backslash J} C'$. By Proposition 1.7(g), $I\backslash \bar{J}$ and $\bar{J}\backslash J$ are not connected in the basic graph. Therefore, there exists a chamber C'' such that $C \sim_{I\backslash \bar{J}} C'' \sim_{\bar{J}\backslash J} C'$. Then $C'' \in \mathscr{A}'$, and $C_{\bar{J}} = C''_{\bar{J}}$, as desired.

(b) Suppose that $F(\mathscr{B}, I\backslash K) \subseteq F(\mathscr{A}, I\backslash J)$. Choose some $D \in \mathscr{B}$ and an interior point p in $M_{\bar{K}}$. There exists $C \in \mathscr{A}$ such that $[D, p] \in C_{\bar{J}}$. This means firstly that $p \in M_{\bar{J}}$, that is, $\bar{K} = I(p) \subseteq \bar{J}$, and secondly that $C \sim_{I\backslash \bar{K}} D$, *a fortiori* $C \sim_{I\backslash K} D$, and therefore $C \in \mathscr{B}$, so that indeed $\mathscr{A} \cap \mathscr{B} \neq \varnothing$. Conversely, suppose that $\bar{K} \subseteq \bar{J}$ and $\mathscr{A} \cap \mathscr{B} \neq \varnothing$. By (a), we have

$$F(\mathscr{B}, I\backslash K) = \bigcup_{D' \in \mathscr{B}'} D'_{\bar{K}},$$

for some $(\bar{K}\backslash K)$-class $\mathscr{B}'$ such that $\mathscr{A} \cap \mathscr{B}' \neq \varnothing$. By Proposition 1.7(h), $\bar{K}\backslash K \subseteq \bar{J}\backslash J$ holds, and therefore $\mathscr{B}' \subseteq \mathscr{A}$, and therefore

$$F(\mathscr{B}, I\backslash K) \subseteq \bigcup_{D \in \mathscr{A}} D_{\bar{K}} \subseteq \bigcup_{D \in \mathscr{A}} D_{\bar{J}} = F(\mathscr{A}, I\backslash J).$$

The non-empty $F(\mathscr{A}, I\backslash J)$ described in Proposition 2.3 are sometimes called *cells*, the family of all cells is called the *geometrical realization of the shadow geometry $\mathscr{S}(\mathscr{C}, I_0)$ in the space $E(\mathscr{C}, M)$*. Strictly speaking, only a part of $\mathscr{S}(\mathscr{C}, I_0)$ which should systematically be called $\mathscr{S}(\mathscr{C}, I_0; M)$ is represented or realized in $E(\mathscr{C}, M)$. If one identifies according to 2.1 an $(I\backslash J)$-class $\mathscr{A}$ with the face $C_J \subseteq E(\mathscr{C}, M)$, for some $C \in \mathscr{A}$, then the corresponding cell $F(C_J) := F(\mathscr{A}, I\backslash J)$ is the union of all faces of type $\bar{J}$ which contain C_J.

We illustrate this by the simple example of the shadow geometry,

realized in the Euclidean plane E'. The original simplicial complex is the barycentric subdivision of the regular square tessellation, the vertices of types 0, 1, 2 are the vertices, midpoints of the edges, midpoints of the faces, respectively, of the square tessellation. The family of cells $F(\mathscr{A}, I\backslash J)$ looks as shown in Fig. 6.

The fact that the $F(\mathscr{A}, I\backslash J)$ again form a 'tessellation' of E (in a precise sense, to be defined below) is the subject of the following § 3.

So far, the typical chamber M of a space $E(\mathscr{C}, M)$ has had to be just a set. In the applications, M usually is a topological space. In this case, $E(\mathscr{C}, M)$ is

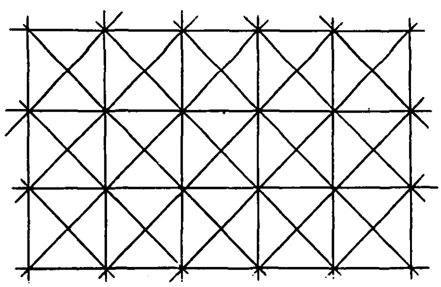


FIG. 6

equipped with the quotient topology of the product topology on $\mathscr{C} \times M$, where $\mathscr{C}$ is equipped with the discrete topology. This topology on $E(\mathscr{C}, M)$ coincides with the weak topology with respect to the chambers, each chamber being equipped with the given topology of M via the canonical bijection. We shall always assume that the panels M^i, and therefore all the faces M_J, are closed in M. Then (and only then) all chambers are closed in $E(\mathscr{C}, M)$, and the topology of a chamber induced from $E(\mathscr{C}, M)$ coincides with the original topology of the chamber. If M is Hausdorff, then every $E(\mathscr{C}, M)$ is Hausdorff, no matter what $\mathscr{C}$ looks like. If $\mathscr{C}$ is the chamber system of a properly discontinuous reflection group on E' and M is a chamber of that group, then the bijection from $E(\mathscr{C}, M)$ onto E' given above is a homeomorphism, by [**25**, Chapter III, § 3].

3. *Interpreting the shadow geometries as tessellations*

In the second section of this paper we started from an abstract chamber system $\mathscr{C}$ and a space with panels M, and we have constructed a space $E = E(\mathscr{C}, M)$ by pasting copies of M according to the data given in $\mathscr{C}$. Then certain decompositions of such a space were defined, each of which was determined by a specified subset I_0 of the common type set I of $\mathscr{C}$ and M. These decompositions could be regarded as geometrical realizations of certain abstract shadow geometries of $\mathscr{C}$.

In the present section I shall prove that, provided that the model chamber M is a compact convex polytope which is non-degenerate in the sense of § 2, and assuming a certain compatibility between M and the basic graph of $\mathscr{C}$, each of our decompositions of E possesses a 'barycentric subdivision'. More precisely, it is shown that the standard geometrical realization of the abstract poset of all cells of the decomposition can be regarded as a subdivision of the original decomposition of E into the chambers and their faces. The reader should notice that in the case of polytopes, our notion of non-degenerateness coincides with the well-studied [**13, 29, 36**] notion of a simple polytope, that is, all vertex figures are simplices.

The following definition is due to A. Dress [**16**]:

DEFINITION. Let E be a topological space and $\mathscr{F}$ a family of subsets of E. We say that $\mathscr{F}$ admits a *barycentric subdivision*, or that $\mathscr{F}$ is a *tessellation* of E, if the following conditions hold:

(a) $\mathscr{F}$ is a pure poset with respect to inclusion;

(b) there exists a homeomorphism from the standard geometrical realization $\|\mathscr{F}, \subseteq\|$ of the poset $\mathscr{F}$ onto E which, for every $F \in \mathscr{F}$, maps the subcomplex $\|F_*\| := \|\{G \in \mathscr{F} \mid G \subseteq F\}\|$ onto F.

In the case of two-dimensional manifolds, this definition agrees with the definition generally used (except for the fact that in our definition, the intersection of two members of $\mathcal{F}$ need not be connected; cf. **[20]**). The present paper shows that the definition is also reasonable in higher dimensions.

Notice that the geometrical realization of $(\mathcal{F}, \subseteq)$ (or any pure poset of dimension n) has the following properties. It comes along with a filtration $\|\mathcal{F}\|_d := \|\mathcal{F}_0 \cup \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_d\|$, for $d = 0, \ldots, n$, where $\mathcal{F}_d \subseteq \mathcal{F}$ are the elements of dimension d, and each d-dimensional 'cell' $\|F_*\|$ $(F \in \mathcal{F}_d)$ is a cone with basis its boundary $\partial \|F_*\| := \|F_*\| \cap \|\mathcal{F}\|_{d-1} = \|F_* \setminus \{F\}\|$, that is, the union of the segments $\overline{qF}$, for $q \in \partial \|F_*\|$, which have pairwise only the vertex F in common. If $\|\mathcal{F}, \subseteq\|$ is a manifold, then the $\partial \|F_*\|$ are necessarily spheres, and $\|\mathcal{F}\|_{d+1}$ is constructed from $\|\mathcal{F}\|_d$ by adjoining the $(d+1)$-cells $\|F_*\|$, where $F \in \mathcal{F}_{d+1}$. So a tessellation of a manifold is in particular a CW-complex (cf. **[16, 17]**).

In the following main theorem of this paper and its proof, the notions J_{red} and $\bar{J}$ are relative to the basic graph of the chamber system and the distinguished type set I_0, as in § 2 and in Proposition 1.7.

3.1. THEOREM. *Let $(\mathcal{C}, \sim_i, i \in I)$ be a chamber system, $(M, M^i, i \in I)$ a space with panels, and I_0 a non-empty subset of the type set I. The following conditions imply that the geometrical realization in $E(\mathcal{C}, M)$ of the I_0-shadow geometry of $\mathcal{C}$ is a tessellation of $E(\mathcal{C}, M)$:*

(a) *$\mathcal{C}$ is strongly connected, that is, from $C \sim_J D \sim_K C$, $C, D \in \mathcal{C}$, $J, K \subseteq I$ it follows that $C \sim_{J \cap K} D$;*

(b) *M is a compact convex polytope, and M^i are its faces of codimension* 1, *and M is a non-degenerate;*

(c) *if J is a cospherical (with respect to M) type set, then J_{red} is also cospherical;*

or

(c′) (i) *I_0 is cospherical,*

(ii) *the basic graph of $\mathcal{C}$ is* compatible *with the spherical type sets in the following sense: if J, K are spherical and not connected in the basic graph, then also $J \cup K$ is cospherical.*

Proof. We denote by $\mathcal{F}$ the geometrical realization in question, this is the set of all cells $F(\mathcal{A}, I \setminus J)$ considered in Proposition 2.3(b).

I first show that (c) follows from (c′), and I shall use in the remainder of the proof only the conditions (a), (b), (c). The stronger condition (c′) has been included in the theorem because it subdivides the condition (c) into two parts which can often be verified more easily. Notice that (i) is a special case of (c), and (ii) is independent of the particular I_0.

Let J be cospherical; then $\bar{J}$ is *a fortiori* cospherical. By Proposition 1.7(g), $I \setminus \bar{J}$ and $\bar{J} \setminus J_{\text{red}}$ are not connected in the basic graph. By (c′) (ii), it is therefore sufficient to show that $\bar{J} \setminus J_{\text{red}}$ is spherical. But $\bar{J} \cap I_0 \subseteq J_{\text{red}}$, that is, $\bar{J} \setminus J_{\text{red}} \subseteq I \setminus I_0$, and $I \setminus I_0$ is spherical by the assumption (c′) (ii).

The following first step of the proper proof relies on the assumption that M is non-degenerate in the sense of § 2. This implies that $\dim(M^J) = \dim(M) - |J|$ for each spherical type set J.

(1) *If $\bar{J}$ is cospherical and minimal among the closed cospherical type sets, then $M_{\bar{J}}$ is a point and $\bar{J} = J_{\text{red}}$.*

We first notice that the second part of the claim follows from the first. For, if $M_{\bar{J}}$ is a point, then $\bar{J}$ is minimal among the cospherical type sets; on the other hand, the subset $J_{\text{red}} \subseteq \bar{J}$ is also cospherical, by (c). Now assume that $M_{\bar{J}}$ contains more than one point, and let $M_{\bar{J}} \cap M^k$, with $k \in K$, denote the panels of $M_{\bar{J}}$. Then $M_{\bar{J}} \cap M^K = \emptyset$, that is, $(I \backslash \bar{J}) \cup K$ is not spherical. This implies that $K \nsubseteq I \backslash J_{\text{red}}$. Choose a $k \in K \cap J_{\text{red}}$. Then, on the one hand, $\bar{J} \backslash k$ is closed, and, on the other hand, $C_{\bar{J} \backslash k} = C_{\bar{J}} \cap C^k \neq \emptyset$, so $\bar{J} \backslash k$ is a cospherical. This implies that $\bar{J}$ is not minimal among the closed, cospherical type sets, which was to be shown.

(2) *The closed cospherical type sets form a pure poset of dimension $n =$ $\dim(M)$.*

All J such that M_J is a point have the same cardinality $|I| - \dim(M)$. Therefore, (2) is an immediate consequence of (1), and the considerations following 1.7.

(3) *The flags of the shadow geometry $\mathcal{S}(\mathscr{C}, I_0)$ are all of the form $(\mathscr{A}_1, I \backslash J_1), \ldots, (\mathscr{A}_s, I \backslash J_s)$, where $J_1 < J_2 < \ldots < J_s$ and the intersection of all $\mathscr{A}_t$ is non-empty.*

To prove this, let $s \geqslant 3$. Choose some $C \in \mathscr{A}_1 \cap \mathscr{A}_2$ by assumption, and some $C' \in \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_s$ by induction. Since $J_1 < J_2 < \bar{J}_3$, it follows that J_1 and J_3 are separated by J_2 (see the proof of 1.2). So there exist type sets $\bar{J}_1 \supseteq J_1$, $\bar{J}_3 \supseteq J_3$ such that $I \backslash J_2 = I \backslash \bar{J}_1 \,\dot{\cup}\, I \backslash \bar{J}_3$, and $I \backslash \bar{J}_1$ and $I \backslash \bar{J}_3$ are not connected. Now consider the $(I \backslash \bar{J}_1)$-class $\mathscr{B}_1$ of C and the $(I \backslash \bar{J}_3)$-class $\mathscr{B}_3$ of C'. We have $\mathscr{B}_1 \subseteq \mathscr{A}_1 \cap \mathscr{A}_2$, because $I \backslash \bar{J}_1 \subseteq I \backslash J_1 \cap I \backslash J_2$, and $\mathscr{B}_3 \subseteq \mathscr{A}_2 \cap \mathscr{A}_3 \cap \ldots \cap \mathscr{A}_s$, because

$$I \backslash \bar{J}_3 \subseteq I \backslash J_2 \cap I \backslash \bar{J}_3 \subseteq (I \backslash J_2) \cap (I \backslash J_3) \cap \ldots \cap (I \backslash J_s).$$

Applying Proposition 2.2 to $\mathscr{A} = \mathscr{A}_2$ gives $\mathscr{B}_1 \cap \mathscr{B}_3 \neq \emptyset$, so *a fortiori*

$$\mathscr{A}_1 \cap \mathscr{A}_2 \cap \ldots \cap \mathscr{A}_s \neq \emptyset.$$

The properties (2) and (3) immediately imply that $\mathscr{F}$ is a pure poset, so it satisfies Condition (a) in the definition of a tessellation. In order to prepare the proof of (b), that is, the construction of the barycentric subdivision of $\mathscr{F}$, we introduce a piecewise linear structure on E. For this, we verify that for any two points x, y lying in one chamber C, so $x = [C, p]$, $y = [C, q]$, $p, q \in M$, and for $r \in \mathbb{R}$, $0 \leqslant r \leqslant 1$, the definition

$$(1-r)x + ry := [C, (1-r)p + rq]$$

makes sense. So let C' be such that $[C, p] = [C', p]$, $[C, q] = [C', q]$. For $0 < r < 1$, we obviously have $J((1-r)p + rq) = J(p) \cap J(q)$, and the strong connectedness of $\mathscr{C}$ indeed implies that $[C, (1-r)p + rq] = [C', (1-r)p + rq]$. Each cell $F = F(\mathscr{A}, I \backslash J)$ possesses a 'barycentre' $x(F)$, namely the point $[C, p_J]$ where $C \in \mathscr{A}$ is arbitrarily chosen and p_J is the barycentre of M_J.

If $F_0 \subset F_1 \subset \ldots \subset F_d$ is a flag in $\mathscr{F}$, then, by (3), there exists a chamber $C \subseteq E$ which contains all $x(F_i)$. So we can consider the simplex spanned in E by all the

$x(F_t)$:

$$\Sigma(F_0, \ldots, F_d) := \Sigma(x(F_0), \ldots, x(F_d)) = \left\{ \sum r_t x(F_t) \middle| \ r_t \geq 0, \sum r_t = 1 \right\},$$

and there exists a simplicial map

$$\varphi: \ \|\mathcal{F}\| \to E$$

given by $\varphi(F) = x(F)$. I shall now prove that this φ is a homeomorphism mapping each $\|F_*\|$ onto F.

Following the remarks after the definition of a tessellation, we first show that each F is a cone over a suitably defined 'boundary' ∂F. We set $E_d := \bigcup \mathcal{F}_d := \bigcup_{F \in F_d} F$ $(d = 1, \ldots, n)$, and for $F \in \mathcal{F}_d$, we set

$$\partial F := \bigcup \{G \in \mathcal{F} | \ G \subset F\} = \bigcup (F_* \setminus \{F\}).$$

More precisely, ∂F should be called the (combinatorial) $\mathcal{F}$-boundary of F. We want to show the following:

(4) $F = \bigcup \{\overline{x(F)y} | \ y \in \partial F\}$, *for each* $F \in \mathcal{F}$, *and the segments* $\overline{x(F)y}$ *have pairwise only the point* $x(F)$ *in common. That is, each* $x \in F$ *has a representation* $x = (1 - r)y + rx(F)$, *for a unique* $r \in [0, 1]$ *and* $y \in \partial F$, *unique if* $r \neq 1$.

The proof of (4) uses the fundamental property of a compact convex polytope being a cone with vertex chosen arbitrarily from its points and with basis the union of all panels not containing this point. In particular,

$$M_J = \bigcup \left\{ \overline{p_J q} \middle| \ q \in \bigcup_{j \in J} M_{\bar{J} \setminus j} \right\},$$

and therefore

$$C_{\bar{J}} = \bigcup \left\{ \overline{x(F)y} \middle| \ y \in \bigcup_{j \in J} C_{\bar{J} \setminus j} \right\}.$$

The statement (4) is an immediate consequence once we know the following:

(5) $\partial F = \bigcup \{C_{\bar{J} \setminus j} | \ C \in \mathcal{A}, j \in J\}$, *for* $F = F(\mathcal{A}, I \setminus J)$.

For the proof of (5), we first notice that $C_{\bar{J} \setminus j}$, for $C \in \mathcal{A}$, $j \in J$, is contained in $F(\mathcal{A}', I \setminus J') \subset F$, where $J' := (J \setminus j)_{\mathrm{red}}$ and $\mathcal{A}'$ is the $(I \setminus J')$-class containing C. So $C_{\bar{J} \setminus j}$ is indeed contained in ∂F. Conversely, consider some $F' = F(\mathcal{B}, I \setminus K) \subseteq \partial F$, where K is cospherical, reduced, $\bar{K} \subset \bar{J}$, $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. Choose a $(\bar{K} \setminus K)$-class $\mathcal{B}'$ contained in $\mathcal{B}$ and such $\mathcal{A} \cap \mathcal{B}' \neq \emptyset$, that is, $\mathcal{B}' \subseteq \mathcal{A}$. By Proposition 2.3(a),

$$F' = \bigcup \{C_{\bar{K}} | \ C \in \mathcal{B}'\} \subseteq \bigcup \{C_{\bar{K}} | \ C \in \mathcal{A}\} \subseteq \bigcup \{C_{\bar{J} \setminus j} | \ C \in \mathcal{A}\}$$

for some appropriate $j \in J$, as desired.

From (4) it immediately follows, by induction on d, that

$$F = \{\Sigma(F_0, F_1, \ldots, F_{d-1}, F) | \ F_0 \subset F_1 \subset \ldots \subset F_{d-1} \subset F\},$$

that is, $F = \varphi \ \|F_*\|$, as soon as we know that $F \in \mathcal{F}_0$ consists of the single point $x(F)$, only. This follows from (1) and Proposition 2.3(a).

In order to derive the injectivity of φ, we first show the following:

(6) *If $F, G \in \mathcal{F}$, with $F \nsubseteq G$, then $F \cap G \subseteq \partial F$. In particular, $\partial F = F \cap E_{d-1}$.*

For the proof, let $F = F(\mathcal{A}, I\backslash J)$, $G = F(\mathcal{B}, I\backslash K)$. Choose a point $x \in F \cap G$, so $x \in C_{\bar{J}} \cap D_{\bar{K}}$ for some $C \in \mathcal{A}$, $D \in \mathcal{B}$. Notice that $C_{I(x)} = D_{I(x)}$. We want to show that $x \in C_{\bar{J}\backslash j}$ for an appropriate $j \in J$. If this is not the case, then $x \notin C^j$, for $j \in J$, that is, $J \subseteq I(x)$. Then $J \subseteq \bar{K}$, so $\bar{J} \subseteq \bar{K}$, and $C_J = D_J$, so $D \in \mathcal{A}$. So altogether $F \subseteq G$ holds, a contradiction.

Property (6) means that, for $x \in E_d \backslash E_{d-1}$, there is a unique $F \in \mathcal{F}_d$ containing F. From this fact and (4) it follows easily, by induction on d, that φ is injective on all $\|\mathcal{F}\|_d$ and therefore injective on $\|\mathcal{F}\| = \|\mathcal{F}\|_n$.

We finally have to show that φ is a homeomorphism. For every chamber C, the union Z_C of all maximal simplices in $\|\mathcal{F}\|$ which are mapped into C, is mapped bijectively and therefore, being compact, homeomorphically onto C. So we only need to know that $\|\mathcal{F}\|$ has the weak topology with respect to the Z_C. This is trivial.

As a first application of Theorem 3.1, we consider the situation of Theorem 1.5. That is, $\mathcal{C}$ is the chamber system of a strongly connected I-numbered complex Δ and M is the standard simplex $\|I\|$. Then the chambers J-equivalent to a fixed chamber C are precisely the chambers which contain the simplex of type $I\backslash J$ contained in C. Therefore, $\mathcal{C}$ is strongly connected in the sense of (a). The cospherical type sets are precisely the non-empty subsets of I. So the assumption (c) of 3.1 is fulfilled as soon as we assume that J being non-empty implies that J_{red} is non-empty. But this is equivalent to the assumption made in 1.5 that I_0 meets every connected component of the basic graph. So 1.5 is proved. The more precise statement at the end of 1.5 is not formulated in 3.1, but it is easily read off from the proof of 3.1.

The following second corollary of Theorem 3.1 is the main motivation for our general framework of chamber systems.

3.2. COROLLARY. *Let W be a fixed point free reflection group on a sphere E, or a reflection group with compact chambers on a Euclidean or hyperbolic space E. Let I_0 be a subset of the type set I of W which meets every connected component of the Coxeter graph and such that the subgroup corresponding to $I\backslash I_0$ is finite. Then the geometrical realization in E of the I_0-shadow geometry of the chamber system of W is a tessellation of E.*

Proof. First, remember the identification of E with the appropriate $E(\mathcal{C}, M)$, given in § 2. The chamber system of W is strongly connected, by the same reason (with 'face' replacing 'simplex') as in the simplicial case treated above.

In the fixed point free spherical case, a chamber is a (spherical) simplex; therefore we are in the simplicial case (Theorem 1.5). In the Euclidean case, a compact chamber is a product of simplices; in particular, it is non-degenerate. Also in the hyperbolic case, a chamber is a convex polytope (regarded as space with panels; in a suitable model it is even an ordinary polytope). By [**1**, Theorem 1; cf. **44**, Theorem 7], these polytopes are non-degenerate. (The reason is that their vertex figures are spherical polytopes with angles at most $\frac{1}{2}\pi$, and therefore are spherical simplices.) In the Euclidean and hyperbolic cases, the spherical type sets are precisely those for which the corresponding subgroup of W is finite. Therefore the assumptions (c′) (i) and (ii) of the Theorem 3.1 are also satisfied. (Notice that in the Euclidean case, the cospherical type sets are precisely those which meet every connected component of the Coxeter graph. So the two assumptions on I_0 are equivalent, the assumption (c) of Theorem 3.1 obviously holds, not using (c′).)

In the hyperbolic case, the reflection groups whose chambers have finite volume form a reasonable class, whereas it is rather limiting to restrict oneself to compact chambers. This is indicated, for instance, by the examples and results of [**27**, **26**, **43**, **45**, **31**, **46**, **30**, **23**, **34**]. The chambers of finite volume of hyperbolic reflection groups can be compactified by adding certain 'improper vertices' or 'cusps' in such a way that they become compact convex polytopes. However, in general these polytopes are not simple any more. For instance, in dimension 3, in a cusp four edges may meet; this happens if the stabilizer of the cusp is a product of two infinite dihedral groups each acting on the affine line, with diagram

$$\circ\overset{\infty}{\text{——}}\circ \quad \circ\overset{\infty}{\text{——}}\circ.$$

Therefore, the theory developed in this paper does not directly apply to hyperbolic reflection groups with a non-compact fundamental domain of finite volume. We shall show in future work, in the context of tessellations of hyperbolic space by ideal Archimedian solids, how to modify the notion of shadow geometries in order to include those reflection groups as well.

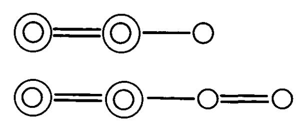
We conclude this section with a general application of Theorem 3.1, independent of chamber systems, which may illustrate the definition of a tessellation adopted here.

3.3. Proposition. *Let $\mathcal{F}_n$ be a locally finite family of n-dimensional compact convex polytopes (in some $\mathbb{R}^N$) and $\mathcal{F} \supseteq \mathcal{F}_n$ be the set of all faces of members of $\mathcal{F}$. Suppose that $F, G \in \mathcal{F}$, $F \nsubseteq G$ implies that $F \cap G$ is contained in the boundary of F. Then $\mathcal{F}$ is a tessellation of the union of its members.*

Proof. In the proof of Theorem 3.1, the construction of the bijection φ only used the following facts: the poset $\mathcal{F}$ is pure of dimension n, the minimal members of $\mathcal{F}$ are points, the cone property (4) holds, and the members of $\mathcal{F}$ can only meet in their combinatorial $\mathcal{F}$-boundaries (6). All these four ingredients are trivially satisfied in our present situation. For the last one, notice that the above assumption on the intersections $F \cap G$ implies that the $\mathcal{F}$-boundary of a member of $\mathcal{F}$ coincides with the ordinary boundary. From the local finiteness of $\mathcal{F}$, it easily follows that the bijection φ is actually a homeomorphism.

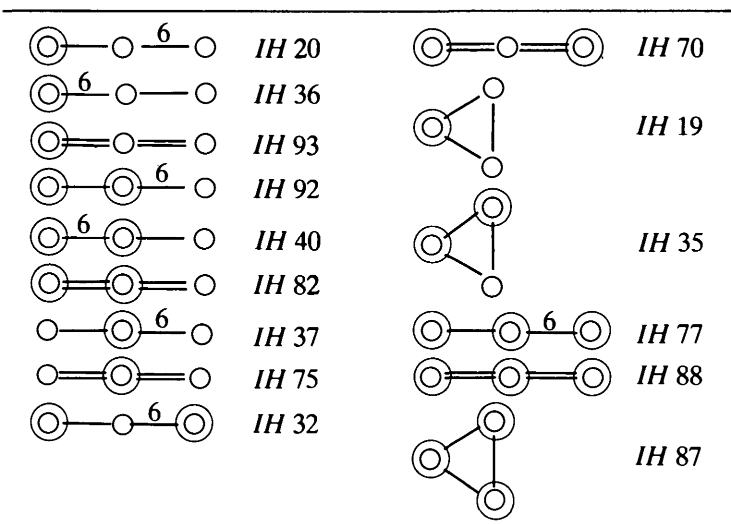
4. *Transitive tessellations for reflection groups*

In this section, some extensions of the results of § 2 and § 3 will be given in the case of (chamber systems of) reflection groups. If W is a reflection group on some space E, then the shadow geometries of the chamber system of W will be called shadow geometries of W, for simplicity, and all geometrical realizations in the sense of § 2 will be in E. In order to illustrate what one can say in addition to the general result that the realization of shadow geometries are tessellations (Corollary 3.2), we come back to the examples



which we have already used. We wish to emphasize the fact that the lower-dimensional faces of the corresponding tessellations as described after Proposition 2.3, are precisely the faces in the sense of convexity of the maximal tiles. It is the purpose of the first Theorem 4.1 of this section to prove this in general for the geometrical realizations of shadow geometries of reflection groups. The complementary Proposition 4.5 shows that, in a sense, the tessellations described in Corollary 3.2 and Theorem 4.1 are the only ones with a reflection group acting transitively on the maximal tiles. More precisely, the maximal tiles are necessarily as in the case of geometrical realizations of shadow geometries. In the two-dimensional Euclidean case, this result can be checked by direct inspection, using the list of all isohedral tilings of the Euclidean plane given in [**21**], cf. [**19**, **15**], and restricting this list to the groups $p6m$, $p4m$, $p3m1$, in the notation of [**21**, **12**]. The correspondence is as shown in Table 1.

TABLE 1. *The shadow geometries of the irreducible reflection groups in the Euclidean plane*



In this section, a reflection group is a 'linear reflection group' in the sense of [**44**]. That is, E is a subset of a vector space, W acts by restriction of linear mappings, and the hyperplanes are ordinary linear hyperplanes. The chambers are full intersections of halfspaces. So E consists of rays with endpoint the origin, and will often be replaced by its intersection with an appropriate hypersurface.

The new E then has the 'right' dimension. A linear reflection group in general does not act properly discontinuously, but it has the usual combinatorial properties of a reflection group. This, in particular, means that the chamber system $\mathscr{C}$ of W and, after the choice of a chamber M, the canonical bijection $E(\mathscr{C}, M) \xrightarrow{\cong} E$ exist. Thus we can indeed apply the notions and results of § 2. However, the canonical bijection is in general not a homeomorphism. The spherical, Euclidean and hyperbolic reflection groups can be considered as linear reflection groups. Here, the hypersurface is the unit sphere, a hyperplane $x_0 = 1$ or a hyperquadric $x_0^2 - x_1^2 - \dots - x_n^2 = 1$, with $x_0 > 0$, respectively. The hyperquadric can be transformed onto the open unit ball of the $(x_1, \dots, x_n)$-space by dividing by x_0. In addition to the groups just mentioned, the 'geometrical representations' of abstract Coxeter groups $(W, s_i, i \in I)$ introduced in [**39**; **2**, chapitre V, § 2] form an important class of linear reflection groups. Here, the chambers are simplices (strictly speaking, simplicial cones, see above), and the poset of the chambers and their faces is isomorphic to the (abstract) Coxeter–Tits complex of $(W, s_i, i \in I)$, whose simplices are by definition the cosets of subgroups $\langle s_j | \ j \in J \rangle$, where $J \subseteq I$ [**41**, Chapter 2; **2**, chapitre IV, § 1, Exercices 15 à 23].

4.1. THEOREM. *The geometrical realization of the shadow geometry of a reflection group W on E with respect to a cospherical type set I_0 has the following properties:*

(a) *the cells are convex, and the cells contained in a cell F are exactly the faces in the sense of convexity of F;*
(b) *each cell is an intersection of maximal cells.*

For the proof, I first give an explicit representation of an arbitrary cell as an intersection of reflecting hyperplanes and halfspaces. If C is a chamber and J a reduced, cospherical type set, then $F(C, J)$ denotes the cell belonging to the $(I \backslash J)$-class containing C. Recall that this is the union of all faces of type $\bar{J}$ containing C_J. For each $i \in I$, we denote by $H_i(C)$ the hyperplane whose intersection with C is the panel C^i, and by $H_i^+(C)$ and $H_i^-(C)$ the closed halfspace of $H_i(C)$ which contains, respectively does not contain, C. The reflection at $H_i(C)$ is denoted by $s_i(C)$ or simply by s_i, and $\langle J \rangle_C$ or $\langle J \rangle$ denotes the subgroup of W generated by the s_j, for $j \in J$. We have the formula

$$F(C, J) = \langle I \backslash J \rangle C_{\bar{J}} = \langle \bar{J} \backslash J \rangle C_{\bar{J}};$$

the second equality holds by Proposition 2.3(a).

4.2. LEMMA.

$$F(C, J) = \bigcap_{i \in I \backslash \bar{J}} H_i(C) \cap \bigcap_{\substack{j \in J \\ D \in \langle \bar{J} \backslash J \rangle C}} H_j^+(D).$$

Proof. We shall first derive the formula

$$(*) \qquad \langle I \backslash J \rangle C = \bigcap_{\substack{j \in J \\ D \in \langle I \backslash J \rangle C}} H_j^+(D)$$

(which in the case where $J = I_0$ is already the full Lemma). Given a chamber wC, with $w \in \langle I \backslash J \rangle$, and given $j \in J$ and $D = vC$, $v \in \langle I \backslash J \rangle$, we have $C_J = v^{-1}wC_J \subseteq$

$v^{-1}wC$ and $C_J \nsubseteq H_j(C) = H_j^-(C) \cap H_j^+(C)$ and therefore $C_J \nsubseteq H_j^-(C)$, *a fortiori* $v^{-1}wC \nsubseteq H_j^-(C)$. Thus $wC \nsubseteq H_j^-(D)$ and therefore $wC \subseteq H_j^+(D)$, as desired.

To prove the converse inclusion we notice that the right-hand side F in (*) is a union of full chambers. Let C' be a chamber contained in F and $(C_0 = C, C_1, \ldots, C_m = C')$ be a shortest gallery. As F is an intersection of halfspaces, all C_t are contained in F (cf. [**41**, Proposition 1.10]). Assuming $C' \nsubseteq \langle I \backslash J \rangle C$, let the index t be such that $C_{t-1} \subseteq \langle I \backslash J \rangle C$, $C_t \nsubseteq \langle I \backslash J \rangle C$. Now $C_{t-1} = wC$ for some $w \in \langle I \backslash J \rangle$, and the adjacent chamber C_t necessarily is of the form ws_jC for some $j \in I$. Then $j \in J$, for otherwise $C_t \subseteq \langle I \backslash J \rangle C$. But $s_jC \subseteq H_j^-(C)$, so $C_t \subseteq H_j^-(wC)$, contradicting the assumption $C_t \subseteq F$.

We now prove the formula given in the lemma; the right-hand side is again denoted by F. By (*), $F(C, J) \subseteq H_j^+(D)$ for every $j \in J$, $D \in \langle \bar{J} \backslash J \rangle$. Furthermore, each s_k, with $k \in I \backslash \bar{J}$, fixes each $wC_{\bar{J}}$, with $w \in \langle \bar{J} \backslash J \rangle$, because $s_k w = w s_k$. Combining these two facts we see that $F(C, J) \subseteq F$.

Let, conversely, $q \in F$ be given. For each $v \in \langle I \backslash \bar{J} \rangle$ and $D \in \langle \bar{J} \backslash J \rangle C$, we have $q = vq \in H_j^+(vD)$. From the decomposition $\langle I \backslash J \rangle = \langle I \backslash \bar{J} \rangle \langle \bar{J} \backslash J \rangle$, it then follows that $q \in H_j^+(D)$ even for all $D \in \langle I \backslash J \rangle C$ and $j \in J$. So, by (*), $q = wp$ for some $p \in C$ and $w \in \langle I \backslash J \rangle$. Write $w = uv = vu$, $u \in \langle I \backslash \bar{J} \rangle$, $v \in \langle \bar{J} \backslash J \rangle$. We now use the length function l on W with respect to the generating set s_i, for $i \in I$. The shortest element u' in the coset $u\langle K \rangle$, with $\langle K \rangle$ the stabilizer of p, is *a fortiori* contained in $\langle I \backslash \bar{J} \rangle$ (cf. [**2**, chapitre IV, § 1, Exercice 3]). Therefore we may suppose that $u = u'$. Then our assumption $s_i q = q$ for $i \in \langle I \backslash \bar{J} \rangle$ implies that $s_i up = up$, that is, $s_i u \langle K \rangle = u \langle K \rangle$, and therefore $l(s_i u) > l(u)$. This can only hold for $u = 1$. So $s_i p = p$ for all $i \in I \backslash \bar{J}$, that is, $p \in C_{\bar{J}}$, and furthermore $q = vp \in \langle \bar{J} \backslash J \rangle C_{\bar{J}}$, as desired.

We shall now describe the faces of codimension 1 of $F(C, J)$ which are obtained by replacing one of the halfspaces $H_j^+(D)$ occurring in Lemma 4.2 by its boundary $H_j(D)$.

4.3. Lemma. *Let J be cospherical, reduced, and $j \in J$ such that $\bar{J} \backslash j$ is cospherical. Then*

$$F(C, J) \cap H_j(C) = F(C, (\bar{J} \backslash j)_{\text{red}}).$$

Proof. Notice that $\bar{J} \backslash j$ is closed, set $J_j := (\bar{J} \backslash j)_{\text{red}}$. We use 1.7(g) and the fact that the boundary of $\bar{J} \backslash j$ in the basic graph consists of the points of the boundary of $\bar{J}$ distinct from j and of the neighbours of j in $\bar{J}$. It follows that

$$(\bar{J} \backslash j) \backslash J_j = \{ i \in \bar{J} \backslash J \mid m_{ij} = 2 \}$$

and therefore

$$F(C, J_j) = ZC_{\bar{J} \backslash j}, \quad Z = \langle i \in \bar{J} \backslash J \mid m_{ij} = 2 \rangle_C.$$

In particular, $F(C, J_j) \subseteq \langle \bar{J} \backslash J \rangle C_{\bar{J}} = F(C, J)$, and $F(C, J_j)$ is fixed by $s_j = s_j(C)$, because Z commutes with s_j. Conversely, consider a face wC_K, with $w \in \langle \bar{J} \backslash J \rangle$, $K \subseteq \bar{J}$, contained in $F(C, J)$, and fixed by s_j. Then

$$s_j w = wv \quad \text{for some } v \in \langle I \backslash K \rangle.$$

As in the proof of 4.2, we can assume that w is the shortest element in the coset $w\langle I \backslash K \rangle$. Then $l(s_j w) = l(wv) = l(w) + l(v)$, so $l(v) = 1$, and s_j occurs in every shortest word representing wv. But $j \in J$ and $w \in \langle \bar{J} \backslash J \rangle$, so this can only happen for $s_j = w$, and w centralizes s_j. Thus C_K is fixed by s_j, that is, $K \subseteq \bar{J}\backslash j$. We now use the fact that the above group Z is the exact centralizer of s_j in $\langle \bar{J}\backslash J \rangle$. (This is an easy consequence of théorème 3 in [40].) Thus $w \in Z$, and $wC_K \subseteq ZC_{\bar{J}\backslash j} \subseteq F(C, J_j)$, as desired.

Proof of Theorem 4.1. Each of the faces $F(C, J) \cap H_j(D)$ in Lemma 4.3 has indeed no larger codimension than 1, for it contains $C_{\bar{J}\backslash j}$. So it follows from 4.2 and 4.3 that the faces of codimension 1 of an $F(C, J)$ are precisely the $F(D, (\bar{J}\backslash j)_{\text{red}})$, where j is as in 4.3 and D runs through the chambers containing C_J. By 2.3(b), these are precisely the maximal ones among the cells properly contained in $F(C, J)$. This proves (a).

For the proof of (b), I show that every non-maximal $F(C, K)$, with $K \neq I_0$, is the intersection of two cells of the next highest dimension. Let $j \in I\backslash \bar{K}$ be such that $\bar{K} \cup j$ is closed. Set $J := (\bar{K} \cup j)_{\text{red}}$, and $C' = s_k(C)$. C. I claim that

$$F(C, K) = F(C, J) \cap F(C', J).$$

The inclusion '$\subseteq$' is trivial, because $C_K = C'_K$ and therefore $F(C, K) = F(C', K)$. Conversely, it follows from Lemma 4.2 that $F(C, J) \subseteq H_j^+(C)$, $F(C', J) \subseteq H_j^+(C') = H_j^-(C)$; therefore the intersection $F(C, J) \cap F(C', J)$ is indeed contained in $H_j(C) \cap F(C, J) = F(C, K)$ (see 4.3).

The general assumption of a linear reflection group is not essential for the Theorem 4.1. It merely has the advantage that one can speak of convexity without comment. The two Lemmata and 4.1(b) are true for arbitrary properly discontinuous reflection groups.

4.4. Corollary (Tits). *The shadow geometries of buildings are faithful. That is, for two simplices A, B of reduced type, $S_{I_0}(A) \subseteq S_{I_0}(B)$ is equivalent to $A \preccurlyeq B$.*

Proof. As usual, one can verify the claim inside one apartment, so it is enough to prove the corollary for the Coxeter–Tits complex of a Coxeter group $(W, s_i, i \in I)$. The result immediately follows from 4.1(b), applied to the geometrical representation of W (or, more elementarily, to the barycentric subdivision $E(W, P(I))$ of the Coxeter–Tits complex).

The following final proposition of this section is a sort of converse to Corollary 3.2.

4.5. Proposition. *Let E be a spherical, Euclidean or hyperbolic space and W a discrete reflection group on E whose chambers have finite volume. Each W-invariant tessellation of E such that the tiles have finite volume and W acts transitively on the maximal tiles has the same maximal tiles as the geometrical realization of an appropriate shadow geometry of W.*

Proof. A maximal tile of any tessellation is always the closure of its interior, the interior is connected, and the interiors of two distinct maximal tiles are disjoint. Therefore it is enough to show the following:

Let F be an open connected subset of finite volume of E such that

(i) $$E = \bigcup_{w \in W} w\bar{F},$$

(ii) $$\bar{F} \cap wF \neq \varnothing \;\Rightarrow\; F = wF, \quad \text{for } w \in W.$$

Then $F = F(p_0)$ for an appropriate point p_0, where $F(p)$ denotes the interior of the union of all chambers containing p.

For the proof, we first remark that a domain of the form $F(p_0)$ satisfies (ii) and that its stabilizer $W_{F(p_0)}$ coincides with the point stabilizer W_{p_0}. (This is easy and well known.) Therefore it is sufficient to find p_0 such that $F \subseteq F(p_0)$ and $W_{p_0} \subseteq W_F$. Indeed, if $q \in F(p_0)$ is arbitrary, then by (i), we can choose w such that $wq \in \bar{F}$, so $\overline{F(p_0)} \cap wF(p_0) \neq \varnothing$, so $w \in W_{F(p_0)} = W_{p_0} \subseteq W_F$, and so $q \in F$. Therefore, $F(p_0) \subseteq \bar{F}$; consequently $\bar{F} \subseteq \overline{F(p_0)} \subseteq \bar{F}$ and therefore $F = F(p_0)$.

Consider the group $V_F := \langle s_H |\; H \cap F \neq \varnothing \rangle$ (with H running through all reflecting hyperplanes). By (ii), $V_F \subseteq W_F$. By the assumption of finite volume, W_F, and *a fortiori* V_F, is finite. So we can consider the mapping

$$f\colon E \to E,$$

$$p \to \frac{1}{|V_F|} \sum_{v \in V_F} vp.$$

Here, E is for the moment supposed to be Euclidean or hyperbolic. In the hyperbolic case, the definition of $f(p)$ makes sense in the 'linear model', that is, the open unit ball.

We first show that the desired inclusion $F \subseteq F(f(p))$ holds for every $p \in F$. It is sufficient to know that for each hyperplane H not containing $f(p)$, the domain F is fully contained in the open halfspace of H containing $f(p)$. Now $H \cap F = \varnothing$, for otherwise $s_H \in V$ and therefore $s_H f(p) = f(p)$. So F, being connected, is contained in one open halfspace of H. This halfspace contains all vp, with $v \in V_F$, and therefore contains $f(p)$.

We shall now prove the desired inclusion $W_{f(p)} \subseteq W_F$ by showing that $W_{f(p)} = V_F$ for an appropriate $p \in F$. Notice that f is an open mapping from E onto the fixed point set $E' = \bigcap \{H |\; H \cap F \neq \varnothing\}$ of V_F. For all points q contained in the dense subset $E'' = \{q \in E' |\; q \notin H' \text{ for all } H' \text{ such that } H' \cap F = \varnothing\}$, the stabilizer W_q is equal to V_F. Now F is open, and therefore there exists indeed a point $p \in F$ such that $f(p) \in E''$.

If E is the sphere, we have to modify the definition of f by dividing the vector on the right-hand side by its norm. Thus, f is not defined on a certain subsphere $E_F \subseteq E$. One easily verifies that $V_F \neq W$ (otherwise $F = E$). This implies that V_F has a fixed point, that is, $E_F \neq E$, and the domain $E \backslash E_F$ of f is open and dense in E. Thus, the argument given in the Euclidean and hyperbolic case still holds.

If a W-invariant tessellation of E with transitivity on the maximal tiles is given, one can construct lots of new tessellations of that kind by subdividing the non-maximal tiles in a W-compatible way. Therefore one cannot improve on the fact that Proposition 4.5 only gives a conclusion about the maximal tiles.

5. *The Delaney symbols of the tessellations with a transitive reflection group*

A part of the results of the present paper can be roughly summarized by saying that the tessellations of a spherical, Euclidean, or hyperbolic n-space together with an action of a reflection group, transitive on the maximal tiles, are completely characterized by specifying the Coxeter diagram of a reflection group (with chambers of finite volume), together with a distinguished subset, belonging to a finite subgroup. In [16], A. Dress has quite generally defined a so called *Delaney symbol* (or generalized Schläfli symbol) $(\mathcal{D}(\mathcal{T}, G); r_1, r_2, \ldots, r_n)$ for any pair $(\mathcal{T}, G)$ consisting of a tessellation $\mathcal{T}$ of some n-manifold and a group G of homeomorphisms respecting $\mathcal{T}$. This symbol depends only on G and the underlying combinatorial tessellation, that is, the poset $\mathcal{T}$. In the case where the poset $\mathcal{T}$ comes from a simply connected manifold, the pair $(\mathcal{T}, G)$ is determined up to isomorphism (in the obvious sense) by $(\mathcal{D}(\mathcal{T}, G); r_1, \ldots, r_n)$.

The first part of the symbol, the '*Delaney set*' $\mathcal{D}(\mathcal{T}, G)$ is just the quotient of the chamber system of $\mathcal{T}$ (that is, the chamber system of the flag complex of the poset $(\mathcal{T}, \subseteq)$) by G:

$$\mathcal{D}(\mathcal{T}, G) := G \backslash \mathcal{C}(\mathcal{T}).$$

Recall from [16] or [17, 18, 19] that geometrically, the chambers (elements of $\mathcal{C}(\mathcal{T})$) are the maximal simplices of the barycentric subdivision of $\mathcal{T}$.

EXAMPLE 1. In our standard example of the Archimedian tiling by squares and octagons (Fig. 3), the chambers C, D, E as indicated in Fig. 7 are obviously representatives for the action of W on $\mathcal{C}(\mathcal{T})$. Thus, the Delaney set $\mathcal{D}$ can be identified with $\{C, D, E\}$. The chamber system structure is

$$\mathcal{D}: \quad C \overset{2}{—} D \overset{1}{—} E$$

where $\overset{i}{—}$ denotes i-adjacency.

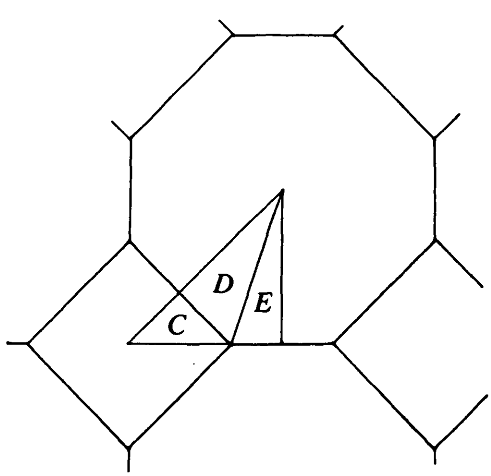


FIG. 7

It is known [16] that the chamber system of a tessellation of a manifold is *thin*, that is, for any chamber C and any dimension $d \in \{0, \ldots, n\}$, there is a unique chamber D which is d-adjacent to C. We denote this chamber by $C \,.\, \sigma_d$:

$$C \sim_d D,\ C \neq D \quad :\Leftrightarrow \quad D = C \,.\, \sigma_d.$$

Obviously, $(C \,.\, \sigma_d) \,.\, \sigma_{d'} = (C \,.\, \sigma_{d'}) \,.\, \sigma_d$ if $|d - d'| \geq 2$. Therefore, the chamber system structure on $\mathscr{C}(\mathscr{T})$ as well as on $\mathscr{D}(\mathscr{T}, G)$ can be described by an action of the 'universal' Coxeter group,

$$\Sigma(n) := \langle \sigma_d \mid d = 0, \ldots, n\,;\ \sigma_d^2 = 1\,;\ \sigma_d \sigma_{d'} = \sigma_{d'} \sigma_d \text{ for } |d - d'| \geq 2 \rangle.$$

We can now define the r_d which are functions from $\mathscr{D}(\mathscr{T}, G)$ into $\{2, 3, 4, \ldots\}$. For $C \in \mathscr{C}(\mathscr{T})$, and $d \in \{1, \ldots, n\}$, we set

$$r_d(C) = \min\{r \geq 1 \mid C \,.\, (\sigma_{d-1}\sigma_d)^r = C\}.$$

Geometrically, the definition of $r_d(c)$ implies that $C \,.\, \langle \sigma_{d-1}, \sigma_d \rangle$, which is the $\{d-1, d\}$-class of C, has $2r_d(C)$ elements, and in fact is an $2r_d(C)$-gon. Considering more concretely the case where $d = 1$, we can identify a $\{0, 1\}$-class with a partial flag $(t_2, \ldots, t_n)$ of elements of $\mathscr{T}$, and the value of r_1 on that flag gives the number of vertices and edges of the two-dimensional face t_2. This number depends only on the orbit $G \,.\, C$; therefore r_d can indeed be viewed as a function on $\mathscr{D}(\mathscr{T}, G)$. The $r_d(C)$ are called the *ramification numbers* of $(\mathscr{T}, G)$.

EXAMPLE 1 (continued). For the Archimedian tiling by squares and octagons, the ramification numbers are as follows (C, D, E are as in Fig. 7):

	C	D	E
r_1	4	8	8
r_2	3	3	8

For the remainder of this section, we fix the following notation: W is a discrete reflection group on spherical, Euclidean, or hyperbolic n-space E whose chambers have finite volume; I is the type set of W and $M = (m_{ij})_{i,j \in I}$ the Coxeter matrix of W. By $\mathscr{C}$ we denote the chamber system of W. We fix some cospherical $I_0 \subseteq I$; that is, the subgroup belonging to $I \setminus I_0$ is finite. By $\mathscr{S} = \mathscr{S}(\mathscr{C}, I_0\,; E)$ we denote that part of the abstract shadow geometry $\mathscr{S}(\mathscr{C}, I_0)$ (see § 2) that can be realized (in the sense of 2.3) in the space E. By 2.1, this poset $\mathscr{S}$ can be redefined as

$$\mathscr{S} = \{C_J \mid C \in \mathscr{C},\ J \text{ reduced with respect to } I_0,\ C_J \neq \varnothing\}$$

and

$$C_J \leq D_K \quad \Leftrightarrow \quad J \leq K,\ C_J \cup D_K \text{ is contained in a chamber.}$$

We know from 2.3(b) and 3.1 that the poset $\mathscr{S}$ comes from a W-equivariant tessellation of E. Notice that $\mathscr{S}$ is combinatorially dual to the tessellation defined in § 2 (the ordering is reversed in 2.3(b)), but it coincides with what has been obtained in special cases by Wythoff's construction [**7**, **8**, **9**, **10**, **11**].

The goal of the present section is simply an explicit determination of the Delaney symbol $(\mathscr{D}(\mathscr{S}, W)\,; r_1, \ldots, r_n)$ in terms of the Coxeter matrix M and I_0. Since, by definition, the underlying set $\mathscr{D}(\mathscr{S}, W) = W \setminus \mathscr{C}(\mathscr{S})$ can be identified with those maximal simplices of the barycentric subdivision of the geometrical realization of $\mathscr{S}$ that lie inside a fixed chamber (fundamental domain) of W, it is clear that $\mathscr{D}(\mathscr{S}, W)$ can be identified with the set of maximal flags of cospherical, reduced type sets (cf. 1.3(c)). To see this concretely, recall Figs 4 and 5 of § 1. For the determination of the r_d we shall need an explicit description of the $\Sigma(n)$-action on $\mathscr{C}(\mathscr{S})$ which we shall now derive.

5.1. LEMMA. *Let* $\mathscr{D}'$ *denote the set of all injective mappings* $\mathbf{i} = (i(0), i(1), \ldots, i(n-1))$ *from* $\{0, 1, \ldots, n-1\}$ *into* I *such that*

(i) $i(0) \in I_0$,

(ii) $i(t) \in I_0$, *or* $i(t)$ *is connected with* $\{i(0), \ldots, i(t-1)\}$ *in the Coxeter graph, for* $t = 0, \ldots, n-1$,

(iii) $\{i(0), \ldots, i(t)\}$ *is spherical, for all* $t = 0, \ldots, n-1$.

(a) *There is a bijective correspondence between* $\mathscr{D}'$ *and the set of all maximal flags of cospherical, reduced type sets given by*

$$\mathbf{i} \to (J_0, J_1, \ldots, J_n),$$

where $J_t = (I \setminus \{i(0), \ldots, i(t-1)\})_{\text{red}}$.

(b) *For* $\mathbf{i} \in \mathscr{D}'$, *there is a unique* $i(n) \in I$ *such that* $\{i(0), \ldots, i(n-2), i(n)\}$ *is spherical.*

Proof. From the very first part of the proof of Theorem 3.1 we know that, if $\bar{J}$ is cospherical, then J is cospherical itself. Therefore, we only have to describe the flags of closed, cospherical type sets with respect to inclusion (see Proposition 1.7(d)).

If $J \subseteq I$ is closed, then the elements $j \in J$ such that $J \setminus j$ are closed are precisely the elements in $J \cap I_0$ and the elements connected with the complement $I \setminus J$. We know this from Proposition 1.7(f) and (g). Therefore, the flags of complements of closed, cospherical type sets are indeed the sets $\{i(0), \ldots, i(t)\}$, where $\mathbf{i}$ is as above. The cardinality of the maximal flags of this kind is $n+1$, by (2) in the proof of Theorem 3.1. Thus (a) is proved.

As for (b), notice that $C^{\{i(0), \ldots, i(n-2)\}}$ is an edge of the chamber C, which has precisely two vertices, namely $C^{\{i(0), \ldots, i(n-1)\}}$ and a vertex of the form $C^{\{i(0), \ldots, i(n-2), i(n)\}}$ for some $i(n) \notin \{i(0), \ldots, i(n-1)\}$.

With the notation introduced in Lemma 5.1, we are now able to describe the chamber system of our specified shadow geometry $\mathscr{S}$ of W explicitly. For this purpose, we set

$$\mathscr{D} = \{\mathbf{i}\colon \{0, \ldots, n\} \to I \mid (i(0), \ldots, i(n-1)) \in \mathscr{D}', i(n) \text{ is as in 5.1(b)}\}.$$

For $\mathbf{i} \in \mathscr{D}$ or, more generally, for any $\mathbf{i}\colon \{0, \ldots, n\} \to I$, we denote by $\mathbf{i}' := (i(0), \ldots, i(n-1))$ its restriction to $\{0, \ldots, n-1\}$.

In the following proposition, $(d-1, d)$ denotes the transposition of $d-1$, d, and for $C \in \mathscr{C}$ and $i \in I$, the unique chamber i-adjacent to C is denoted by $C.i$.

5.2. PROPOSITION. *There is a canonical bijection from the set* $\mathscr{C} \times \mathscr{D}$ *onto the set*

$\mathscr{C}(\mathscr{S})$ of all maximal flags in $\mathscr{S}$, as follows: to $C \in \mathscr{C}$ and $\mathbf{i} \in \mathscr{D}$ we associate the flag

$$(C_{J_0}, C_{J_1}, \ldots, C_{J_n}) =: [C, \mathbf{i}],$$

where $J_t = (I \setminus \{i(0), \ldots, i(t-1)\})_{\mathrm{red}}$. Under this bijection, the $\Sigma(n)$-action on the thin chamber system $\mathscr{C}(\mathscr{S})$ corresponds to the action given by

$$(C, \mathbf{i}) \,.\, \sigma_0 = (C \,.\, i(0), \mathbf{i}),$$
$$(C, \mathbf{i}) \,.\, \sigma_d = \begin{cases} (C \,.\, i(d), \mathbf{i}) & \text{if } \mathbf{i} \circ (d-1, d) \notin \mathscr{D}, \\ (C, \mathbf{i} \circ (d-1, d)) & \text{if } \mathbf{i} \circ (d-1, d) \in \mathscr{D}, \end{cases}$$

for $d = 1, \ldots, n$, $d \neq n-1$,

$$(C, \mathbf{i}) \,.\, \sigma_{n-1} = \begin{cases} (C \,.\, i(n-1), \mathbf{i}) & \text{if } \mathbf{i}' \circ (n-2, n-1) \notin \mathscr{D}', \\ (C, \bar{\mathbf{i}}), & \text{if } \mathbf{i}' \circ (n-2, n-1) \in \mathscr{D}', \end{cases}$$

where $\bar{\mathbf{i}}$ is the unique element in $\mathscr{D}$ such that $\bar{\mathbf{i}}' = \mathbf{i}' \circ (n-2, n-1)$.

Proof. From statement (3) in the proof of Theorem 3.1 we know that our map $\mathscr{C} \times \mathscr{D} \to \mathscr{D}(\mathscr{S})$ is surjective.

To prove the injectivity, it is sufficient to show the uniqueness of C, for a given maximal flag $(C_{J_0}, \ldots, C_{J_n})$. Let D be another chamber such that $C_{J_t} = D_{J_t}$ for all t. We know from statement (1) in the proof of Theorem 3.1 that J_n is closed, that is, $J_n = I \setminus \{i(0), \ldots, i(n-1)\}$. Furthermore, $i(t) \in J_t$, for $i = 0, \ldots, n-1$, by 1.7(f). Therefore the union of the J_t, for $t = 0, \ldots, n$, is all of I, and the intersection of the $I \setminus J_t$ is empty. But $C \sim_{I \setminus J_t} D$ for all $t = 0, \ldots, n$, so $C = D$.

We now determine the $\Sigma(n)$-action on $\mathscr{C} \times \mathscr{D}$. So let a maximal flag $[C, \mathbf{i}]$ be given, and let $d \in \{0, \ldots, n\}$. We are looking for a flag $[D, \mathbf{k}] = (D_{K_0}, \ldots, D_{K_n})$ such that $C_{J_t} = D_{K_t}$ for $t \neq d$ and $C_{J_d} \neq D_{K_d}$. If we set $J' = \bigcup_{t \neq d} J_t$, then $C_{J'} = D_{J'}$, that is, $C \sim_{I \setminus J'} D$. Arguing as in the proof of the injectivity, we see that $J' \supseteq I \setminus \{i(d)\}$; therefore $D = C$ or $D = C \,.\, i(d)$. On the other hand, it is clear from the definition of $\mathbf{i}$ and $\mathbf{k}$ that $\mathbf{i} = \mathbf{k}$ in the case where $d = 0$, and $i(t) = k(t)$ for $t \neq d-1, d, n$, $\{i(d-1), i(d)\} = \{k(d-1), k(d)\}$ in the cases where $0 < d < n$. The same also holds in the case where $d = n$, because $i(d-1)$ and $i(d)$ are the only elements $i \in I \setminus \{i(0), \ldots, i(n-2)\}$ such that $\{i(0), \ldots, i(n-2), i\}$ is spherical. Therefore $\mathbf{k} = \mathbf{i}$ or $\mathbf{k} = \mathbf{i} \circ (d-1, d)$ except possibly for $d = n-1$, where it may happen that $i(n) \neq k(n)$. In the case where $d = 0$, obviously $J' = I \setminus \{i(0)\}$; therefore $[D, \mathbf{k}] = [C \,.\, i(0), \mathbf{i}]$, as claimed.

Now let $d > 0$, and assume first that $d \neq n-1$. We have to distinguish two cases. If $\mathbf{i} \circ (d-1, d) \notin \mathscr{D}$, that is, $i(d)$ is not contained in I_0 and not connected to $\{i(0), \ldots, i(d-2)\}$, then $i(d)$ is *a fortiori* not connected to $\{i(0), \ldots, i(t-1)\} = J \setminus \bar{J}_t$, for $t < d$, and therefore $i(d)$ is not contained in J_t (by 1.7(f) and (g)). For $t > d$, $i(d)$ is by definition not contained in J_t. Therefore $J' = I \setminus \{i(d)\}$, and $[C \,.\, i(d), \mathbf{i}]$ is the unique d-neighbour of $[C, \mathbf{i}]$. If $\mathbf{i} \circ (d-1, d) \in \mathscr{D}$, that is, $i(d) \in I_0$ or $i(d)$ is connected to $\{i(0), \ldots, i(d-2)\}$, then $i(d) \in J_{d-1}$, by 1.7(f) and (g); therefore $J' = I$ and necessarily $C = D$. So $[C, \mathbf{i} \circ (d-1, d)]$ is the unique d-neighbour of $[C, \mathbf{i}]$, as claimed. For $d = n-1$, this argument applies with the obvious modifications.

5.3. Corollary. *The Delaney set $\mathscr{D}(\mathscr{S}, W)$ is canonically isomorphic to the set $\mathscr{D}$, the $\Sigma(n)$-action being given by*

$$\mathbf{i}\,.\,\sigma_0 = \mathbf{i}, \quad \textit{for all } i \in \mathscr{D},$$

$$\mathbf{i}\,.\,\sigma_d = \begin{cases} \mathbf{i} & \textit{if } \mathbf{i}' \circ (d-1, d) \notin \mathscr{D}',\ 0 < d < n, \\ & \textit{or if } \mathbf{i} \circ (n-1, n) \notin \mathscr{D},\ d = n, \\ \mathbf{i} \circ (d-1, d) & \textit{if } \mathbf{i} \circ (d-1, d) \in \mathscr{D},\ 0 < d < n,\ d \neq n-1, \\ \bar{\mathbf{i}} & \textit{if } \mathbf{i}' \circ (n-2, n-1) \in \mathscr{D}',\ d = n-1, \end{cases}$$

where $\bar{\mathbf{i}}$ denotes the unique element in $\mathscr{D}$ such that $\bar{\mathbf{i}}' = \mathbf{i}' \circ (n-2, n-1)$.

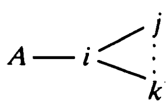
5.4. Proposition. *The ramification numbers $r_d(\mathbf{i})$, where $\mathbf{i} \in \mathscr{D}$, of the pair $(\mathscr{S}, W)$* (cf. 5.3) *are given as follows.*

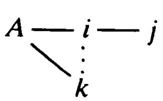
In the case where $d = 1$,

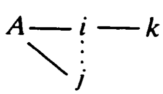
$$r_1(\mathbf{i}) = \begin{cases} m_{i(0)i(1)} & \textit{if } i(1) \notin I_0, \\ 2m_{i(0)i(1)} & \textit{if } i(1) \in I_0. \end{cases}$$

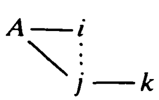
In the cases where $d \geq 2$, and $d \neq n$, if the fundamental domain of W is not a simplex, we proceed as follows. Consider the following graph on four points A, $i = i(d-2)$, $j = i(d-1)$, $k = i(d)$: the edges inside $\{i, j, k\}$ are as in the Coxeter graph, and $l \in \{i, j, k\}$ is connected to A if it is connected to some $i(t)$, for $t < d-2$, or is an element of I_0. According to the possibilities for this graph, the values of r_d are as follows:

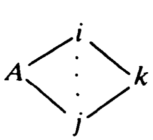
Case 1. $A — i — j — k$ $\quad r_d(\mathbf{i}) = m_{jk}$.

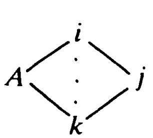
Case 2.  $\quad r_d(\mathbf{i}) = 2m_{jk}$.

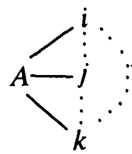
Case 3. (a) 

(b)  $\quad r_d(\mathbf{i}) = 3$.

(c) 

Case 4. (a) 

(b)  $\quad r_d(\mathbf{i}) = 4$.

Case 5.  $r_d(\mathbf{i}) = 3.$

(*A dotted line means that this edge may be absent.*)

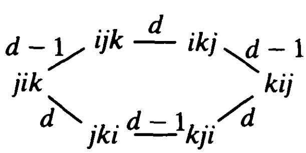
Proof. It is readily checked that the graphs in the list are indeed all the graphs on $\{A, i, j, k\}$ such that A and i are connected, j and $\{A, i\}$ are connected, k and $\{A, i, j\}$ are connected. In each case, and also in the case where $d = 1$, the claim about $r_d(\mathbf{i})$ easily follows from Proposition 5.2. It is helpful to distinguish the five cases by the shape of the $\langle \sigma_{d-1}, \sigma_d \rangle$-orbit of $\mathbf{i}$, that is, the $\{d-1, d\}$-class in the thin chamber system $\mathscr{D}$. This looks as follows, with the obvious abbreviations:

Case 1. the trivial orbit $\mathbf{i} \simeq ijk$

Case 2. $ijk \overset{d}{\text{——}} ikj$

Case 3(a). $ijk \overset{d}{\text{——}} ikj \overset{d-1}{\text{———}} kij$

Case 4. $jki \overset{d}{\text{——}} jik \overset{d-1}{\text{———}} ijk \overset{d}{\text{——}} ikj$

Case 5. 

For instance, in Case 2, the relevant part of the $\langle \sigma_{d-1}, \sigma_{d-2} \rangle$-orbit of some $[C, \mathbf{i}]$ looks as follows:

$$\ldots [C, \mathbf{i}] \overset{d-1}{\text{———}} [C.j, ijk] \overset{d}{\text{———}} [C.j, ikj] \overset{d-1}{\text{———}}$$

$$\text{———} [C.jk, ikj] \overset{d}{\text{———}} [C.jk, ijk] \text{———} \ldots .$$

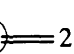
Now notice that the action from the right of the symbols $l \in I$ on the chambers $C \in \mathscr{C}$ factors through a faithful action of the abstract Coxeter group $\langle l \in I \mid (ll')^{m_{ll'}} = 1 \rangle$. (This action has to be distinguished from the left action of W on $\mathscr{C}$; it is the 'action by galleries'.) Therefore

$$[C, \mathbf{i}] \,.\, (\sigma_{d-1}\sigma_d)^{2r} = [C \,.\, (jk)^r, \mathbf{i}],$$

for all $r \geqslant 1$, and $2m_{jk}$ is indeed the smallest r such that $(\sigma_{d-1}\sigma_d)^r$ fixes $[C, \mathbf{i}]$. The other cases are similar.

We illustrate Proposition 5.4 by a series of Examples. We start with the Archimedian tiling of the plane by squares and octagons (Fig. 3, Example 1 of this section) where we already know the values of r_1 and r_2.

EXAMPLE 1 (continued). Let $(\mathscr{S}, W)$ be given by the diagram



(E is the Euclidean plane). The three maximal flags of reduced types as given in § 1, Example 1 read as follows in our present notation:

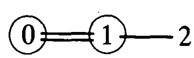
$$\{01, 02, 0\} \simeq 120,$$
$$\{01, 02, 2\} \simeq 102,$$
$$\{01, 1, 2\} \ \simeq 012.$$

(Recall that this correspondence is obtained by first passing to the corresponding flags of closed sets of types $\{012, 02, 0\}$, $\{012, 02, 2\}$, $\{012, 12, 2\}$.) Comparing Fig. 4 in § 1 with Fig. 7, one sees that they are to be identified with the flags denoted above by C, D, E, respectively. According to Proposition 5.4, we can now recalculate the values of r_1 and r_2 on $\mathcal{D}$ as follows:

	120	102	012
r_1	4	8	8
r_2	3	3	3

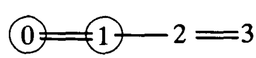
(For r_2, Cases 3(a), 3(b), 3(c), respectively, of 5.4 apply.)

If we replace the diagram by



and the Euclidean plane by the 2-sphere, the same discussion applies, except that $r_1(120) = 3$ now. The tessellation is given by the truncated cube as shown in Fig. 2 in the Introduction.

EXAMPLE 2. Let $(\mathcal{S}, W)$ be given by the diagram



(E is Euclidean 3-space). Before passing to a formal treatment, let us briefly describe this tessellation intuitively. Consider the group W as the symmetry groups of the regular tessellation by cubes. The vertices of types 0, 1, 2, 3 of the simplicial complex belonging to W are the vertices of the cube tessellation, the centres of the edges, the centres of the square faces, and the centres of the cubes, respectively. Distinguishing the set of types $I_0 = \{0, 1\}$ means that the new vertices are all centres of 'half-edges' (or some other points specified on the half-edges in a uniform way). Thus, it is intuitively clear that the tessellation $\mathcal{S}$ consists of truncated cubes and of octahedra (the octahedra sitting around the vertices of the original cube tessellation). We shall now derive this result in a formal way, analogous to Example 1, which is equally applicable in less intuitive (e.g. higher-dimensional) cases.

The four maximal flags of reduced types as given in § 1, Example 2 now read as follows:

$$\{01, 02, 03, 0\} \simeq 1230 =: C,$$
$$\{01, 02, 03, 3\} \simeq 1203 =: D,$$
$$\{01, 02, 2, 3\} \ \simeq 1023 =: E,$$
$$\{01, 1, 2, 3\} \ \ \simeq 0123 =: F.$$

The chamber system structure of the Delaney set $\mathscr{D} = \{C, D, E, F\}$ obviously is

$$C \overset{3}{\text{———}} D \overset{2}{\text{———}} E \overset{1}{\text{———}} F.$$

If we calculate the ramification numbers by means of Proposition 5.4, we get the following table:

	C	D	E	F
r_1	3	3	8	8
r_2	4	3	3	3
r_3	3	3	3	4

(For r_2, Cases 1, 3(a), 3(b), 3(c), and for r_3, Cases 3(a), 3(b), 3(c), 1, respectively, of Proposition 5.4 apply.)

In order to visualize the tessellation, we describe its maximal tiles and its vertex figures. It is clear from the definitions that the classes of maximal tiles (with respect to W) are in one-to-one correspondence with the $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$-orbits on $\mathscr{D} = \{C, D, E, F\}$. There are two such orbits: $\mathscr{D}' = \{C\}$, $\mathscr{D}'' = \{D, E, F\}$. If we restrict r_1 and r_2 to $\mathscr{D}'$ and $\mathscr{D}''$, we get the Delaney symbols of the two kinds of maximal tiles. The group acting on the respective tile and defining $\mathscr{D}'$, $\mathscr{D}''$ is its stabilizer in W which is (conjugate to) $\langle s_1, s_2, s_3 \rangle$, respectively $\langle s_0, s_1, s_2 \rangle$. Thus the tile corresponding to $\mathscr{D}'$ is platonic with $r_1 = 3$, $r_2 = 4$, that is, is an octahedron. The tile corresponding to $\mathscr{D}''$ has the Delaney symbol

	D	E	F
r_1	3	8	8
r_2	3	3	3

We know from the previous Example 1 that this describes the truncated cube.

We now look at the vertex figures. By the general construction of the tessellation realizing a shadow geometry (or rather, its dual, as considered in the present section), there is only one orbit of vertices with respect to W. This corresponds to the fact that already the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \subseteq \Sigma(3)$ acts transitively on the Delaney set $\mathscr{D}$. The Delaney symbol of a typical vertex figure is the original $\mathscr{D}$, neglecting the function r_1, and shifting the dimensions by one:

$$\mathscr{D}\colon\ C \overset{0}{\text{——}} D \overset{1}{\text{——}} E \overset{2}{\text{——}} F$$

	C	D	E	F
r_1	4	3	3	3
r_2	3	3	3	4

Figure 8 immediately shows that this Delaney symbol describes the cone over a square (considered as a spherical tiling) together with the symmetry group of the square.

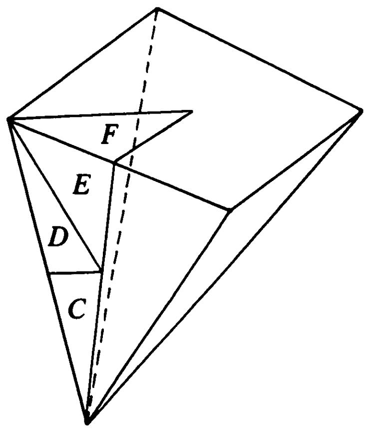


FIG. 8

In passing we remark that this is the tiling denoted by $2HTS31(4)$ in Grünbaum's and Shephard's classification of all spherical tilings with only two classes of edges [22]. This tiling is even 'minimal non-transitive', that is, has precisely two orbits of vertices, edges, and faces. The plane analogues (with the same Delaney set

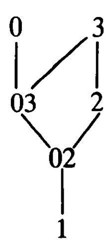
$$\circ \overset{0}{\text{——}} \circ \overset{1}{\text{——}} \circ \overset{2}{\text{——}} \circ,$$

but different ramification numbers) are the tilings 1.1 to 1.8 of [20]. Returning to the tessellation by truncated cubes and octahedra described at the beginning of this example, the reader will immediately convince himself that, combinatorially, the vertex figures of that tiling are precisely what we have just described.

EXAMPLE 3. Let $(\mathscr{S}, W)$ be given by

$$0 \text{——} (1) \text{——} 2 = 3.$$

The poset of reduced types is the following:



The maximal flags are

$$\begin{aligned}
\{1, 02, 03, 0\} &\simeq \{0123, 023, 03, 0\} \simeq 1230 =: C,\\
\{1, 02, 03, 3\} &\simeq \{0123, 023, 03, 3\} \simeq 1203 =: D,\\
\{1, 02, 2, 3\} &\simeq \{0123, 023, 23, 3\} \simeq 1023 =: E.
\end{aligned}$$

The Delaney set is

$$C \overset{3}{\text{——}} D \overset{2}{\text{——}} E.$$

The values of r_1, r_2, r_3 on $\mathcal{D}$ are as follows:

	1230	1203	1023
r_1	3	3	3
r_2	4	4	4
r_3	3	3	3

(For r_2, Cases 1, 2, 2, and for r_3, Cases 3(a), 3(b), 3(c), respectively, of Proposition 5.4 apply.)

We see that the functions r_1, r_2, r_3 are constant on $\mathcal{D}$. This implies that the automorphism group Aut($\mathcal{S}$) is a quotient $W(F_4)/N$ of the Coxeter group of type F_4:

o—o═o—o

(see Theorem 2 in [**16**]). By our main result Theorem 3.1 (or 3.2), the geometrical realization $\|\mathcal{S}\|$ is the 3-sphere, and in particular, is simply connected. This fact and Theorem 3 of [**16**] imply that actually $N = 1$, that is, $\mathrm{Aut}(\mathcal{S}) \simeq W(F_4)$. Thus $\mathcal{S}$ 'is' the 24-cell, and we get an embedding

$$W \simeq W(C_4) \hookrightarrow W(F_4).$$

This construction is of course well known from the classical theory of regular polytopes in Euclidean 4-space; see [**7**, § 8.3]. In the context of shadow geometries, the above embedding occurs in a systematic way.

If we replace the group by

0 ═ 1 — 2 ═ 3

but keep $I_0 = \{1\}$, we get

	1230	1203	1023
r_1	3	3	4
r_2	4	4	4
r_3	3	3	3

The choice of the new vertices as the centres of the edges of the regular tessellation by cubes suggests that $\mathcal{S}$ is a tessellation by cubeoctahedra and prisms over squares (combinatorially regular, but not metrically). This can be derived formally by looking at the Delaney symbols of the maximal tiles and vertex figures in a way completely analogous to Example 2. Notice that the Delaney symbol of the cubeoctahedron (with the maximum possible group acting) is indeed

$$\mathcal{D}\colon\ D \overset{2}{\text{——}} E,$$

	D	E
r_1	3	4
r_2	4	4

The faces are triangles and squares, the vertices are regular of valency 4.

EXAMPLE 4. Let $(\mathscr{S}, W)$ be given by the diagram

$$\textcircled{1} = 2 — 3 \quad \cdots \quad (n-1) — n,$$

$$\textcircled{1'} = 2' — 3' \quad \cdots \quad (m-1)' — m',$$

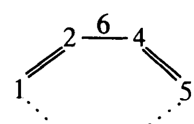
that is, W has the Coxeter diagram $C_n \times C_m$, $n \geq 1$, $m \geq 1$. It is clear how the reduced types, the closed types and our set $\mathscr{D}$ look like. The value of r_1 is always $4 = 2m_{11'} = m_{12} = m_{1'2'}$. The value of r_d, for $d \geq 2$, is always 3 (by Case 1 or Case 3 of Proposition 5.4). By the same argument as is used in Example 2, the flag complex $\Delta(\mathscr{S})$ is actually the Coxeter–Tits complex of type C_{n+m}.

Now let Δ_1, Δ_2 be buildings of type C_n, C_m, respectively, and consider the shadow geometry $\mathscr{S} = \mathscr{S}(\Delta_1 \times \Delta_2, \{1, 1'\})$ as above. It is readily checked that the products of the apartments in Δ_1 and Δ_2 give rise to a system of apartments (in the sense of Conditions (B3), (B4) in [**41**, Chapter 3]) in the flag complex $\Delta(\mathscr{S})$. We have just seen that these apartments in $\Delta(\mathscr{S})$ are indeed Coxeter–Tits complexes. Therefore, $\Delta(\mathscr{S})$ is a weak (that is, not necessarily thick) building of type C_{n+m}.

If we consider this building as the flag complex of a polar space S in the usual way (see, for instance, [**41**]), we have to dualize $\mathscr{S}$. The points are the elements of highest dimension $n+m-1$ in $\mathscr{S}$, that is, the vertices of type n in Δ_1 and the vertices of type m' in Δ_2. The lines correspond to the elements of dimension $n+m-2$ in $\mathscr{S}$, that is, the vertices of type $n-1$ in Δ_1, the vertices of type $(m-1)'$ in Δ_2, and the flags of type $\{n, m'\}$. If we also consider Δ_1, Δ_2 as flag complexes of polar spaces S_1, S_2, this can be rephrased as follows: the new polar space S has as point set the disjoint union of S_1 and S_2. Its lines are the original lines and, furthermore, all pairs (p_1, p_2), with $p_1 \in S_1$, $p_2 \in S_2$. This is exactly the polar space considered by Buekenhout and Sprague in [**6**, p. 226; cf. **33**, p. 78].

EXAMPLE 5. Let W be an arbitrary reflection group whose fundamental domain is a simplex, and set $I_0 = I$. Then all values of all r_d, with $d \geq 2$, are equal to 3, because Case 5 of Proposition 5.4 always applies. This means that all 'vertex figures' $\{t \in \mathscr{S} \mid t > s\}$, where s is a minimal element of $\mathscr{S}$, are isomorphic to a simplex $P\{0, \ldots, \dim E\}$. Indeed, the tessellation of E corresponding to $\mathscr{S}$ is the dual of the decomposition of E into the chambers and faces of W.

EXAMPLE 6. In this example, the underlying space is hyperbolic 3-space H^3, and for the group W we take the reflection group with compact fundamental domain described by Mennicke in [**30**], using explicit coordinates for the bounding planes and vertices. If we keep Mennicke's original labelling of the planes, the Coxeter diagram is as follows:



Following a convention of Vinberg's, we note that a dotted edge $i \ldots j$ denotes a pair of planes which are non-intersecting and non-parallel. In particular, the order of the product of the corresponding reflections is infinity. The spherical subsets of $I = \{1, 2, 3, 4, 5, 6\}$ are precisely the subsets such that the corresponding subgroup is finite. In particular, ij is spherical if and only if i and j are not joined by a dotted edge. (Notice that pairs of parallel bounding planes do not occur.) If we denote the faces corresponding to the above labels by $S_1, \ldots, S_6$, the 2-element spherical subsets ij correspond to the edges $S_i \cap S_j$, and the 3-element spherical subsets ijk correspond to the vertices $S_i \cap S_j \cap S_k$ of the fundamental chamber. Obviously, the spherical subsets are the following; we list the edges for each face.

Edges

12	13	14	15	
21	23	24	25	26
31	32	34		
41	42	43	45	46
51	52	54	56	
62	64	65		

Vertices

123 125 134 145 234 246 256 456

Thus, the polytope is bounded by two triangles 3 and 6, two 4-gons 1 and 5, and two pentagons 2 and 4, and has eight vertices. Figure 9 does not take care of the hyperbolic metrical properties; it is just supposed to show the combinatorial shape and the combinatorial symmetry of the fundamental domain.

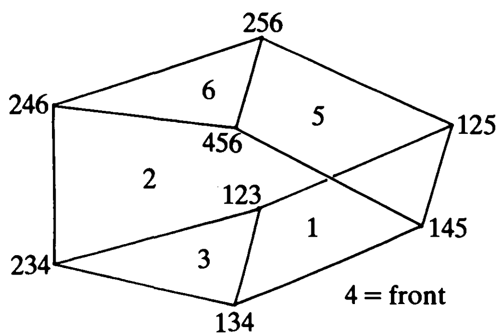


FIG. 9

We now discuss two distinct shadow geometries and tessellations for W. The first one is the tessellation considered by Mennicke, where a typical maximal tile is the union of the six chambers containing the edge 24. In our language, this is the geometrical realization by cells $F(C, J)$ of the shadow geometry with respect to $I_0 = I \setminus \{2, 4\} = \{1, 3, 5, 6\}$ as studied in §§ 2–4 above. We shall now discuss at the same time the shadow geometry with respect to $I_0 = \{1, 5, 6\}$. That is, the maximal tiles are the union of the twelve chambers containing the vertex 234 (forming one orbit under the 12-element subgroup $\langle s_2, s_3, s_4 \rangle \subseteq W$) and all its transforms under W. As in the previous examples, we shall write down the Delaney set by means of Corollary 5.3, and we shall calculate the ramification numbers.

In order to derive the flags of reduced (or closed) cospherical type sets, it is not suitable to write down the totality of reduced sets and to restrict to the cospherical ones afterwards. In all cases where the number of cospherical sets is small (compared to $2^{|I|}$), it works faster to calculate the closure for the complements of all spherical sets (which we have listed anyway), then to suppress sets which occur several times, and to order the remaining ones by reversed inclusion. If we carry this out for $I_0 = \{1, 3, 5, 6\}$, we immediately arrive at the following poset of closed, cospherical types:

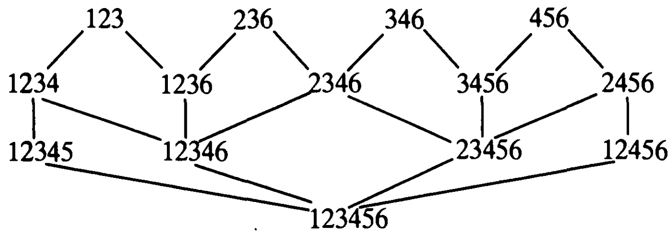


FIG. 10

In the case where $I_0 = \{1, 5, 6\}$, the three sets to the right 12456, 2456, 456 have to be omitted; all other sets remain closed. The first three symbols $ijk \in \mathscr{D}'$ of a maximal flag $ijkl \in \mathscr{D}$ (cf. Lemma 5.1) are now directly read off from this figure; recall that $ijk \in \mathscr{D}'$ corresponds to the flag $\{I, I\setminus\{i\}, I\setminus\{i, j\}, I\setminus\{i, j, k\}\}$ of closed sets. Recall, furthermore, that the last symbol l is such that $\{i, j, l\}$ is spherical. (Here, the definition of l becomes crucial for the first time; in the examples given up to now, the last symbol $i(n)$ was the unique 'missing' symbol, i.e. trivially determined by the equality $I = \{i(0), \ldots, i(n-1), i(n)\}$.

For $I_0 = \{1, 5, 6\}$, we obtain the Delaney set:

$$(6542) \overset{1}{\text{---}} (5642) \overset{2}{\text{---}} (5461) \overset{3}{\text{---}} (5416) \overset{2}{\text{---}} (5142) \overset{1}{\text{---}} (1542)$$

$$(5142) \overset{3}{|} (5124), \quad (1542) \overset{3}{|} (1524)$$

$$(5124) \overset{1}{\text{---}} (1524) \overset{2}{\text{---}} (1253)$$

For $I_0 = \{1, 3, 5, 6\}$, this figure has to be completed to the right by

$$\cdots (1253) \overset{3}{\text{---}} (1235) \overset{2}{\text{---}} (1324) \overset{1}{\text{---}} (3124).$$

Thus $\mathscr{D}$ becomes symmetric with respect to interchanging 24, 15, 36, as is to be expected from the fact that this renumbering is a diagram automorphism respecting I_0 of the Coxeter diagram.

The ramification numbers $r_1(ijkl)$, $r_2(ijkl)$, $ijkl \in \mathscr{D}$ can now be calculated by means of Proposition 5.4 as before. They turn out to be constant: $r_1 = 4$, $r_2 = 3$. For the calculation of r_3, the fact becomes crucial that in an adjacency

$$(ijkl) \overset{2}{\text{---}} (ikjl')$$

we usually have $l \neq l'$. Recall that this was the reason why r_n had to be excluded in Proposition 5.4 in the non-simplicial case. In each concrete case like the given one, we can however calculate r_3 by determining the $\langle \sigma_2, \sigma_3 \rangle$-orbits of chambers directly from Proposition 5.2.

We first treat Mennicke's 'symmetric case' $I_0 = \{1, 3, 5, 6\}$. There are four cases corresponding to the four $\langle\sigma_2, \sigma_3\rangle$-orbits on $\mathcal{D}$, which are reduced to two cases (6542) and (5642) by the diagram automorphism.

If C is any chamber for W, the $\langle\sigma_2, \sigma_3\rangle$-orbit of the chamber $[C, 6542]$ of the shadow geometry looks as follows:

$$[C, 6542] \overset{2}{\text{——}} [C.4, 6542] \overset{3}{\text{——}} [C.42, 6542] \overset{2}{\text{——}} [C.424, 6542] \ldots,$$

and therefore $r_3(6542) = m_{24} = 6$.

For 5642, we have

$$[C, 5642] \overset{2}{\text{——}} [C, 5461] \overset{3}{\text{——}} [C, 5416] \overset{2}{\text{——}} [C, 5142] \overset{3}{\text{——}}$$
$$[C, 5124] \overset{2}{\text{——}} [C.2, 5124] \overset{3}{\text{——}} [C.2, 5142] \overset{2}{\text{——}} [C.2, 5416] \overset{3}{\text{——}}$$
$$[C.2, 5461] \overset{2}{\text{——}} [C.4, 5642] \overset{3}{\text{——}} [C, 5642],$$

and therefore $r_3(5642) = 5$.

For the geometrical interpretation, we have to recall that Mennicke's tessellation is dual to what we have considered in this section for defining the r_i, that is, we have to reverse the rôles of r_1 and r_3. Thus, the values of r_3 reconfirm the result that the faces of a maximal tile are hexagons and pentagons (the plane 6 supports a hexagon, the plane 5 supports a pentagon; see [**30**, Figure 4]). The values $r_1 = 4$, $r_2 = 3$ show that, combinatorially, the vertex figures are all octahedra.

In the case where $I_0 = \{1, 5, 6\}$, the element $1542 \in \mathcal{D}$ represents a different $\langle\sigma_2, \sigma_3\rangle$-orbit which now is not equivalent to 5642 by a diagram automorphism. A calculation similar to that above yields the result $r_3(1542) = 6$.

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