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Symplectic Shifted Tableaux and Deformations of Weyl's Denominator Formula for $sp(2n)$

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Abstract. A determinantal expansion due to Okada is used to derive both a deformation of Weyl's denominator formula for the Lie algebra $sp(2n)$ of the symplectic group and a further generalisation involving a product of the deformed denominator with a deformation of flagged characters of $sp(2n)$. In each case the relevant expansion is expressed in terms of certain shifted $sp(2n)$-standard tableaux. It is then re-expressed, first in terms of monotone patterns and then in terms of alternating sign matrices.

Keywords: alternating sign matrices, symplectic shifted tableau, monotone triangle, Weyl's denominator formula

Introduction

By considering the trivial identity representation of a semisimple Lie algebra, Weyl's character formula yields Weyl's denominator formula [20]. This formula expresses a certain product taken over the positive roots of the Lie algebra as a sum taken over the elements of the corresponding Weyl group of the Lie algebra. Writing the roots in a standard euclidean basis and replacing formal exponentials, e^{ϵ_i}, of the basis elements by indeterminates x_i, for $i = 1, 2, \ldots, n$ gives rise to an identity which for some Lie algebras, most notably $A_{n-1} = sl(n)$ or $gl(n)$, has a combinatorial interpretation [4].

In this setting it is natural to ask to what extent Weyl's denominator formula may be deformed through the introduction of a parameter t which generalises the sign factor -1 which is so crucial a feature of the original formula. Tokuyama [18] derived just such a deformation in the case of the Lie algebra $gl(n)$ of the general linear group. By using certain strict Gelfand patterns he expressed the product form of the denominator as a sum of terms whose coefficients have an explicit, very simple, t-dependence. This deformation was inspired in part by the work of Mills et al. [9], which used both alternating sign matrices and certain shifted plane partitions.

Since that time, deformations of Weyl's denominator formula have been derived for each of the other classical Lie algebras, $B_n = so(2n+1)$, $C_n = sp(2n)$ and $D_n = so(2n)$ by Okada [11] and more recently by Simpson [13, 14]. In each case use is made of a variety of combinatorial constructs such as partitions, Ferrers diagrams, plane partitions, alternating sign matrices or weighted digraphs. The particular deformations studied are not all identical, and some differ from the most natural deformation of Weyl's denominator formula in that long and short roots are not treated in precisely the same way.

Remarkably, Tokuyama's key result [18] for $gl(n)$ went further and gave an explicit formula for the expansion of not only the natural deformation of the product form of Weyl's denominator but also a product of this with the character of any irreducible representation of $gl(n)$ labelled by a partition λ. Such a character has combinatorial realisations in terms of both Gelfand patterns and standard Young tableaux. The original proof offered by Tokuyama exploited some representation theoretic methods, but a combinatorial proof has since been provided by Okada [10]. This used shifted plane partitions, monotone triangles and lattice paths.

Here, the intention is to consider the most natural deformation of Weyl's denominator formula in the case of the Lie algebra $C_n = sp(2n)$ and to derive the direct generalisation of Tokuyama's result, and to do so by means of an extension of Okada's methods.

1. Tokuyama's result and its extension to $sp(2n)$

For any simple Lie algebra g of a Lie group G, Weyl's denominator formula [20] takes the form:

$$e^{\delta} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) = \sum_{w \in W} \mathrm{sgn}(w) e^{w\delta}, \tag{1.1}$$

where the product on the left is over all α in the set, Δ_+, of positive roots of g and the sum on the right is over all elements w of the Weyl group, W, of g. The notation is such that δ is half the sum of the positive roots and $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ where $\ell(w)$ is the length of w when expressed as a word in the generators of W.

One particularly simple deformation of the left hand side of (1.1) takes the form

$$D_g(t) = e^{\delta} \prod_{\alpha \in \Delta_+} (1 + t e^{-\alpha}) \tag{1.2}$$

where t is the deformation parameter.

In the case of the Lie algebra $A_{n-1} = sl(n)$ of the Lie group $SL(n)$

$$\Delta_+ = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\}, \tag{1.3}$$

with $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$. It follows that $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$. The corresponding Weyl group is $W = S_n$, the symmetric group. This acts naturally on the basis vectors ϵ_i, that is each $w = \pi \in S_n$ maps ϵ_i to ϵ_{π_i} for $i = 1, 2, \ldots, n$. Setting $x_i = e^{\epsilon_i}$ for $i = 1, 2, \ldots, n$ and $x = (x_1, x_2, \ldots, x_n)$ this implies that

$$D_{sl(n)}(x; t) = \prod_{1 \le i \le n} x_i^{n-i+1} \prod_{1 \le i < j \le n} \left(1 + t x_i^{-1} x_j\right). \tag{1.4}$$

Each finite-dimensional irreducible representation of $sl(n)$ is specified by a highest weight vector λ, which in the ϵ-basis takes the form $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$ with λ_i an integer for $i = 1, 2, \ldots, n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Equivalently, we may specify this irreducible representation by the corresponding partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and its character is given by the Schur function [8, 15]:

$$s_\lambda(x) = s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in T^\lambda(sl(n))} x^{\mathrm{wgt}(T)}, \tag{1.5}$$

where the sum is over all $sl(n)$-standard tableaux T of shape λ and

$$x^{\mathrm{wgt}(T)} = x_1^{m_1(T)} x_2^{m_2(T)} \ldots\ x_n^{m_n(T)} \tag{1.6}$$

with $m_k(T)$ equal to the number of entries k in T.

Tokuyama [18] has established an explicit formula for the expansion of the product $D_{sl(n)}(x;t)\, s_\lambda(x)$ which thanks to the connection between strict Gelfand patterns and shifted Young tableaux can be recast in the form:

Theorem 1.1 ([18]) *Let λ be a partition into no more than n parts, and let δ be the partition $(n, n-1, \ldots, 1)$ then*

$$D_{sl(n)}(x;t)\, s_\lambda(x) = \sum_{S \in ST^{\lambda+\delta}(sl(n))} t^{\mathrm{hgt}(S)}\, (1+t)^{\mathrm{str}(S)-n}\, x^{\mathrm{wgt}(S)} \tag{1.7}$$

where the summation is taken over all $sl(n)$-standard shifted tableaux S of shape $\lambda+\delta$. The notation is such that str(S) *is the total number of connected components of all the ribbon strips of S,*

$$\mathrm{hgt}(S) = \sum_{k=1}^{n} (\mathrm{row}_k(S) - \mathrm{con}_k(S)) \tag{1.8}$$

where $\mathrm{row}_k(S)$ *is the numbers of rows of S containing an entry k, and* $\mathrm{con}_k(S)$ *is the number of connected components of the ribbon strip of S consisting of all the entries k, while $x^{\mathrm{wgt}(S)}$ is defined as in* (1.6) *with the tableau T replaced by the shifted tableau S.*

The main result of the present paper is the derivation of an analogue of (1.7) in the case of the Lie algebra $C_n = sp(2n)$ of the Lie group $Sp(2n)$. In the case of $sp(2n)$:

$$\Delta_+ = \{2\epsilon_i \mid 1 \leq i \leq n\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}. \tag{1.9}$$

Once again $\delta = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$, The corresponding Weyl group is $W = H_n = S_2 \wr S_n$, the hyperoctohedral group. This acts naturally on the basis vectors ϵ_i by sign changes and permutations, that is each $w = \bar{\pi} \in H_n$ maps ϵ_i to $\pm\epsilon_{\pi_i}$ for $i = 1, 2, \ldots, n$. Setting $x_i = e^{\epsilon_i}$ for $i = 1, 2, \ldots, n$ gives

$$D_{sp(2n)}(x;t) = \prod_{1 \leq i \leq n} x_i^{n-i+1} \prod_{1 \leq i \leq n} \left(1+tx_i^{-2}\right) \prod_{1 \leq i < j \leq n} \left(1+tx_i^{-1}x_j\right)\left(1+tx_i^{-1}x_j^{-1}\right). \tag{1.10}$$

As for $sl(n)$, each finite-dimensional irreducible representation of $sp(2n)$ is specified by its highest weight vector λ which in the ϵ-basis again takes the form $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$ with λ_i an integer for $i = 1, 2, \ldots, n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Equivalently, we may again specify the irreducible representation by the corresponding partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and its character may be defined in terms of tableaux [5, 6, 17] by:

$$sp_\lambda(x) = sp_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathcal{T}^\lambda(sp(2n))} x^{\mathrm{wgt}(T)}, \tag{1.11}$$

where the sum is now over all $sp(2n)$-standard tableaux T of shape λ and

$$x^{\mathrm{wgt}(T)} = x_1^{m_1(T)-m_{\bar{1}}(T)}\, x_2^{m_2(T)-m_{\bar{2}}(T)} \ldots\, x_n^{m_n(T)-m_{\bar{n}}(T)} \tag{1.12}$$

with $m_k(T)$ and $m_{\bar{k}}(T)$ equal to the number of entries k and $\bar{k}$, respectively, in T. Thus each entry k or $\bar{k}$ contributes a factor x_k or $x_{\bar{k}} = x_k^{-1}$ to $x^{\mathrm{wgt}(S)}$ for all $k = 1, 2, \ldots, n$. It is convenient in the case of $sp(2n)$ to deform not just the denominator, as in (1.10), but also the character (1.11) by allowing each entry $\bar{k}$ to contribute not just a factor $x_{\bar{k}}$ but $t^2 x_{\bar{k}}$. This leads to the definition

$$sp_\lambda(x;t) = \sum_{T \in \mathcal{T}^\lambda(sp(2n))} t^{2\mathrm{bar}(T)}\, x^{\mathrm{wgt}(T)}, \tag{1.13}$$

where $\mathrm{bar}(T)$ is the number of barred entries in T, that is

$$\mathrm{bar}(T) = \sum_{k=1}^{n} m_{\bar{k}}(T). \tag{1.14}$$

With this notation we find:

Theorem 1.2 *Let λ be a partition into no more that n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$\begin{aligned} &D_{sp(2n)}(x;t)\, sp_\lambda(x;t) \\ &\quad = \sum_{S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\, (1+t)^{\mathrm{str}(S)-n}\, x^{\mathrm{wgt}(S)} \end{aligned} \tag{1.15}$$

where the summation is taken over all $sp(2n)$-standard shifted tableaux S of shape $\lambda + \delta$. The notation is such that $\mathrm{bar}(S)$ *is the total number of barred entries in S,* $\mathrm{str}(S)$ *is the total number of connected components of all the ribbon strips of S and*

$$\mathrm{hgt}(S) = \sum_{k=1}^{n} (\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\bar{k}}(S)), \tag{1.16}$$

where $\mathrm{row}_k(S)$ *and* $\mathrm{row}_{\bar{k}}(S)$ *are the numbers of rows of S containing an entry k and $\bar{k}$, respectively, and* $\mathrm{con}_k(S)$ *is the number of connected components of the ribbon strip of S consisting of all the entries k, while $x^{\mathrm{wgt}(S)}$ is defined as in* (1.12) *with the tableau T replaced by the shifted tableau S.*

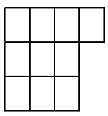
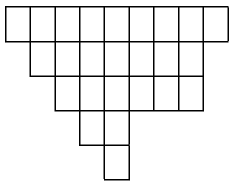
The precise meaning of the terminology used in Theorems 1.1 and 1.2 regarding standard tableaux and ribbon strip subtableaux is explained in Section 2.

2. Young diagrams and tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition, that is a weakly decreasing sequence of non-negative integers λ_i. The weight, $|\lambda|$, of the partition λ is the sum of its parts, and its length, $\ell(\lambda) \leq n$, is the number of its non-zero parts. Each such partition λ defines a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i that are left adjusted to a vertical line. Formally, $F^\lambda = \{(i, j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$.

Recalling that δ is the partition $\delta = (n, n-1, \ldots, 1)$, then $\mu = \lambda + \delta$ is a partition all of whose parts $\mu_i = \lambda_i + n - i + 1$ are distinct and non-zero. Thus μ is a strongly decreasing sequence of n positive integers. More generally, any partition μ all of whose parts are distinct, defines a shifted Young diagram SF^μ consisting of $|\mu|$ boxes arranged in $\ell(\mu)$ rows of lengths μ_i that are left adjusted to a diagonal line. To be precise, $SF^\mu = \{(i, j) \mid 1 \leq i \leq \ell(\mu), i \leq j \leq \mu_i + i - 1\}$.

For example, when $\lambda = (4, 3, 3)$ and $\mu = (9, 7, 6, 2, 1)$ we have

$F^\lambda =$  and $SF^\mu =$  (2.1)

There exists a variety of useful sets of tableaux associated with F^λ and SF^μ. The tableaux are all formed by placing entries from some totally ordered set, or alphabet, into the boxes of the relevant diagram subject to certain rules. The notation adopted here is that in forming each tableau the entry in the box in the ith row and jth column of either F^λ or SF^μ, as appropriate, is signified by η_{ij}.

First, let A be a totally ordered set and let A^r be the set of all sequences $a = (a_1, a_2, \ldots, a_r)$ of elements of A of length r. In addition, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of length r. The set $\mathcal{T}^\lambda(A; a)$ consists of all those standard tableaux, T, with respect to A, of profile a and shape λ, formed by placing an entry from A in each of the boxes of F^λ in such a way that the entries are weakly increasing from left to right across each row, and strictly increasing from top to bottom down each column, with the entries in the first column being given by the components of a, that is:

$$\begin{array}{lll} \text{(T1)} & \eta_{ij} \in A & \text{for all } (i, j) \in F^\lambda; \\ \text{(T2)} & \eta_{i1} = a_i \in A & \text{for all } (i, 1) \in F^\lambda; \\ \text{(T3)} & \eta_{ij} \leq \eta_{i,j+1} & \text{for all } (i, j), (i, j+1) \in F^\lambda; \\ \text{(T4)} & \eta_{ij} < \eta_{i+1,j} & \text{for all } (i, j), (i+1, j) \in F^\lambda. \end{array} \tag{2.2}$$

Second, as before let A be a totally ordered set and let A^r be the set of all sequences $a = (a_1, a_2, \ldots, a_r)$ of elements of A of length r, but now let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set $\mathcal{ST}^\mu(A;a)$ is defined to be the set of all standard shifted tableaux, S, with respect to A, of profile a and shape μ, formed by placing an entry from A in each of the boxes of SF^μ in such a way that the entries are weakly increasing from left to right across each row and from top to bottom down each column, and strictly increasing from top-left to bottom-right along each diagonal, with the entries in the leading diagonal being given by the components of a, that is:

$$\begin{array}{lll} \text{(S1)} & \eta_{ij} \in A & \text{for all } (i,j) \in SF^\mu; \\ \text{(S2)} & \eta_{ii} = a_i \in A & \text{for all } (i,i) \in SF^\mu; \\ \text{(S3)} & \eta_{ij} \leq \eta_{i,j+1} & \text{for all } (i,j), (i,j+1) \in SF^\mu; \\ \text{(S4)} & \eta_{ij} \leq \eta_{i+1,j} & \text{for all } (i,j), (i+1,j) \in SF^\mu; \\ \text{(S5)} & \eta_{ij} < \eta_{i+1,j+1} & \text{for all } (i,j), (i+1,j+1) \in SF^\mu. \end{array} \tag{2.3}$$

Third, let D be a totally ordered set such that $D = A \cup B$ with $A \cap B = \emptyset$, let D^r be the set of all sequences $d = (d_1, d_2, \ldots, d_r)$ of elements of D of length r, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set $\mathcal{PST}^\mu(A, B; d)$ is defined to be the set of all standard shifted supertableaux, P, formed by placing an entry from $D = A \cup B$ in each of the boxes of SF^μ in such a way that the entries are weakly increasing from left to right across each row and from top to bottom down each column. In addition, any entry from A appears at most once in each column, and any entry from B appears at most once in each row, with the entries in the leading diagonal being the components of d. These constraints take the form:

$$\begin{array}{lll} \text{(P1)} & \eta_{ij} \in D = A \cup B & \text{for all } (i,j) \in SF^\mu; \\ \text{(P2)} & \eta_{ii} = d_i \in D & \text{for all } (i,i) \in SF^\mu; \\ \text{(P3)} & \eta_{ij} \leq \eta_{i,j+1} \text{ if } \eta_{ij} \in A & \text{for all } (i,j), (i,j+1) \in SF^\mu; \\ \text{(P4)} & \eta_{ij} < \eta_{i+1,j} \text{ if } \eta_{ij} \in A & \text{for all } (i,j), (i+1,j) \in SF^\mu; \\ \text{(P5)} & \eta_{ij} < \eta_{i,j+1} \text{ if } \eta_{ij} \in B & \text{for all } (i,j), (i,j+1) \in SF^\mu; \\ \text{(P6)} & \eta_{ij} \leq \eta_{i+1,j} \text{ if } \eta_{ij} \in B & \text{for all } (i,j), (i+1,j) \in SF^\mu. \end{array} \tag{2.4}$$

As a consequence of the conditions (P3)–(P6), the entries are strictly increasing from top-left to bottom-right along each diagonal.

With these definitions we are now in a position to specify all the standard tableaux of interest in the present context:

Definition 2.1 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of length r. Then the set of all $sl(n)$-standard tableaux of shape λ is defined by:

$$\mathcal{T}^\lambda(sl(n)) = \{T \in \mathcal{T}^\lambda(A;a) \mid A = [n], a \in [n]^r\}, \tag{2.5}$$

where the entries η_{ij} of each $sl(n)$-standard tableau T are subject to the conditions (T1)–(T4) of (2.2), with $A = [n] = \{1, 2, \ldots, n\}$ and the elements of $[n]$ subject to the order relations $1 < 2 < \cdots < n$.

Definition 2.2 Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set of all $sl(n)$-standard shifted tableaux of shape μ is defined by:

$$\mathcal{ST}^{\mu}(sl(n)) = \{S \in \mathcal{ST}^{\mu}(A; a) \mid A = [n], a \in [n]^r\}, \tag{2.6}$$

where the entries η_{ij} of each $sl(n)$-standard shifted tableau S are subject to the conditions (S1)–(S5) of (2.3), with $A = [n] = \{1, 2, \ldots, n\}$ and the elements of $[n]$ subject to the order relations $1 < 2 < \cdots < n$.

Definition 2.3 ([10]) Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set of all $sl(n)$-standard primed shifted tableaux of shape μ is defined by:

$$\begin{aligned}\mathcal{PST}^{\mu}(sl(n)) = \{P \in \mathcal{PST}^{\mu}(A, B; d) \mid A = [n], B = [n'], d \in [n, n']^r \\ \text{with } d_i \in \{i, i'\} \text{ for } i = 1, 2, \ldots, r\},\end{aligned} \tag{2.7}$$

where the entries η_{ij} in each $sl(n)$-standard primed shifted tableau P subject to the conditions (P1)–(P6) of (2.4), with $A = [n] = \{1, 2, \ldots, n\}$, $B = [n'] = \{1', 2', \ldots, n'\}$ and the elements of $D = [n, n'] = [n] \cup [n']$ subject to the order relations $1' < 1 < 2' < 2 < \cdots < n' < n$.

By way of illustration, in the case $n = 5$, $\lambda = (4, 3, 3)$ and $\mu = (9, 7, 6, 2, 1)$ we have typically:

$$T = \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & \\ 4 & 4 & 5 & \end{array} \in \mathcal{T}^{433}(sl(5)), \quad S = \begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 \\ & 2 & 2 & 2 & 3 & 4 & 4 & 4 & \\ & & 3 & 4 & 5 & 5 & 5 & 5 & \\ & & & 4 & 5 & & & & \\ & & & & 5 & & & & \end{array} \in \mathcal{ST}^{97621}(sl(5)) \tag{2.8}$$

and

$$P = \begin{array}{ccccccccc} 1 & 1 & 1 & 2' & 2 & 3 & 3 & 4' & 5 \\ & 2' & 2 & 2 & 3' & 4' & 4 & 4 & \\ & & 3 & 4' & 5' & 5 & 5 & 5 & \\ & & & 4' & 5' & & & & \\ & & & & 5' & & & & \end{array} \in \mathcal{PST}^{97621}(sl(5)). \tag{2.9}$$

The structure of each $T \in T^{\lambda}(sl(n))$ is that of a sequence of horizontal strips [8]. Each horizontal strip, $\mathrm{str}_k(T)$, which may or may not be connected, is the subtableau of T consisting of all boxes of T for which the entries η_{ij} take the same value k. The rules (2.2) are such that there are no two boxes of a horizontal strip in the same column. In the same way the structure of each $S \in ST^{\mu}(sp(2n))$ is that of a sequence of what we shall call ribbon strips. They appear in the literature as boundary strips [19] where they are used to calculate characters of Hecke algebras. In that context they are a generalisation of the more familiar border strips [8], also known as skew hooks or rim hooks [12], that are used to calculate characters of the symmetric group by means of the Murnaghan-Nakayama rule. Here, each ribbon strip, $\mathrm{str}_k(S)$, which may or may not be connected, is the subtableau of S consisting of all boxes of S for which the entries η_{ij} take the same value k. In this case the rules (2.3) are such that there are no two boxes of a ribbon strip on the same diagonal. These two types of strip are illustrated by the following subtableaux of the tableaux of (2.8):

$$\mathrm{str}_4(T) = \begin{array}{cccc} & & & \boxed{4} \\ \boxed{4} & \boxed{4} & & \end{array} \quad \text{and} \quad \mathrm{str}_4(S) = \begin{array}{ccccc} & & & & \boxed{4} \\ & & \boxed{4} & \boxed{4} & \boxed{4} \\ \boxed{4} & & & & \\ \boxed{4} & & & & \end{array} \tag{2.10}$$

Similarly, the structure of each $P \in PST^{\mu}(sl(n))$ is that of a sequence of ribbon strips. This time each ribbon strip, $\mathrm{str}_{k,k'}(T)$, which may or may not be connected, is the subtableau of P consisting of all boxes of P for which the entries η_{ij} take the value k or k'. The rules (2.4) are such that there are no two boxes of a ribbon strip on the same diagonal. These primed ribbon strips are illustrated by the following subtableau of the primed tableau (2.9):

$$\mathrm{str}_{4,4'}(P) = \begin{array}{ccccc} & & & & \boxed{4'} \\ & & \boxed{4'} & \boxed{4} & \boxed{4} \\ \boxed{4'} & & & & \\ \boxed{4'} & & & & \end{array} \tag{2.11}$$

All of the above can be extended from the case of $sl(n)$ to that of $sp(2n)$. The essential steps are to replace $A = [n] = \{1, 2, \ldots, n\}$ by $A = [n, \bar{n}] = [n] \cup [\bar{n}] = \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$, and to identify the appropriate order relations and constraints on the relevant profiles. The required definitions are as follows:

Definition 2.4 ([5, 6, 17]) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of length r. Then the set of all $sp(n)$-standard tableaux of shape λ is defined by

$$\begin{aligned} \mathcal{T}^{\lambda}(sp(2n)) = \{T \in \mathcal{T}^{\lambda}(A; a) \mid A = [n, \bar{n}], a \in [n, \bar{n}]^r \\ \text{with } a_i \geq i \text{ for } i = 1, 2, \ldots, r\}, \end{aligned} \tag{2.12}$$

where the entries η_{ij} of each $sp(2n)$-standard tableau T satisfy the conditions (T1)–(T4) of (2.2), with $A = [n, \bar{n}] = \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$, and the elements of $[n, \bar{n}]$ subject to the order relations $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{n} < n$.

Definition 2.5 Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set of all $sp(2n)$-standard shifted tableaux of shape μ is defined by:

$$\begin{aligned} \mathcal{ST}^{\mu}(sp(2n)) = \{S \in \mathcal{ST}^{\mu}(A; a) \mid A = [n, \bar{n}], a \in [n, \bar{n}]^r \\ \text{with } a_i \in \{i, \bar{i}\} \text{ for } i = 1, 2, \ldots, r\}, \end{aligned} \tag{2.13}$$

where the entries η_{ij} of each $sp(2n)$-standard shifted tableau S satisfy the conditions (S1)–(S5) of (2.3), with $A = [n, \bar{n}] = \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$, and the elements of $[n, \bar{n}]$ subject to the order relations $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{n} < n$.

Definition 2.6 Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length r, all of whose parts are distinct. Then the set of all $sp(2n)$-standard primed shifted tableaux of shape μ is defined by:

$$\begin{aligned}\mathcal{PST}^{\mu}(sp(2n)) = \{P \in \mathcal{PST}^{\mu}(A, B; d) \mid A = [n, \bar{n}],\ B = [n', \bar{n}'],\ d \in [n, \bar{n}, n', \bar{n}']^r \\ \text{with } d_i \in [i, \bar{i}, i', \bar{i}']\} \text{ for } i = 1, 2, \ldots, r\},\end{aligned} \tag{2.14}$$

where the entries η_{ij} of each $sp(2n)$-standard primed shifted tableau P satisfy the conditions (P1)–(P6) of (2.4), with $A = [n, \bar{n}] = \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ and $B = [n', \bar{n}'] = \{1', 2', \ldots, n'\} \cup \{\bar{1}', \bar{2}', \ldots, \bar{n}'\}$, and the elements of $D = [n, \bar{n}, n'\bar{n}'] = [n, \bar{n}] \cup [n', \bar{n}']$ subject to the order relation

$$\bar{1}' < \bar{1} < 1' < 1 < \bar{2}' < \bar{2} < 2' < 2 < \cdots < \bar{n}' < \bar{n} < n' < n. \tag{2.15}$$

Typically, for $n = 5$, $\lambda = (4, 3, 3)$ and $\mu = (9, 7, 6, 2, 1)$ we have

$$T = \begin{array}{cccc} \bar{1} & 1 & \bar{2} & 4 \\ 3 & \bar{4} & \bar{4} & \\ \bar{4} & 4 & 4 & \end{array} \in \mathcal{T}^{433}(sp(10)), \quad S = \begin{array}{ccccccccc} \bar{1} & 1 & \bar{2} & 2 & \bar{3} & \bar{3} & \bar{4} & 4 & 5 \\ & \bar{2} & \bar{2} & 2 & 3 & \bar{4} & \bar{4} & 4 & \\ & & 3 & \bar{4} & 4 & 4 & 4 & 4 & \\ & & & 4 & 4 & & & & \\ & & & & \bar{5} & & & & \end{array} \in \mathcal{ST}^{97621}(sp(10)) \tag{2.16}$$

and

$$P = \begin{array}{ccccccccc} \bar{1} & 1 & \bar{2}' & 2' & \bar{3}' & \bar{3} & \bar{4}' & 4' & 5 \\ & \bar{2}' & \bar{2} & 2 & 3 & \bar{4}' & \bar{4} & 4' & \\ & & 3' & \bar{4}' & 4' & 4 & 4 & 4 & \\ & & & 4' & 4 & & & & \\ & & & & \bar{5}' & & & & \end{array} \in \mathcal{PST}^{97621}(sp(10)). \tag{2.17}$$

As before all $T \in \mathcal{T}^{\lambda}(sp(2n))$, $S \in \mathcal{ST}^{\mu}(sp(2n))$ and $P \in \mathcal{PST}^{\mu}(sp(2n))$ are made up of sequences of horizontal or ribbon strips, as appropriate. These strips, now associated with entries all k, or all $\bar{k}$, or all k and k', or all $\bar{k}$ and $\bar{k}'$ are exemplified by

$$\mathrm{str}_{\bar{4}}(T) = \begin{array}{ccc} & \bar{4} & \bar{4} \\ \bar{4} & & \end{array} \qquad \mathrm{str}_{4}(T) = \begin{array}{ccc} & & 4 \\ & & \\ 4 & 4 & \end{array}, \tag{2.18}$$

$$\mathrm{str}_{\bar{4}}(S) = \begin{array}{cccc} & & & \bar{4} \\ & & \bar{4} & \bar{4} \\ \bar{4} & & & \end{array} \qquad \mathrm{str}_4(S) = \begin{array}{ccccc} & & & & 4 \\ & & & & 4 \\ & 4 & 4 & 4 & 4 \\ 4 & 4 & & & \end{array} \tag{2.19}$$

and

$$\mathrm{str}_{\bar{4},\bar{4}'}(P) = \begin{array}{cccc} & & & \bar{4}' \\ & & \bar{4}' & \bar{4} \\ \bar{4}' & & & \end{array} \qquad \mathrm{str}_{4,4'}(P) = \begin{array}{ccccc} & & & & 4' \\ & & & & 4' \\ & 4' & 4 & 4 & 4 \\ 4 & 4 & & & \end{array}. \tag{2.20}$$

3. Okada's Theorem

The proof of Tokuyama's Theorem 1.1 offered by Okada [10] depends crucially on the following:

Theorem 3.1 ([10]) *Let $D = A \cup B$ with $A \cap B = \emptyset$ be a totally ordered set, and let $d = (d_1, d_2, \ldots, d_r)$ be a strictly increasing sequence of elements of D. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition of length $\ell(\mu) = r$, all of whose parts are distinct. Then*

$$\sum_{P \in \mathcal{PST}^{\mu}(A,B;d)} z^{\mathrm{wgt}(P)} = \left| \tilde{q}^{(d_j)}_{\mu_i}(z) - \tilde{q}^{(d_j+1)}_{\mu_i}(z) \right|_{1 \le i,j \le r}, \tag{3.1}$$

where the summation on the left hand side is taken over all primed shifted tableaux, P, such that the entries η_{ij} satisfy the conditions (2.4), *and*

$$z^{\mathrm{wgt}(P)} = \prod_{(i,j) \in SF^{\mu}} z_{\eta_{ij}}, \tag{3.2}$$

while on the right hand side the $\tilde{q}^{(m)}_k(z)$'s are determined by the following generating function in the indeterminate s

$$\sum_{k=0}^{\infty} \tilde{q}^{(m)}_k(z)\, s^k = \prod_{a \in A; a \ge m} (1 - z_a s)^{-1} \prod_{b \in B; b \ge m} (1 + z_b s). \tag{3.3}$$

In order to derive Theorem 1.1 from Theorem 3.1 it is necessary to set $r = n$, $\mu = \lambda + \delta$, to identify A with $[n]$ and B with $[n']$, to restrict d_i to be either i or i' for $i = 1, 2, \ldots, r$, as in (2.7), and to set $z_a = x_k$ for $a = k \in A = [n]$ and $z_b = tx_k$ for $b = k' \in B = [n']$. Provided that we make analogous assignments we can use precisely the same technique, due to Okada [10], to derive Theorem 1.2 from Theorem 3.1. First we require

Lemma 3.2 *In the notation of Theorem* 3.1, *let* $r = n$, $A = [n] \cup [\bar{n}]$, $B = [n'] \cup [\bar{n}']$, *and let* $\mathcal{D}(n, n', \bar{n}, \bar{n}') = \{d = (d_1, d_2, \ldots, d_n) \,|\, d_i \in \{i, i', \bar{i}, \bar{i}'\}$ *for* $i = 1, 2 \ldots, n\}$. *Let* μ *be a partition of length* $\ell(\mu) = n$, *all of whose parts are distinct. If*

$$z_k = x_k, \quad z_{k'} = tx_k, \quad z_{\bar{k}} = t^2 x_{\bar{k}} = t^2 x_k^{-1} \quad \textit{and} \quad z_{\bar{k}'} = tx_{\bar{k}} = tx_k^{-1} \tag{3.4}$$

for all $k = 1, 2, \ldots, n$, *then*

$$\sum_{\substack{P \in \mathcal{PST}^\mu([n,\bar{n}],[n',\bar{n}'];d)\\ d \in \mathcal{D}(n,n',\bar{n},\bar{n}')}} z^{\mathrm{wgt}(P)} = \sum_{S \in \mathcal{ST}^\mu(sp(2n))} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)} (1+t)^{\mathrm{str}(S)}\, x^{\mathrm{wgt}(S)}, \tag{3.5}$$

where

$$x^{\mathrm{wgt}(S)} = \sum_{(i,j) \in SF^\mu} x_{\eta_{ij}}, \tag{3.6}$$

while $\mathrm{bar}(S)$ *is the total number of barred entries in* S, $\mathrm{str}(S)$ *is the total number of connected components of all ribbon strips of unbarred and barred entries of* S, *and*

$$\mathrm{hgt}(S) = \sum_{k=1}^{n} (\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\bar{k}}(S)). \tag{3.7}$$

Proof: The requirement that $d_i \in \{i, i', \bar{i}, \bar{i}'\}$ for all $i = 1, 2 \ldots, n$, when coupled with the condition $\ell(\mu) = n$, is sufficient to ensure that for each $P \in \mathcal{PST}^\mu([n, \bar{n}], [n', \bar{n}']; d)$ the removal of all primes from the entries of P will yield some $S \in \mathcal{ST}^\mu(sp(2n))$. Moreover, every such S is obtained in this way. If we let $\mathcal{P}(S)$ be the set of primed tableaux $P \in \mathcal{PST}^\mu([n, \bar{n}], [n', \bar{n}']; d)$ such that the deletion of primes from P yields the tableau $S \in \mathcal{ST}^\mu(sp(2n))$, then the x-dependence of $z^{\mathrm{wgt}(P)}$ is the same for all $P \in \mathcal{P}(S)$. In fact, by virtue of the assignments (3.4) this x-dependence is just $x^{\mathrm{wgt}(S)}$. Moreover, these same assignments imply that the t-dependance of $z^{\mathrm{wgt}(P)}$ is just $t^{n'(P)+2\bar{n}(P)+\bar{n}'(P)}$, where $n'(P)$, $\bar{n}(P)$ and $\bar{n}'(P)$ denote the numbers of entries η_{ij} in P that belong to $[n']$, $[\bar{n}]$ and $[\bar{n}']$, respectively. It follows that

$$\begin{aligned}\sum_{\substack{P \in \mathcal{PST}^\mu([n,\bar{n}],[n',\bar{n}'];d)\\ d \in \mathcal{D}(n,n',\bar{n},\bar{n}')}} z^{\mathrm{wgt}(P)} &= \sum_{S \in \mathcal{ST}^\mu(sp(2n))} \sum_{P \in \mathcal{P}(S)} z^{\mathrm{wgt}(P)} \\ &= \sum_{S \in \mathcal{ST}^\mu(sp(2n))} x^{\mathrm{wgt}(S)} \sum_{P \in \mathcal{P}(S)} t^{n'(P)+2\bar{n}(P)+\bar{n}'(P)},\end{aligned} \tag{3.8}$$

To explore the t-dependence further it is worth considering the mapping from P to S in more detail. The constraints (2.3) and (2.4) on S and P, respectively, are such that all $P \in \mathcal{P}(S)$ giving rise to a particular S through the removal of primes are identical, save for the entries of P in the bottom left hand box of each connected component of each ribbon strip subtableau. These entries may be either primed or unprimed, as shown below in typical connected components of the ribbon strips $\mathrm{str}_{k,k'}(P)$ and $\mathrm{str}_{\bar{k},\bar{k}'}(P)$:

$$\begin{array}{|c|c|c|c|c|c|} & & & k' & k & k \\ & & & k' & & \\ & & & k' & & \\ & & & k' & & \\ k' & k & k & k & & \\ k & & & & & \end{array}\;, \quad \begin{array}{cccccc} & & & k' & k & k \\ & & & k' & & \\ & & & k' & & \\ & & & k' & & \\ k' & k & k & k & & \\ k' & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{cccccc} & & & k & k & k \\ & & & k & & \\ & & & k & & \\ & & & k & & \\ k & k & k & k & & \\ k & & & & & \end{array} \tag{3.9}$$

and

$$\begin{array}{cccccc} & & & \bar{k}' & \bar{k} & \bar{k} \\ & & & \bar{k}' & & \\ & & & \bar{k}' & & \\ & & & \bar{k}' & & \\ \bar{k}' & \bar{k} & \bar{k} & \bar{k} & & \\ \bar{k} & & & & & \end{array}\;, \quad \begin{array}{cccccc} & & & \bar{k}' & \bar{k} & \bar{k} \\ & & & \bar{k}' & & \\ & & & \bar{k}' & & \\ & & & \bar{k}' & & \\ \bar{k}' & \bar{k} & \bar{k} & \bar{k} & & \\ \bar{k}' & & & & & \end{array} \quad \longrightarrow \quad \begin{array}{cccccc} & & & \bar{k} & \bar{k} & \bar{k} \\ & & & \bar{k} & & \\ & & & \bar{k} & & \\ & & & \bar{k} & & \\ \bar{k} & \bar{k} & \bar{k} & \bar{k} & & \\ \bar{k} & & & & & \end{array}\;. \tag{3.10}$$

The t-dependence of the left hand sides of (3.9) and (3.10) is completely determined by the assignments (3.4) which imply that entries k, $\bar{k}$, k' and $\bar{k}'$ in P give rise to factors 1, t^2, t and t, respectively. Combining the contributions from the pairs of terms on the left hand sides then fixes the contribution on the right hand sides as follows:

$$\begin{array}{cccccc} & & & t & 1 & 1 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & 1 & 1 & 1 & & \\ 1 & & & & & \end{array} \quad + \quad \begin{array}{cccccc} & & & t & 1 & 1 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & 1 & 1 & 1 & & \\ t & & & & & \end{array} \quad \longrightarrow \quad (1+t) \quad \begin{array}{cccccc} & & & t & 1 & 1 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & 1 & 1 & 1 & & \\ 1 & & & & & \end{array} \tag{3.11}$$

and

$$\begin{array}{cccccc} & & & t & t^2 & t^2 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & t^2 & t^2 & t^2 & & \\ t^2 & & & & & \end{array} \quad + \quad \begin{array}{cccccc} & & & t & t^2 & t^2 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & t^2 & t^2 & t^2 & & \\ t & & & & & \end{array} \quad \longrightarrow \quad (1+t) \quad \begin{array}{cccccc} & & & t & t^2 & t^2 \\ & & & t & & \\ & & & t & & \\ & & & t & & \\ t & t^2 & t^2 & t^2 & & \\ t & & & & & \end{array}\;. \tag{3.12}$$

The right hand side of the latter can equally well be rewritten in the form:

$$(1+t)\, t^{2\bar{b}} \qquad \begin{array}{|c|c|c|c|c|c|}\cline{4-6} \multicolumn{3}{c|}{} & \bar{t} & 1 & 1 \\ \cline{4-6} \multicolumn{3}{c|}{} & \bar{t} & \multicolumn{2}{c}{} \\ \cline{4-4} \multicolumn{3}{c|}{} & \bar{t} & \multicolumn{2}{c}{} \\ \cline{4-4} \multicolumn{3}{c|}{} & \bar{t} & \multicolumn{2}{c}{} \\ \cline{1-4} \bar{t} & 1 & 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-4} \bar{t} & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array} \quad , \tag{3.13}$$

where $\bar{t} = t^{-1}$ and $\bar{b}$ is the total number of boxes in the relevant connected component of the barred ribbon strip.

Consideration of the general structure of these examples shows that each connected component of a ribbon strip of S contributes a factor $(1+t)$, each barred entry contributes a factor t^2, while connected components of ribbon strips of unbarred and barred entries contribute factors t^{r-1} and t^{-r}, respectively, where r is the number of rows occupied by the connected component.

By way of example, the $sp(2n)$-standard shifted tableau S displayed in (2.16) consists of 6 connected components of ribbon strips of unbarred and 6 of barred entries. Applying (3.11) and (3.12) to obtain the t-dependence of each connected components of type (3.9) and (3.10), respectively, gives rise to the following t-dependence of (2.16):

$$(1+t)^{12} \qquad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline t & 1 & t & t & t & t^2 & t & t & 1 \\ \hline \multicolumn{1}{c|}{} & t & t^2 & 1 & 1 & t & t^2 & t & \multicolumn{1}{c}{} \\ \cline{2-8} \multicolumn{2}{c|}{} & 1 & t & t & 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{3-8} \multicolumn{3}{c|}{} & 1 & 1 & \multicolumn{4}{c}{} \\ \cline{4-5} \multicolumn{4}{c|}{} & t & \multicolumn{4}{c}{} \\ \cline{5-5} \end{array} \quad = (1+t)^{12}\, t^{18}. \tag{3.14}$$

More generally, these considerations lead, in the notation of Lemma 3.2 to the identity:

$$\sum_{P \in \mathcal{P}(S)} t^{n'(P)+2\bar{n}(P)+\bar{n}'(P)} = t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\, (1+t)^{\mathrm{str}(S)}. \tag{3.15}$$

Using this result (3.15) in (3.8) then completes the proof of Lemma 3.2. □

Turning to the right hand side of Okada's identity (3.1) allows one to derive the following:

Lemma 3.3 *In the notation of Theorem* 3.1, *let* $r = n$, $A = [n] \cup [\bar{n}]$, $B = [n'] \cup [\bar{n}']$, *and let* μ *be a partition of length* $\ell(\mu) = n$, *all of whose parts are distinct. If*

$$z_i = x_i, \quad z_{i'} = t x_i, \quad z_{\bar{i}} = t^2 x_{\bar{i}} = t^2 x_i^{-1} \quad \text{and} \quad z_{\bar{i}'} = t x_{\bar{i}} = t x_i^{-1} \tag{3.16}$$

for all $i = 1, 2, \ldots, n$, *then*

$$\sum_{d \in \mathcal{D}(n,n',\bar{n},\bar{n}')} \left| \tilde{q}_{\mu_i}^{(d_j)}(z) - \tilde{q}_{\mu_i}^{(d_j+1)}(z) \right|_{1 \le i,j \le n} = \left| q_{\mu_i}^{(j)}(x,t) - q_{\mu_i}^{(j+1)}(x,t) \right|_{1 \le i,j \le n}, \tag{3.17}$$

where

$$\sum_{k=0}^{\infty} q_k^{(m)}(x,t)\, s^k = \prod_{i=m}^{n} \frac{(1+tx_i s)\big(1+tx_i^{-1}s\big)}{(1-x_i s)\big(1-t^2x_i^{-1}s\big)}. \tag{3.18}$$

Proof: Our ordering (2.14) and the definition of $\mathcal{D}(n, n', \bar{n}, \bar{n}')$ involves four independent choices from $\{j, j', \bar{j}, \bar{j}'\}$ for each d_j, with $\bar{j}'+1=\bar{j}$, $\bar{j}+1=j'$, $j'+1=j$ and $j+1=\overline{j+1}'$. It follows that

$$\begin{aligned}
&\sum_{d\in\mathcal{D}(n,n',\bar{n},\bar{n}')} \big|\tilde{q}_{\mu_i}^{(d_j)}(z)-\tilde{q}_{\mu_i}^{(d_j+1)}(z)\big|_{1\le i,j\le n}\\
&\quad= \big|\tilde{q}_{\mu_i}^{(\bar{j}')}(z)-\tilde{q}_{\mu_i}^{(\bar{j})}(z)+\tilde{q}_{\mu_i}^{(\bar{j})}(z)-\tilde{q}_{\mu_i}^{(j')}(z)+\tilde{q}_{\mu_i}^{(j')}(z)-\tilde{q}_{\mu_i}^{(j)}(z)\\
&\qquad+\tilde{q}_{\mu_i}^{(j)}(z)-\tilde{q}_{\mu_i}^{(\overline{j+1}')}(z)\big|_{1\le i,j\le n}\\
&\quad= \big|\tilde{q}_{\mu_i}^{(\bar{j}')}(z)-\tilde{q}_{\mu_i}^{(\overline{j+1}')}(z)\big|_{1\le i,j\le n},
\end{aligned} \tag{3.19}$$

where, from (3.3),

$$\sum_{k=0}^{\infty} \tilde{q}_k^{(\bar{j}')}(z)\, s^k = \prod_{i=j}^{n}(1-z_i s)^{-1}(1-z_{\bar{i}}s)^{-1}\prod_{i=j}^{n}(1+z_{i'}s)(1+z_{\bar{i}'}s). \tag{3.20}$$

The use of the specialisation (3.16) linking z to x and t, and comparison with the definition (3.18) then completes the proof of Lemma 3.3. □

This brings us to our final Lemma, namely:

Lemma 3.4 *Let $X_k^{(m)}$ be defined by*

$$X_k^{(m)} = \frac{1}{1+t}\big(q_k^{(m)}-q_k^{(m+1)}\big) \quad \textit{for } 1\le k,m\le n, \tag{3.21}$$

and let $Y_k^{(m,p)}$ be defined by the generating function

$$\sum_{k=0}^{\infty} Y_k^{(m,p)}s^k = s^p\prod_{i=m+p}^{n}(1+tx_i s)\big(1+tx_i^{-1}s\big)\prod_{i=m}^{n}(1-x_i s)^{-1}\big(1-t^2x_i^{-1}s\big)^{-1}$$
$$\textit{for } 1\le k,m\le n \quad \textit{and} \quad 1\le p\le n-m+1. \tag{3.22}$$

Then

$$\big|X_k^{(m)}\big|_{1\le k,m\le n} = D_{sp(2n)}(x;t)\big|Y_k^{(m,n-m+1)}\big|_{1\le k,m\le n}, \tag{3.23}$$

where $D_{sp(2n)}(x;t)$ is defined by (1.10)

Proof: From (3.18)

$$\begin{aligned}
&\sum_{k=0}^{\infty} X_k^{(m)} s^k \\
&\quad = \frac{1}{1+t}\big((1+tx_m s)\big(1+tx_m^{-1}s\big) - (1-x_m s)\big(1-t^2x_m^{-1}s\big)\big) \\
&\qquad \times \prod_{i=m+1}^{n}(1+tx_i s)\big(1+tx_i^{-1}s\big)\prod_{i=m}^{n}(1-x_i s)^{-1}\big(1-t^2x_i^{-1}s\big)^{-1} \\
&\quad = \big(x_m + tx_m^{-1}\big)s \prod_{i=m+1}^{n}(1+tx_i s)\big(1+tx_i^{-1}s\big)\prod_{i=m}^{n}(1-x_i s)^{-1}\big(1-t^2x_i^{-1}s\big)^{-1} \\
&\quad = x_m\big(1+tx_m^{-2}\big)\sum_{k=0}^{\infty} Y_k^{(m,1)}\, s^k.
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\infty}\big(Y_k^{(m,p)} - Y_k^{(m+1,p)}\big)s^k \\
&\quad = \big((1+tx_{m+p}s)\big(1+tx_{m+p}^{-1}s\big) - (1-x_m s)\big(1-t^2x_m^{-1}s\big)\big) \\
&\qquad \times s^p \prod_{i=m+p+1}^{n}(1+tx_i s)\big(1+tx_i^{-1}s\big)\prod_{i=m}^{n}(1-x_i s)^{-1}\big(1-t^2x_i^{-1}s\big)^{-1} \\
&\quad = \big(x_m + tx_{m+p} + tx_{m+p}^{-1} + t^2x_m^{-1}\big) \\
&\qquad \times s^{p+1} \prod_{i=m+p+1}^{n}(1+tx_i s)\big(1+tx_i^{-1}s\big)\prod_{i=m}^{n}(1-x_i s)^{-1}\big(1-t^2x_i^{-1}s\big)^{-1} \\
&\quad = x_m\big(1+tx_m^{-1}x_{m+p}\big)\big(1+tx_m^{-1}x_{m+p}^{-1}\big)\sum_{k=0}^{\infty} Y_k^{(m,p+1)}\, s^k.
\end{aligned} \tag{3.25}$$

Hence

$$\begin{aligned}
&\big|X_k^{(m)}\big|_{1\le k,m\le n} \\
&\quad = \prod_{i=1}^{n} x_i\big(1-tx_i^{-2}\big)\big|Y_k^{(m,1)}\big|_{1\le k,m\le n} \\
&\quad = \prod_{i=1}^{n} x_i\big(1-tx_i^{-2}\big)\prod_{1\le i<j\le n} x_i\big(1-tx_i^{-1}x_j\big)\big(1-tx_i^{-1}x_j^{-1}\big)\big|Y_k^{(m,n-m+1)}\big|_{1\le k,m\le n} \\
&\quad = D_{sp(2n)}(x;t)\big|Y_k^{(m,n-m+1)}\big|_{1\le k,m\le n},
\end{aligned} \tag{3.26}$$

where the first step follows from (3.24), and the second from (3.25) by the subtraction of column j in the determinant from every column m with $m < j$ succesively for $j = n, n-1, \ldots, 2$. This completes the proof of Lemma 3.4. □

However,

$$
\begin{aligned}
&\sum_{k=0}^{\infty} Y_k^{(m,n-m+1)} s^k \\
&\quad = s^{n-m+1} \prod_{i=m}^{n} (1 - x_i s)^{-1} \left(1 - t^2 x_i^{-1} s\right)^{-1} \\
&\quad = s^{n-m+1} \sum_{\ell=0}^{\infty} h_\ell(x_m, t^2 x_{\bar{m}}, x_{m+1}, t^2 x_{\overline{m+1}}, \ldots, x_n, t^2 x_{\bar{n}})\, s^\ell \\
&\quad = \sum_{k=0}^{\infty} h_{k-n+m-1}(x_m, t^2 x_{\bar{m}}, x_{m+1}, t^2 x_{\overline{m+1}}, \ldots, x_n, t^2 x_{\bar{n}})\, s^k,
\end{aligned}
\tag{3.27}
$$

where h_ℓ is the complete homogeneous symmetric function of degree ℓ of its various arguments. Hence

$$
Y_k^{(m,n-m+1)} = h_{k-n+m-1}(x_m, t^2 x_{\bar{m}}, x_{m+1}, t^2 x_{\overline{m+1}}, \ldots, x_n, t^2 x_{\bar{n}}),
\tag{3.28}
$$

Remarkably, we have the following [2]

Theorem 3.5 ([2]) *Let λ be any partition of length $\ell(\lambda) \leq n$. Then the character $sp_\lambda(x) = sp_\lambda(x_1, x_2, \ldots, x_n)$ of the corresponding irreducible representation of $sp(2n)$ is given by*

$$
sp_\lambda(x) = \left| h_{\lambda_i - i + j}(x_i, x_{\bar{i}}, x_{i+1}, x_{\overline{i+1}}, \ldots, x_n, x_{\bar{n}}) \right|_{1 \leq i, j \leq n} .
\tag{3.29}
$$

However, the deformation of the character $sp_\lambda(x)$, given in terms of $sp(2n)$-standard tableaux by (1.10), to give $sp_\lambda(x; t)$ as in (1.11) is brought about by associating a factor of t^2 with every barred entry $\bar{k}$ in each of the relevant tableaux $T \in \mathcal{T}([n, \bar{n}])$. Applying this additional weighting to each factor of x_k^{-1} arising in the lattice path derivation of (3.29) leads immediately to the identity

$$
sp_\lambda(x; t) = \left| h_{\lambda_i - i + j}(x_i, t^2 x_{\bar{i}}, x_{i+1}, t^2 x_{\overline{i+1}}, \ldots, x_n, t^2 x_{\bar{n}}) \right|_{1 \leq i, j \leq n}.
\tag{3.30}
$$

Combining our sequence of Lemmas with this result gives

$$
\begin{aligned}
&\sum_{S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))} t^{\mathrm{hgt}(S) + 2\mathrm{bar}(S)} (1+t)^{\mathrm{str}(S) - n}\, x^{\mathrm{wgt}(S)} \\
&\quad = (1+t)^{-n} \sum_{\substack{P \in \mathcal{PST}^{\lambda+\delta}([n,\bar{n}],[n',\bar{n}'];d) \\ d \in \mathcal{D}(n,n',\bar{n},\bar{n}')}} z^{\mathrm{wgt}(P)} \\
&\quad = (1+t)^{-n} \sum_{d \in \mathcal{D}(n,n',\bar{n},\bar{n}')} \left| \tilde{q}_{\mu_i}^{(d_j)}(z) - \tilde{q}_{\mu_i}^{(d_j+1)}(z) \right|_{1 \leq i, j \leq n} \\
&\quad = (1+t)^{-n} \left| q_{\mu_i}^{(j)}(x, t) - q_{\mu_i}^{(j+1)}(x, t) \right|_{1 \leq i, j \leq n} = \left| X_{\mu_i}^{(j)} \right|_{1 \leq i, j \leq n}
\end{aligned}
$$

$$
\begin{aligned}
&= D_{sp(2n)}(x;t)\left|Y^{(j,n-j+1)}_{\lambda_i+n-i+1}\right|_{1\le i,j\le n} \\
&= D_{sp(2n)}(x;t)\left|h_{\lambda_i-i+j}(x_j, t^2x_{\bar{j}}, x_{j+1}, t^2x_{\overline{j+1}}, \ldots, x_n, t^2x_{\bar{n}})\right|_{1\le i,j\le n} \\
&= D_{sp(2n)}(x;t)sp_\lambda(x;t),
\end{aligned} \tag{3.31}
$$

where recourse has been made succesively to (3.5) of Lemma 3.2, (3.1) of Theorem 3.1, (3.17) of Lemma 3.3, (3.21) and (3.23) of Lemma 3.4, (3.28) and, finally, (3.30) which is really a corollary of Theorem 3.5. This completes the proof of Theorem 1.2.

4. Specialisations

Theorem 1.2 is rich in corollaries which arise by specialising, both in turn and in various combinations, the parameter t, the partition λ and the indeterminates $x = (x_1, x_2, \ldots, x_n)$.

First of all, the case $t = -1$ and allowing λ to be any fixed partition of length $\ell(\lambda) \le n$, we obtain Weyl's character formula for $sp(2n)$ from Theorem 1.2.

Corollary 4.1 ([20]) *Let W be the Weyl group of $sp(2n)$, let δ be half the sum of the positive roots of $sp(2n)$ and let λ be an arbitrary partition of length $\ell(\lambda) \le n$. Then the irreducible representation of $sp(2n)$ specified by λ has character*:

$$
sp_\lambda(x) = \sum_{w\in W} \operatorname{sgn}(w) e^{w(\lambda+\delta)} \bigg/ \sum_{w\in W} \operatorname{sgn}(w) e^{w(\delta)}. \tag{4.1}
$$

Proof: Setting $t = -1$ in (1.15) of Theorem 1.2 gives

$$
D_{sp(2n)}(x;-1)sp_\lambda(x) = \sum_{\substack{S\in\mathcal{ST}^{\lambda+\delta}(sp(2n))\\ \operatorname{str}(S)=n}} (-1)^{\operatorname{hgt}(S)} x^{\operatorname{wgt}(S)}, \tag{4.2}
$$

with $\delta = (n, n-1, \ldots, 1)$. It should be noted that setting $t = -1$ has reduced $sp_\lambda(x;t)$ to $sp_\lambda(x)$, as can be seen from (1.13) and (1.11), whilst at the same time restricting the summation over S to those S for which $\operatorname{str}(S) = n$.

This condition $\operatorname{str}(S) = n$, together with the fact that $\ell(\lambda+\delta) = n$, implies that there are precisely n ribbon strips in S and that each of these ribbon strips consists of a single connected component. In addition, the conditions (2.13) on the entries in $sp(2n)$-standard shifted tableaux imply that the entry $d_k(S) = a_k$ in the kth box of the main diagonal of each $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ must be either k or $\bar{k}$. Thus, the ribbon strip emanating from this box is either $\operatorname{str}_k(S)$ or $\operatorname{str}_{\bar{k}}(S)$. Moreover, the final condition of (2.3) implies that each diagonal contains distinct entries. The lengths of the diagonals of $SF^{\lambda+\delta}$ now vary from n to 1 in such a way that in moving from one diagonal to the next at most one entry is dropped. If one entry, say k or $\bar{k}$, is dropped then this terminates the corresponding ribbon strip, $\operatorname{str}_k(S)$ or $\operatorname{str}_{\bar{k}}(S)$, and all preceding ribbon strips, $\operatorname{str}_i(S)$ or $\operatorname{str}_{\bar{i}}(S)$ with $i < k$, are extended horizontally by one box, while all succeeding ribbon strips, $\operatorname{str}_j(S)$ or $\operatorname{str}_{\bar{j}}(S)$ with $j > k$, are extended vertically by one box. The length of a ribbon strip is just the number of diagonals it spans. Thus, the ribbon strips of any $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ with $\mathrm{str}(S) = n$ must have distinct lengths equal to the various parts of $\lambda + \delta$.

This is exemplified in the case $n = 5$, $\lambda = (4, 3, 3)$ and $\lambda + \delta = (9, 7, 6, 2, 1)$ by the following tableau $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ with $\mathrm{str}(S) = n$, which is displayed along with its contribution to the right hand side of both (4.2) and the numerator of (4.1),

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\
\hline
\multicolumn{1}{c|}{} & \bar{2} & \bar{2} & 4 & 4 & 4 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{2-8}
\multicolumn{2}{c|}{} & 3 & 4 & \bar{5} & \bar{5} & \bar{5} & \bar{5} & \multicolumn{1}{c}{} \\
\cline{3-8}
\multicolumn{3}{c|}{} & 4 & \bar{5} & \multicolumn{4}{c}{} \\
\cline{4-5}
\multicolumn{4}{c|}{} & \bar{5} & \multicolumn{4}{c}{} \\
\cline{5-5}
\end{array}
\longrightarrow
\begin{cases}
(-1)^{-2+2-3-1+0} x_{\bar{2}}^{9}\, x_{4}^{7}\, x_{\bar{5}}^{6}\, x_{\bar{1}}^{2}\, x_{3}^{1} \\
= \mathrm{sgn}\begin{pmatrix} 1\,2\,3\,4\,5 \\ \bar{2}\,4\,\bar{5}\,\bar{1}\,3 \end{pmatrix} x_1^{-2}\, x_2^{-9}\, x_3^{1}\, x_4^{7}\, x_5^{-6},
\end{cases} \tag{4.3}$$

Quite generally, for each $k = 1, 2, \ldots, n$, the exponent of x_k in $x^{\mathrm{wgt}(S)}$ is either the length $|str_k(S)|$ of $str_k(S)$ or minus the length $|str_{\bar{k}}(S)|$ of $str_{\bar{k}}(S)$, according as $d_k(S)$ is either k or $\bar{k}$, respectively. These lengths are just the parts of $\lambda + \delta$. The corresponding contribution to $hgt(S)$ is either $row_k(S) - 1$ or $-row_{\bar{k}}(S)$, as appropriate.

To make contact with the Weyl group $W = H_n = S_2 \wr S_n$ of $sp(2n)$ as required in (4.1), it is necessary, as in the example (4.3), to identify $w_S \in W$. For given $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ with $\mathrm{str}(S) = n$ the identification of w_S proceeds by noting the sequence formed by the labels of the ribbon strips of S when placed in order of decreasing length $\lambda_1 + n, \lambda_2 + n - 1, \ldots, \lambda_n + 1$. Thus

$$w_S = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix} \tag{4.4}$$

where, for $i = 1, 2, \ldots, n$, w_i is the label k or $\bar{k}$ of the unique ribbon strip of S having length $\lambda_i + \delta_i$, that is

$$w_i = \begin{cases} k & \text{if } |\mathrm{str}_k(S)| = \lambda_i + \delta_i \text{ and } d_k(S) = k; \\ \bar{k} & \text{if } |\mathrm{str}_{\bar{k}}(S)| = \lambda_i + \delta_i \text{ and } d_k(S) = \bar{k}. \end{cases} \tag{4.5}$$

This is exemplified in (4.3) by

$$w_S = \begin{pmatrix} 1\,2\,3\,4\,5 \\ \bar{2}\,4\,\bar{5}\,\bar{1}\,3 \end{pmatrix}. \tag{4.6}$$

It follows from the above definition of w_S that

$$\begin{aligned} x^{\mathrm{wgt}(S)} &= x_{w_1}^{\lambda_1+\delta_1} x_{w_2}^{\lambda_2+\delta_2} \cdots x_{w_n}^{\lambda_n+\delta_n} = w_S\big(x_1^{\lambda_1+\delta_1} x_2^{\lambda_2+\delta_2} \cdots x_n^{\lambda_n+\delta_n}\big) \\ &= w_S(e^{\lambda+\delta}) = e^{w_S(\lambda+\delta)} = x_1^{w_S(\lambda+\delta)_1} x_2^{w_S(\lambda+\delta)_2} \cdots x_n^{w_S(\lambda+\delta)_n}, \end{aligned} \tag{4.7}$$

where in the first line w_S acts naturally on the subscripts of the various x_i in $e^{\lambda+\delta}$, and the transformation of that action in the second line defines the action of w_S on $\lambda + \delta$, and indeed on any vector in the same n-dimensional weight space of $sp(2n)$. It follows from (4.7) and

(4.5) that for all $k = 1, 2, \ldots, n$

$$w_S(\lambda+\delta)_k = \begin{cases} \lambda_i + \delta_i & \text{if } w_i = k; \\ -(\lambda_i+\delta_i) & \text{if } w_i = \bar{k}, \end{cases} = \begin{cases} |\mathrm{str}_k(S)| & \text{if } d_k(S) = k; \\ -|\mathrm{str}_{\bar{k}}(S)| & \text{if } d_k(S) = \bar{k}. \end{cases} \tag{4.8}$$

Since $\mu = \lambda+\delta$ has length $\ell(\mu) = n$, it follows from Definition 2.5 and the conditions (2.3) that the profile $a = (a_1, a_2, \ldots, a_n) = (d_1(S), d_2(S), \ldots, d_n(S))$ of each $S \in \mathcal{ST}^{\mu}(sp(2n))$ is just a signed permutation of $(1, 2, \ldots, n)$. Furthermore the action of w_S by permutation and sign changes on the components of $\mu = \lambda + \delta$ in the ϵ-basis is precisely the action described in Section 1 which ensures that $w_S \in W = H_n = S_2 \wr S_n$.

Turning to the sign factors, we have

$$\begin{aligned}(-1)^{\mathrm{hgt}(S)} &= (-1)^{\sum_{k:d_k(S)\in[n]}(\mathrm{row}_k(S)-1)-\sum_{k:d_k(S)\in[\bar{n}]}\mathrm{row}_{\bar{k}}(S)} \\ &= (-1)^{\#\{k\,|\,d_k(S)\in[\bar{n}]\}}(-1)^{\sum_{k=1}^{n}(\mathrm{row}_k(S)+\mathrm{row}_{\bar{k}}(S)-1)},\end{aligned} \tag{4.9}$$

where use has been made of the fact that if $d_k(S) \in [n]$ then $\mathrm{row}_{\bar{k}}(S) = 0$, whereas if $d_k(S) \in [\bar{n}]$ then $\mathrm{row}_k(S) = 0$. It follows from (4.5) that

$$(-1)^{\#\{k\,|\,d_k(S)\in[\bar{n}]\}} = (-1)^{\#\{k\,|\,w_k\in[\bar{n}]\}}. \tag{4.10}$$

Furthermore the argument regarding vertical steps given prior to the illustrative example (4.3), when coupled with the ordering of the sequence of ribbon strips in the definition of w_S, implies that

$$(-1)^{\sum_{k=1}^{n}(\mathrm{row}_k(S)+\mathrm{row}_{\bar{k}}(S)-1)} = (-1)^{\#\{(k,j)\,|\,1\le k<j\le n, w_k>w_j\}}. \tag{4.11}$$

Using (4.10) and (4.11) in (4.9) gives

$$(-1)^{\mathrm{hgt}(S)} = (-1)^{\#\{k\,|\,w_k\in[\bar{n}]\}}(-1)^{\#\{(k,j)\,|\,1\le k<j\le n, w_k>w_j\}} = \mathrm{sgn}(w_S), \tag{4.12}$$

where the last step involves the recognition that each barred entry in w_S involves a change of sign and hence a single reflection in weight space, while the permutation π_S of the components of vectors in this weight space, which is identified by deleting the bars from w_S, contributes a sign factor given by -1 to a power equal to the number of its inversions.

By way of example, in the case of w_S defined by (4.6) we have

$$\mathrm{sgn}\begin{pmatrix} 1\,2\,3\,4\,5 \\ \bar{2}\,4\,\bar{5}\,\bar{1}\,3 \end{pmatrix} = (-1)^3\mathrm{sgn}\begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,4\,5\,1\,3 \end{pmatrix} = (-1)^3\,(-1)^{0+1+0+2+2} = +1. \tag{4.13}$$

The first exponent 3 is just the number of barred ribbon strips of S in (4.3), while the other exponents 0, 1, 0, 2, 2 enumerated by counting the inversions of π_S are nothing other than the numbers of vertical steps of the various ribbon strips of S. The resulting sign factor $+1$ is consistent, as it must be, with the sign factor appearing in (4.3).

It follows from the results (4.7) and (4.12) that for each $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ such that $\mathrm{str}(S) = n$,

$$(-1)^{\mathrm{hgt}(S)}x^{\mathrm{wgt}(S)} = \mathrm{sgn}(w_S)e^{w_S(\lambda+\delta)}, \tag{4.14}$$

with $w_S \in W$.

The next step is to note, conversely, that for all $w \in W$ there exists $S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))$ with $\mathrm{str}(S) = n$, such that $w = w(S)$. Every element w of the Weyl group W of $sp(2n)$ is of the form (4.4) with $w_i = k$ or $\bar{k}$ for some $k \in [n]$ for all $i = 1, 2, \ldots, n$ and $w_j \neq w_i$ or $\bar{w_i}$ for all $j \neq i$. Given such a $w \in W$ the corresponding $sp(2n)$-standard shifted tableau S is created by entering the sequence $(w_1, w_2, \ldots, w_n)$ in the boxes of the main diagonal, subject to the required constraints $\eta_{ii} < \eta_{i+1,i+1}$ for $i = 1, 2, \ldots, n-1$, and then deleting $w_n, \ldots, w_2, w_1$ one at a time in turn as one moves from any diagonal to its immediate successor if and only if that successor is shorter in length. No omissions are required if successive diagonals have the same length, and as already pointed out successive diagonals differ in length by at most one box. This process ensures that S is of shape $SF^{\lambda+\delta}$ and that it consists of precisely n connnected ribbon strips wrapped around one another in such a way as to automatically satisfy all the constraints (2.3).

It follows that (4.2) can be rewritten in the form

$$D_{sp(2n)}(x;-1)sp_\lambda(x) = \sum_{\substack{S\in\mathcal{ST}^{\lambda+\delta}(sp(2n))\\ \mathrm{str}(S)=n}} \mathrm{sgn}(w_S)e^{w_S(\lambda+\delta)} = \sum_{w\in W} \mathrm{sgn}(w)e^{w(\lambda+\delta)}. \quad (4.15)$$

Setting $\lambda = 0$ it follows that

$$D_{sp(2n)}(x;-1) = \sum_{w\in W} \mathrm{sgn}(w)e^{w(\delta)}. \quad (4.16)$$

Taking the ratio of (4.15) and (4.16) immediately gives (4.1), thereby completing the proof of Corollary 4.1. □

From the Definition (1.2), setting $t = -1$ in $D_{sp(2n)}(x;t)$ gives the undeformed Weyl denominator function for $sp(2n)$. It follows that (4.16) is just the original denominator formula of Weyl [20] for the case of the Lie algebra $sp(2n)$:

Corollary 4.2 ([20]) *Let Δ_+ be the set of positive roots of $sp(2n)$, let δ be half the sum of these positive roots, and let W be the Weyl group of $sp(2n)$. Then*

$$e^\delta \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha}) = \sum_{w\in W} \mathrm{sgn}(w)\, e^{w(\delta)}. \quad (4.17)$$

It might be thought that the case $t = 0$ of Theorem 1.2 might provide something new. This is not the case however, since it leads only to the confirmation of the fact that

$$sp_\lambda(x;0) = s_\lambda(x). \quad (4.18)$$

Turning to a more interesting case, setting $t = 1$ in Theorem 1.2 leads directly to:

Corollary 4.3 *Let λ be a partition into no more than n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$D_{sp(2n)}(x;1)\, sp_\lambda(x) = \sum_{S\in\mathcal{ST}^{\lambda+\delta}(sp(2n))} 2^{\mathrm{str}(S)-n}\, x^{\mathrm{wgt}(S)}. \quad (4.19)$$

This represents what might be called a symplectic version of the formula given by Stanley [16] for the $t = -1$ case of the Hall-Littlewood function $P_\mu(x;t)$ with $\mu = \lambda + \delta$.

Making the further specialisation $x_i = 1$ for all $i = 1, 2, \ldots, n$ in Corollary 4.3, and using (1.10) to note that under the same specialisation $D_{sp(2n)}(x;1)$ is just 2^{n^2}, leads to:

Corollary 4.4 *Let λ be a partition into no more than n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$2^{n^2} \dim_{2n}(sp_\lambda) = \sum_{S \in \mathcal{ST}^{\lambda+\delta}(sp(2n))} 2^{\mathrm{str}(S)-n}, \tag{4.20}$$

where $\dim_{2n}(sp_\lambda) = sp_\lambda(1, 1, \ldots, 1)$ *is the dimension of the irreducible representation of* $sp(2n)$ *having character* $sp_\lambda(x)$.

This, in turn, is the symplectic version of a formula due to Mills et al. [9]. The relevant formula is that of their Theorem 2 which involves a factor that can be identified as nothing other than $\dim_n(sl_\lambda)$ where $\lambda_i = a_{n-i+1} - n + i - 1$ for $i = 1, 2, \ldots, n$.

Alternatively, making the further specialisation $\lambda = 0$ in Corollary 4.3, and employing (1.10) in Theorem 1.2, leads to:

Corollary 4.5 *Let* $\delta = (n, n-1, \ldots, 1)$. *Then*

$$\begin{aligned}\prod_{1\le i\le n} x_i^{n-i+1} \prod_{1\le i\le n} \left(1 + x_i^{-2}\right) \prod_{1\le i<j\le n} \left(1 + x_i^{-1}x_j\right)\left(1 + x_i^{-1}x_j^{-1}\right) \\ = \sum_{S\in\mathcal{ST}^\delta(sp(2n))} 2^{\mathrm{str}(S)-n}\, x^{\mathrm{wgt}(S)}.\end{aligned} \tag{4.21}$$

Generalising this for all t, if one sets $\lambda = 0$ in Theorem 1.2 and replaces x_i by its inverse x_i^{-1} for all $i = 1, 2, \ldots, n$, then one obtains an identity which is reminiscent of, but somewhat simpler than, the deformations of Weyl's denominator formulae due to Okada [11] and Simpson [13, 14]:

Corollary 4.6 *Let* $\delta = (n, n-1, \ldots, 1)$. *Then*

$$\begin{aligned}\prod_{1\le i\le n} \left(1 + t x_i^2\right) \prod_{1\le i<j\le n} (1 + t x_i x_j)\left(1 + t x_i x_j^{-1}\right) \\ = \sum_{S\in\mathcal{ST}^\delta(sp(2n))} t^{\mathrm{hgt}(S)+2\mathrm{bar}(S)}\, (1+t)^{\mathrm{str}(S)-n}\, x^{\delta-\mathrm{wgt}(S)},\end{aligned} \tag{4.22}$$

where $x^{\delta-\mathrm{wgt}(S)} = x^\delta x^{-\mathrm{wgt}(S)}$ *with* $x^\delta = x_1^n x_2^{n-1} \ldots x_n$.

5. Monotonic patterns

One of the combinatorial devices used by Tokuyama [18] was that of Gelfand patterns [3]. While we have favoured here the use of Young tableaux, there exist bijections between

$sl(n)$-standard tableaux and Gelfand patterns whose rows are specified by partitions satisfying certain betweenness conditions [1, 5] and between $sl(n)$-standard shifted tableaux and certain monotone triangles. Such monotone triangles are strict Gelfand patterns for which each row is specified by a partition all of whose parts are distinct [10, 18]. Similar bijections may be established for both $sp(2n)$-standard tableaux and $sp(2n)$-standard shifted tableaux. The first of these between $sp(2n)$-standard tableaux and a symplectic version of Gelfand patterns has already been identified [5]. The symplectic Gelfand patterns involve interweaving consecutive rows of pairs of patterns due to Zhelobenko [21] to give a single pattern. The bijection is exemplified in the case of the $sp(2n)$-standard tableau T of (2.16) by:

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & 1 & \bar{2} & 4 \\ \hline 3 & \bar{4} & \bar{4} \\ \cline{1-3} \bar{4} & 4 & 4 \\ \cline{1-3}\end{array} \quad \longleftrightarrow \quad \begin{array}{ccccccccccccccc}
4 && 3 && 3 && 0 && 0 \\
& 4 && 3 && 3 && 0 && 0 \\
&& 4 && 3 && 3 && 0 \\
&&& 3 && 3 && 1 && 0 \\
&&&& 3 && 1 && 0 \\
&&&&& 3 && 0 && 0 \\
&&&&&& 3 && 0 \\
&&&&&&& 3 && 0 \\
&&&&&&&& 2 \\
&&&&&&&&& 1
\end{array} \tag{5.1}$$

The non-zero entries in the topmost row of the pattern Z are just the parts of the partition λ defining the shape of T. In this example $n = 5$, $\lambda = (4, 3, 3)$ and the non-zero entries in the mth row of the pattern Z, counted from the top, are the parts of the partition defining the shape of the subtableau of T consisting of entries $\eta_{ij} \leq k$ with $k = (2n + 1 - m)/2 = (11 - m)$ if m is odd, and $\eta_{ij} \leq \bar{k}$ with $k = (2n + 2 - m)/2 = (12 - m)$ if m is even, for $m = 1, 2, \ldots, 2n = 1, 2, \ldots, 10$.

More generally, we specify and display the entries z_{ki} and $z_{\bar{k}i}$ of a symplectic Gelfand pattern, Z, as follows:

$$\begin{array}{cccccccccccc}
z_{n1} && z_{n2} && \cdots && \cdots & \cdots && z_{nn} \\
& z_{\bar{n}1} && z_{\bar{n}2} && \cdots & \cdots & \cdots &&& z_{\bar{n}n} \\
&& z_{n-1,1} && z_{n-1,2} && \cdots & \cdots & z_{n-1,n-1} \\
&&& z_{\overline{n-1},1} && z_{\overline{n-1},2} & \cdots & \cdots &&& z_{\overline{n-1},n-1} \\
&&&& \cdots && \cdots & \cdots && \cdots \\
&&&&& \cdots & \cdots & \cdots &&& \cdots \\
&&&&&&& z_{21} && z_{22} \\
&&&&&&&& z_{\bar{2}1} && z_{\bar{2}2} \\
&&&&&&&&& z_{11} \\
&&&&&&&&&& z_{\bar{1}1}
\end{array} \tag{5.2}$$

With this notation, we have:

Definition 5.1 [5] Let λ be a partition with length $\ell(\lambda) \leq n$. Then Z is said to belong to the set, $\mathcal{Z}^\lambda(sp(2n))$, of $sp(2n)$-patterns with top-most row λ if and only if all the entries, z_{ki} and $z_{\bar{k}i}$, of Z are non-negative integers satisfying the boundary and betweenness conditions:

$$\begin{array}{lll} \text{(Z1)} & z_{ni} = \lambda_i & \text{for } 1 \leq i \leq n \text{ with } \lambda_i = 0 \text{ for } i > \ell(\lambda); \\ \text{(Z2)} & z_{ki} \geq z_{\bar{k}i} \geq z_{k,i+1} & \text{for } 1 \leq i < k \leq n; \\ \text{(Z3)} & z_{\bar{k}i} \geq z_{k-1,i} \geq z_{\bar{k},i+1} & \text{for } 1 \leq i < k \leq n; \\ \text{(Z4)} & z_{kk} \geq z_{\bar{k}k} \geq 0 & \text{for } 1 \leq k \leq n. \end{array} \tag{5.3}$$

If, in our usual notation, $T \in \mathcal{T}^\lambda(sp(2n))$ is such that the entry in the (i, j)th box of T is η_{ij} for all $(i, j) \in F^\lambda$, then the corresponding $sp(2n)$-pattern $Z \in \mathcal{Z}^\lambda(sp(2n))$ has entries defined by:

$$z_{ki} = \#\{j \mid \eta_{ij} \leq k\} \quad \text{and} \quad z_{\bar{k}i} = \#\{j \mid \eta_{ij} \leq \bar{k}\} \quad \text{for } 1 \leq i \leq k \leq n. \tag{5.4}$$

The conditions on η_{ij}, as given by (T1)–(T4) of (2.2) applied to (2.5) in Definition 2.1, then imply that z_{ki} and $z_{\bar{k}i}$ automatically satisfy (5.3), as required.

Conversely, for given $Z \in \mathcal{Z}^\lambda(sp(2n))$ with entries z_{ki} and $z_{\bar{k}i}$ satisfying (5.3), let the partitions $\zeta(k)$ and $\zeta(\bar{k})$ be defined for $k = 1, 2, \ldots, n$ by $\zeta(k)_i = z_{ki}$ and $\zeta(\bar{k})_i = z_{\bar{k}i}$ for $i = 1, 2, \ldots, k$, and let $\zeta(0) = 0$. Then for all $(i, j) \in F^\lambda$ with $\lambda = \zeta(n)$ the entry in the (i, j)th box of $T \in \mathcal{T}^\lambda(sp(2n))$ is given for all $k \in [n]$ by

$$\eta_{ij} = \begin{cases} k & \text{if } (i, j) \in F^{\zeta(k)/\zeta(\bar{k})}; \\ \bar{k} & \text{if } (i, j) \in F^{\zeta(\bar{k})/\zeta(k-1)}, \end{cases} \tag{5.5}$$

where $F^{\tau/\sigma}$ signifies the skew diagram [8] obtained by deleting all the boxes of F^σ from those of F^τ. This time the conditions (5.3) on all the entries z_{mi} of Z imply that the entries η_{ij} of T automatically satisfy the required conditions (2.2) as applied to (2.5).

Thus for any partition λ with $\ell(\lambda) \leq n$, (5.4) and (5.5) serve to define a bijection between all $T \in \mathcal{T}^\lambda(sp(2n))$ and all $Z \in \mathcal{Z}^\lambda(sp(2n))$.

In exactly the same way there exists a bijection between $sp(2n)$-standard shifted tableaux, S, and certain $sp(2n)$-monotonic patterns, M. This is exemplified in the case of $n = 5$, $\mu = (9, 7, 6, 2, 1)$ and the $sp(2n)$-standard shifted tableau S of (2.16) by:

$$\begin{array}{ccccccccc} \bar{1} & 1 & \bar{2} & 2 & \bar{3} & \bar{3} & \bar{4} & 4 & 5 \\ & \bar{2} & \bar{2} & 2 & 3 & \bar{4} & \bar{4} & 4 & \\ & & 3 & \bar{4} & 4 & 4 & 4 & 4 & \\ & & & 4 & 4 & & & & \\ & & & & \bar{5} & & & & \end{array} \quad \longleftrightarrow \quad \begin{array}{c} 9\ 7\ 6\ 2\ 1 \\ 8\ 7\ 6\ 2\ 1 \\ 8\ 7\ 6\ 2 \\ 7\ 6\ 2\ 0 \\ 6\ 4\ 1 \\ 6\ 3\ 0 \\ 4\ 3 \\ 3\ 2 \\ 2 \\ 1 \end{array} \tag{5.6}$$

The major difference now is that the shifted nature of S implies that the partition associated with each row of the corresponding monotonic pattern M necessarily has distinct parts.

Definition 5.2 Let μ be a partition of length $\ell(\mu) = n$ all of whose parts are distinct. Specifying the entries z_{ki} and $z_{\bar{k}i}$ of M as in (5.2), M is said to belong to the set, $\mathcal{MZ}^\mu(sp(2n))$, of $sp(2n)$-monotonic patterns with top-most row μ if and only if all the entries, z_{ki} and $z_{\bar{k}i}$, of M are non-negative integers satisfying the boundary and betweenness conditions:

$$\begin{array}{lll} \text{(M1)} & z_{ni} = \mu_i & \text{for } 1 \le i \le n; \\ \text{(M2)} & z_{ki} > z_{k,i+1} \quad \text{and} \quad z_{\bar{k}i} > z_{\bar{k},i+1} & \text{for } 1 \le i < k \le n; \\ \text{(M3)} & z_{kk} > 0 \quad \text{and} \quad z_{\bar{k}k} \ge 0 & \text{for } 1 \le k \le n; \\ \text{(M4)} & z_{ki} \ge z_{\bar{k}i} \ge z_{k,i+1} & \text{for } 1 \le i < k \le n; \\ \text{(M5)} & z_{\bar{k}i} \ge z_{k-1,i} \ge z_{\bar{k},i+1} & \text{for } 1 \le i < k \le n; \\ \text{(M6)} & z_{kk} \ge z_{\bar{k}k} \ge 0 & \text{for } 1 \le k \le n. \end{array} \tag{5.7}$$

If, in our usual notation, $S \in \mathcal{ST}^\mu(sp(2n))$ is such that the entry in the (i, j)th box of S is η_{ij} for all $(i, j) \in SF^\mu$, then the corresponding $sp(2n)$-pattern $M \in \mathcal{MZ}^\mu(sp(2n))$ has entries z_{ki} and $z_{\bar{k}i}$ defined once again by (5.4). This time the conditions on η_{ij}, as given by (S1)–(S5) of (2.3) applied to (2.6) of Definition 2.2, then imply that the entries z_{ki} and $z_{\bar{k}i}$ in M automatically satisfy (5.7), as required.

Conversely, given $M \in \mathcal{MZ}^\mu(sp(2n))$ with entries z_{ki} and $z_{\bar{k}i}$ satisfying (5.7), let the partitions $\zeta(k)$ and $\zeta(\bar{k})$ be defined for $k = 1, 2, \ldots, n$ by $\zeta(k)_i = z_{ki}$ and $\zeta(\bar{k})_i = z_{\bar{k}i}$ for $i = 1, 2, \ldots, k$, and let $\zeta(0) = 0$. Then for all $(i, j) \in SF^\mu$ with $\mu = \zeta(n)$ the entry in the (i, j)th box of $S \in \mathcal{ST}^\mu(sp(2n))$ is given for all $k \in [n]$ by

$$\eta_{ij} = \begin{cases} k & \text{if } (i, j) \in SF^{\zeta(k)/\zeta(\bar{k})}; \\ \bar{k} & \text{if } (i, j) \in SF^{\zeta(\bar{k})/\zeta(k-1)}, \end{cases} \tag{5.8}$$

where $SF^{\tau/\sigma}$ signifies the skew shifted diagram obtained by deleting all the boxes of SF^σ from those of SF^τ. Now the conditions (5.7) on all z_{ki} and $z_{\bar{k}i}$ of M imply that the entries η_{ij} of S automatically satisfy the required conditions (2.3) as applied to (2.6).

Thus for any partition μ of length n whose parts are distinct, (5.4) and (5.8) serve to define a bijection between the set of $sp(2n)$-standard shifted tableaux, $\mathcal{ST}^\mu(sp(2n))$ and the set of $sp(2n)$-monotonic patterns, $\mathcal{MZ}^\mu(sp(2n))$.

The significance of this bijection is that Theorem 1.2 and several of its Corollaries involving $sp(2n)$-standard shifted tableaux, S, may be re-expressed in terms of $sp(2n)$-monotonic patterns, M.

First, it is convenient to define $\mathrm{wgt}(M)$ in terms of differences between the weights of the partitions defining the rows of the corresponding monotonic pattern M. To be precise, let

$$x^{\mathrm{wgt}(M)} = x_1^{\mathrm{wgt}_1(M) - \mathrm{wgt}_{\bar{1}}(M)} x_2^{\mathrm{wgt}_2(M) - \mathrm{wgt}_{\bar{2}}(M)} \cdots x_n^{\mathrm{wgt}_n(M) - \mathrm{wgt}_{\bar{n}}(M)}, \tag{5.9}$$

where

$$\mathrm{wgt}_k(M) = |\zeta(k)| - |\,\zeta(\bar{k})| \quad \text{and} \quad \mathrm{wgt}_{\bar{k}} = |\zeta(\bar{k})| - |\,\zeta(k-1)| \ \text{ for } 1 \le k \le n. \tag{5.10}$$

Then in the notation of (5.10), it is convenient to sum over just the barred weights in defining

$$\mathrm{bar}(M) = \sum_{k=1}^{n} \mathrm{wgt}_{\bar{k}}(M). \tag{5.11}$$

The weights of the partitions defining the rows of M in (5.6) are such that in this example $\mathrm{bar}(M) = (24-23)+(15-11)+(9-7)+(5-2)+1 = 11$.

Furthermore, let $\mathrm{btw}(M)$ be the number of instances of strict betweenness that occur in M, that is the number of entries c appearing in triples of the form ${}^{a\,b}_{\;c}$ with $a > c > b \geq 0$ or virtual triples of the form ${}^{a}_{\;c}$ with $a > c > 0$ and b absent. Thus

$$\begin{aligned}\mathrm{btw}(M) = \#\{(k,i)\,|\, & z_{ki} > z_{\bar{k}i} > z_{k,i+1} \text{ for } 1 \leq i < k \leq n\} \\ &+ \#\{(\bar{k},i)\,|\, z_{\bar{k}i} > z_{k-1,i} > z_{\bar{k},i+1} \text{ for } 1 \leq i < k \leq n\} \\ &+ \#\{k\,|\, z_{kk} > z_{\bar{k}k} > 0 \text{ for } 1 \leq k \leq n\}.\end{aligned} \tag{5.12}$$

In the case of the monotonic pattern of (5.6) the relevant entries c at the apex of each betweenness triple may be identified in boldface type as shown below:

$$\begin{array}{cccccccccccccc}
9 & & 7 & & 6 & & 2 & & 1 & & & & & \\
 & \mathbf{8} & & 7 & & 6 & & 2 & & 1 & & & & \\
 & & 8 & & 7 & & 6 & & 2 & & & & & \\
 & & & 7 & & 6 & & 2 & & 0 & & & & \\
 & & & & 6 & & \mathbf{4} & & \mathbf{1} & & & & & \\
 & & & & & 6 & & \mathbf{3} & & 0 & & & & \\
 & & & & & & \mathbf{4} & & 3 & & & & & \\
 & & & & & & & 3 & & \mathbf{2} & & & & \\
 & & & & & & & & 2 & & & & & \\
 & & & & & & & & & \mathbf{1} & & & &
\end{array} \tag{5.13}$$

It follows in this example that $\mathrm{btw}(M) = 7$.

Finally, let the index of M, $\mathrm{ind}(M)$, be defined by the number of entries a appearing in triples of the form ${}^{a\,b}_{\;b}$ with $a > b > 0$ and a in the mth row of M counting from the top with m odd, minus the number of entries a appearing in either a triple of the form ${}^{a\,b}_{\;c}$, with $a > c \geq b \geq 0$ with a in the mth row of M with m even, or in a singlet a with $a > 0$ at the end of the mth row of M with m even. Thus

$$\begin{aligned}\mathrm{ind}(M) = \#\{(k,i)\,|\, & z_{ki} > z_{\bar{k}i} = z_{k,i+1} \text{ for } 1 \leq i < k \leq n\} \\ &- \#\{(\bar{k},i)\,|\, z_{\bar{k}i} > z_{k-1,i} \geq z_{\bar{k},i+1} \text{ for } 1 \leq i < k \leq n\} \\ &- \#\{k\,|\, z_{\bar{k}k} > 0 \text{ for } 1 \leq k \leq n\}.\end{aligned} \tag{5.14}$$

Once again, in the case of the monotonic pattern of (5.6) the relevant entries a contributing to ind(M) may be identified in boldface type, and overlined if their contribution is negative, as shown below:

$$
\begin{array}{ccccccccccc}
9 & & 7 & & 6 & & 2 & & 1 & & \\
& 8 & & 7 & & 6 & & 2 & & \bar{\mathbf{1}} & \\
& & \mathbf{8} & & \mathbf{7} & & \mathbf{6} & & 2 & & \\
& & & \bar{\mathbf{7}} & & \bar{\mathbf{6}} & & \bar{\mathbf{2}} & & 0 & \\
& & & & 6 & & 4 & & 1 & & \\
& & & & & \bar{\mathbf{6}} & & 3 & & 0 & \\
& & & & & & \mathbf{4} & & 3 & & \\
& & & & & & & \bar{\mathbf{3}} & & \bar{\mathbf{2}} & \\
& & & & & & & & 2 & & \\
& & & & & & & & & \bar{\mathbf{1}} &
\end{array}
\tag{5.15}
$$

Hence, in this example ind$(M) = 4 - 8 = -4$.

With this notation it is not difficult to see that Theorem 1.2 implies the validity of the following:

Corollary 5.3 *Let λ be a partition into no more than n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$
\begin{aligned}
&D_{sp(2n)}(x;t)\; sp_\lambda(x;t) \\
&\quad = \sum_{M \in \mathcal{MZ}^{\lambda+\delta}(sp(2n))} t^{\mathrm{ind}(M)+2\mathrm{bar}(M)}\, (1+t)^{\mathrm{btw}(M)}\, x^{\mathrm{wgt}(M)}
\end{aligned}
\tag{5.16}
$$

where the summation is taken over all $sp(2n)$-monotonic patterns with top-most row $\lambda+\delta$.

In the case of the $sp(2n)$-monotonic pattern M displayed in (5.6) it follows from (5.9) to (5.15) that the contribution to the right hand side of (5.16) is given by

$$
\begin{aligned}
&t^{\mathrm{ind}(M)+2\mathrm{bar}(M)}\, (1+t)^{\mathrm{btw}(M)}\, x^{\mathrm{wgt}(M)} \\
&\quad = t^{-4+2\cdot 11}\, (1+t)^7\, x_1^{1-1}\, x_2^{2-3}\, x_3^{2-2}\, x_4^{8-4}\, x_5^{1-1} \\
&\quad = t^{18}\, (1+t)^7\, x_2^{-1}\, x_4^4.
\end{aligned}
\tag{5.17}
$$

Further Corollaries analogous to those of Section 4 may be obtained by specialising various combinations of λ, t and $x = (x_1, x_2, \ldots, x_n)$.

6. Alternating sign matrices

A further combinatorial construct that has been found to be useful in the context of deformations of Weyl's denominator formula is that of alternating sign matrices [11, 14, 18].

Conventionally [9], any square matrix $n \times n$ matrix $A = (a_{ij})$ belongs to the set $\mathcal{A}(n)$ of $n \times n$ alternating sign matrices if and only if all its matrix elements a_{ij} are 1, 0 or -1, every row and column has sum 1, and in every row and column the non-zero matrix elements alternate in sign, that is

$$\begin{aligned}
&a_{ij} \in \{-1, 0, 1\} && \text{for } 1 \le i, j \le n; \\
&\sum_{j=k}^{n} a_{ij} \in \{0, 1\} && \text{for } 1 \le i, k \le n; \\
&\sum_{i=k}^{n} a_{ij} \in \{0, 1\} && \text{for } 1 \le k, j \le n; \\
&\sum_{j=1}^{n} a_{ij} = 1 && \text{for } 1 \le i \le n; \\
&\sum_{i=1}^{n} a_{ij} = 1 && \text{for } 1 \le j \le n.
\end{aligned} \tag{6.1}$$

The generalisation of this notion required in a restatement of Tokuyama's result for $gl(n)$ and $sl(n)$, Theorem 1.1, is that of a μ-alternating sign matrix [11]. For each partition μ, all of whose parts are distinct and for which $\ell(\mu) = n$ and $\mu_1 \le m$, an $n \times m$ matrix $A = (a_{iq})$ belongs to the set $\mathcal{A}^\mu(sp(2n))$ of $n \times m$ μ-alternating sign matrices if the following conditions are satisfied:

$$\begin{aligned}
&a_{iq} \in \{-1, 0, 1\} && \text{for } 1 \le i \le n, 1 \le q \le m; \\
&\sum_{q=p}^{m} a_{iq} \in \{0, 1\} && \text{for } 1 \le i \le n, 1 \le p \le m; \\
&\sum_{i=j}^{n} a_{iq} \in \{0, 1\} && \text{for } 1 \le j \le n, 1 \le q \le m; \\
&\sum_{q=1}^{m} a_{iq} = 1 && \text{for } 1 \le i \le n; \\
&\sum_{i=1}^{n} a_{iq} = \begin{cases} 1 & \text{if } q = \mu_j \text{ for some } j \\ 0 & \text{otherwise} \end{cases} && \text{for } 1 \le q \le m.
\end{aligned} \tag{6.2}$$

The further generalisation needed here in the case of $sp(2n)$ takes the form:

Definition 6.1 Let μ be a partition of length $\ell(\mu) = n$, all of whose parts are distinct, and for which $\mu_1 \le m$. Then the matrix $A = (a_{iq})$ is said to belong to the set $\mathcal{A}^\mu(sp(2n))$ of $sp(2n)$-generalised alternating sign matrices if it is a $2n \times m$ matrix whose elements a_{iq}

satisfy the conditions:

$$
\begin{array}{lll}
\text{(A1)} & a_{iq} \in \{-1, 0, 1\} & \text{for } 1 \le i \le 2n,\ 1 \le q \le m; \\
\text{(A2)} & \displaystyle\sum_{q=p}^{m} a_{iq} \in \{0, 1\} & \text{for } 1 \le i \le 2n,\ 1 \le p \le m; \\
\text{(A3)} & \displaystyle\sum_{i=j}^{2n} a_{iq} \in \{0, 1\} & \text{for } 1 \le j \le 2n,\ 1 \le q \le m. \\
\text{(A4)} & \displaystyle\sum_{q=1}^{m} (a_{2i-1,q} + a_{2i,q}) = 1 & \text{for } 1 \le i \le n; \\
\text{(A5)} & \displaystyle\sum_{i=1}^{2n} a_{iq} = \begin{cases} 1 & \text{if } q = \mu_k \text{ for some } k; \\ 0 & \text{otherwise,} \end{cases} & \text{for } 1 \le q \le m,\ 1 \le k \le n.
\end{array}
\tag{6.3}
$$

In the special case for which $\mu = \delta = (n, n-1, \dots, 1)$ and $m = n$, for which (A5) becomes $\sum_{i=1}^{2n} a_{iq} = 1$ for $1 \le q \le n$, this definition is such that the set $\mathcal{A}^{\delta}(sp(2n))$ coincides with the set of U-turn alternating sign matrices, UASM, defined by Kuperberg [7].

More generally, the connection with what has gone before here comes about through the recognition that there exists a bijection between the monotonic patterns defined by (5.7) and the μ-alternating sign matrices defined by (6.3). Following a route analogous to that described in the $gl(n)$ case by Okada [11], the passage from $M \in \mathcal{MZ}^{\mu}(sp(2n))$ to $A \in \mathcal{A}^{\mu}(sp(2n))$ is accomplished by first constructing the $2n \times m$ matrix B by placing entries 1 in the ith row of B in precisely those columns q for which q is itself a non-vanishing part of the partition whose distinct parts constitute the ith row of M, and setting all the other elements of B to 0. With this convention, $B = (b_{iq})$ with

$$
b_{iq} = \begin{cases} 1 & \text{if } i = 2n+1-2k \text{ and } \zeta(k)_j = z_{kj} = q \text{ for some } j \text{ with } 1 \le j \le k; \\ 1 & \text{if } i = 2n+2-2k \text{ and } \zeta(\bar{k})_j = z_{\bar{k}j} = q \text{ for some } j \text{ with } 1 \le j \le k; \\ 0 & \text{otherwise.} \end{cases}
\tag{6.4}
$$

for $1 \le i \le 2n$ and $1 \le q \le m$. The passage from B to $A \in \mathcal{A}^{\mu}(sp(2n))$ is accomplished by subtracting each row of B from its predecessor to give $A = (a_{iq})$ with

$$
a_{iq} = \begin{cases} b_{iq} - b_{i+1,q} & \text{if } 1 \le i \le 2n-1; \\ b_{iq} & \text{if } i = 2n. \end{cases}
\tag{6.5}
$$

for $1 \le i \le 2n$ and $1 \le q \le m$.

The map Ψ from M to $A = \Psi(M)$ by way of B is illustrated in the case of the example (5.6) by:

$$\begin{array}{r} 9\;7\;6\;2\;1 \\ 8\;7\;6\;2\;1 \\ 8\;7\;6\;2 \\ 7\;6\;2\;0 \\ 6\;4\;1 \\ 6\;3\;0 \\ 4\;3 \\ 3\;2 \\ 2 \\ 1 \end{array} \longrightarrow \begin{bmatrix} 1&1&0&0&0&1&1&0&1 \\ 1&1&0&0&0&1&1&1&0 \\ 0&1&0&0&0&1&1&1&0 \\ 0&1&0&0&0&1&1&0&0 \\ 1&0&0&1&0&1&0&0&0 \\ 0&0&1&0&0&1&0&0&0 \\ 0&0&1&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \longrightarrow \begin{array}{cc} \begin{bmatrix} 0&0&0&0&0&0&0&\bar{1}&1 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ \bar{1}&1&0&\bar{1}&0&0&1&0&0 \\ 1&0&\bar{1}&1&0&0&0&0&0 \\ 0&0&0&\bar{1}&0&1&0&0&0 \\ 0&\bar{1}&0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ \bar{1}&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} \\ 1\;1\;0\;0\;0\;1\;1\;0\;1 & \end{array} \tag{6.6}$$

The row and column sums of A have been indicated on the extreme right and immediately below A, respectively. Those columns q for which the column sum is 1 are those for which $q = 1, 2, 6, 7, 9$. It is clear in this example that the resulting matrix A satisfies (A1)–(A5) of (6.3) with $n = 5$, $m = 9$ and $\mu = (9, 7, 6, 2, 1)$, and thus belongs, as required to $\mathcal{A}^{97621}(sp(10))$.

More generally, we have

Lemma 6.2 *Let μ be a partition of length $\ell(\mu) = n$, all of whose parts are distinct, and with largest part $\mu_1 \leq m$. Let Ψ be the map defined by* (6.4) *and* (6.5) *which takes each monotonic pattern $M \in \mathcal{MZ}^{\mu}(sp(2n))$ with entries z_{ij} for $i \in [n, \bar{n}]$ and $j \in [n]$ to the $2n \times (m + 1)$ matrix $A = \Psi(M)$. Then $A \in \mathcal{A}^{\mu}(sp(2n))$ and Ψ is a bijection from $\mathcal{MZ}^{\mu}(sp(2n))$ to $\mathcal{A}^{\mu}(sp(2n))$.*

The existence of the bijection Ψ implies that the main result Theorem 1.2, which had already been reformulated in terms of the $sp(2n)$-monotonic patterns of $\mathcal{MZ}^{\lambda+\delta}(sp(2n))$ in Corollary 5.1, can now be reformulated in terms of the $sp(2n)$-generalised alternating sign matrices of $\mathcal{A}^{\lambda+\delta}(sp(2n))$.

To this end we require some further definitions. First of all, let $\mathrm{wgt}(A)$ be such that

$$x^{\mathrm{wgt}(A)} = x_1^{m_1(A)-m_{\bar{1}}(A)}\, x_2^{m_2(A)-m_{\bar{2}}(A)} \cdots x_n^{m_n(A)-m_{\bar{n}}(A)} \tag{6.7}$$

with

$$m_k(A) = \sum_{q=1}^{m} q\, a_{2n+1-2k,q} \quad \text{and} \quad m_{\bar{k}}(A) = \sum_{q=1}^{m} q\, a_{2n+2-2k,q}. \tag{6.8}$$

With this notation, let bar(A) be defined by

$$\mathrm{bar}(A) = \sum_{k=1}^{n} m_{\bar{k}}(A) \tag{6.9}$$

and let neg(A) be the total number of elements $-1 = \bar{1}$ in A, that is

$$\mathrm{neg}(A) = \#\{(i, q) \mid a_{iq} = -1 \text{ for } 1 \le i \le 2n, 1 \le q \le m\}. \tag{6.10}$$

Within A there are a number of sites of special interest, namely those sites (i, q) for which $a_{i,q-1}$ is 0 for i odd, and either 0 or -1 for i even, and for which there exist nearest nonzero neighbours to the right and below $(i, q-1)$ which are both equal to 1, together with those sites $(i, 1)$ in the first column of A for which i is even and either $a_{i1} = 1$ or for which there exists a nearest nonzero neighbour to the right which is 1. If the set of all such sites of special interest is denoted by $S(A)$, then let ssi(A) be defined to be the signed sum of sites of special interest of A, that is

$$\mathrm{ssi}(A) = \#\{(2n+1-2k, q) \in S(A)\} - \#\{(2n+2-2k, q) \in S(A)\}. \tag{6.11}$$

These definitions may be illustrated by application to the example (6.6). To find the $2n$ component vector wgt(A) $= (m_i(A))$ one merely multiplies the m-component vector $X = (x_q) = (q)$ by A and reverses the order of its components. In the case of (6.6) one obtains wgt(A) $= (1, 1, 3, 2, 2, 2, 4, 8, 1, 1)$, so that

$$x^{\mathrm{wgt}(A)} = x_1^0 x_2^{-1} x_3^0 x_4^4 x_5^0 = x_2^{-1} x_4^4 \quad \text{and} \quad \mathrm{bar}(A) = 1+3+2+4+1 = 11. \tag{6.12}$$

The various sites (i, q) contributing to neg(A) and ssi(A) as defined by (6.10) and (6.11), respectively, are indicated in the case of the matrix A of example (6.6) by the boldface entries shown below.

$$\begin{bmatrix}
0&0&0&0&0&0&0&\bar{\mathbf{1}}&1\\
1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0\\
\bar{\mathbf{1}}&1&0&\bar{\mathbf{1}}&0&0&1&0&0\\
1&0&\bar{\mathbf{1}}&1&0&0&0&0&0\\
0&0&0&\bar{\mathbf{1}}&0&1&0&0&0\\
0&\bar{\mathbf{1}}&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0\\
\bar{\mathbf{1}}&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0
\end{bmatrix}
\quad
\begin{bmatrix}
0&0&0&0&0&0&0&\bar{1}&1\\
\mathbf{1}&0&0&0&0&0&0&0&0\\
0&0&\mathbf{0}&0&0&0&\mathbf{0}&\mathbf{1}&0\\
\bar{1}&\mathbf{1}&0&\bar{1}&\mathbf{0}&0&\mathbf{1}&0&0\\
1&0&\bar{1}&1&0&0&0&0&0\\
0&0&0&\bar{1}&\mathbf{0}&1&0&0&0\\
0&\bar{1}&0&\mathbf{1}&0&0&0&0&0\\
\mathbf{0}&0&\mathbf{1}&0&0&0&0&0&0\\
\bar{1}&1&0&0&0&0&0&0&0\\
\mathbf{1}&0&0&0&0&0&0&0&0
\end{bmatrix} \tag{6.13}$$

From these, remembering the sign factors in ssi(A) we obtain

$$\mathrm{neg}(A) = 7 \quad \text{and} \quad \mathrm{ssi}(A) = 0-1+3-3+0-1+1-2+0-1 = 4-8 = -4. \tag{6.14}$$

These results are all in accord with those obtained for $x^{\mathrm{wgt}(M)}$, $\mathrm{bar}(M)$, $\mathrm{btw}(M)$ and $\mathrm{ind}(M)$ in the preceding section.

In fact with these definitions we have

Corollary 6.2 *Let λ be a partition into no more than n parts and let δ be the partition $(n, n-1, \ldots, 1)$, then*

$$D_{sp(2n)}(x;t)\, sp_\lambda(x;t) = \sum_{A\in\mathcal{A}^{\lambda+\delta}(sp(2n))} t^{\mathrm{ssi}(A)+2\mathrm{bar}(A)}\,(1+t)^{\mathrm{neg}(A)}\,x^{\mathrm{wgt}(A)} \tag{6.15}$$

where the summation is taken over all $sp(2n)$-generalised alternating sign matrices of shape $2n \times (\lambda_1 + n + 1)$, whose non-vanishing column sums are 1 or 0 according as the column number is or is not a part of $\lambda + \delta$.

Proof: The result follows immediately from Corollary 5.1, by noting that the map from M to A is such that $\mathrm{wgt}(A) = \mathrm{wgt}(M)$, $\mathrm{bar}(A) = \mathrm{bar}(M)$, $\mathrm{neg}(A) = \mathrm{btw}(M)$ and $\mathrm{ssi}(A) = \mathrm{ind}(M)$. The first three are rather obvious identities, while the last is a consequence of the rather careful definition of the set $S(A)$ of sites of special interest, in which each term contributing to $\mathrm{ind}(M)$ in (5.14), together with its sign, has its precise counterpart identified as a signed site of special interest in A. □

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