SEALow Team 2 Presents: More Miles for Your (Sand)Piles

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August 3, 2018

Overview

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2. Results on the Number of Even Invariant factors
3. Reducing the Sandpile Group and Results for Small Cases
4. Largest Cyclic Factors
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Section 1

Definitions and Previous Results

Definitions

Definition

Given $\mathbb{F}_2^r, M$, where $M = \{v_1, \ldots, v_n\}$ is a set of generators, we define the **Cayley graph** $G(F_2^r, M)$ with $V(G) = \mathbb{F}_2^r$ and $u, w \in V(G)$ share an edge if $u - w = v_i$ for some generator. Multiple edges are allowed.

Example

- Let $M = \{e_1, \ldots, e_n\}$. Then $G(\mathbb{F}_2^r, M) = Q_n$, is called the hypercube graph.
- If $M = \{v \in \mathbb{F}_2^r - \{0\}\}$, then $G(\mathbb{F}_2^r, M) = K_{2^r}$ is called the complete graph on 2^r vertices.
- See board for image of Q_2 and K_4 with generators labelled

Definitions

Definition

The Laplacian of a nondirected graph G, denoted $L(G)$ has entries

$$L(G)_{i,j} = \begin{cases} \deg(v_i) & i = j \\ -\#\text{edges from } v_i \text{ to } v_j & i \neq j \end{cases}$$

Definition

Given a connected graph G with $|V(G)| = w$, $L(G)$ is an integer $w \times w$ matrix, so we can view it as map of $\mathbb{Z}$-modules $\mathbb{Z}^w \to \mathbb{Z}^w$. The kernel is $\mathrm{span}(\mathbf{1})$, so $\mathrm{coker}\, L(G) \cong \mathbb{Z} \oplus K(G)$ where $K(G)$ is a finite abelian group. We call $K(G)$ the **sandpile group** of G.

Example

It is well known that $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$. So we can determine at least one case of Cayley graphs.

Previous Results for $\mathbb{F}_2^r$

Lemma

Let $M = \{v_1, \ldots, v_n\}$ be the set of generators. For every $u \in \mathbb{F}_2^r$, let

$$f_u = \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} e_v \qquad \lambda_{u,M} = n - \sum_{i=1}^{n} (-1)^{u \cdot v_i}$$

Then $\{f_u\}$ is an eigenbasis of $\mathbb{R}^{2^r}$ each with eigenvalues $\{\lambda_{u,M}\}$, which is always even. Moreover, $e_v = \frac{1}{2^r} \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} f_v$.

Theorem (Ducey-Jalil)

Let G be a Cayley graph of $\mathbb{F}_2^r$. For all $p \neq 2$,

$$\mathrm{Syl}_p(K(G)) \cong \mathrm{Syl}_p\left(\bigoplus_{k=1}^{2^r} \mathbb{Z}/\lambda_{u,M}\mathbb{Z}\right)$$

More Previous Results

Remark

$L(G)$ is diagonalizable over $\mathbb{Z}[\frac{1}{2}]$, and we can describe the Sylow-p structure for all $p \neq 2$ in terms of the eigenvalues.

What about $p = 2$? Is the Sylow-2 group uniquely determined by the eigenvalues?

Theorem

There is an isomorphism of abelian groups

$$\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \dots, x_r] / \left(x_1^2 - 1, \dots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j} x_j^{(v_i)_j} \right)$$

Previous results for $p = 2$

Theorem (Bai)

For $G = Q_n$, the number of Sylow-2 cyclic factors is $2^{n-1} - 1$. Additionally, the number of $(\mathbb{Z}/2\mathbb{Z})$'s in $K(G)$ is $2^{n-2} - 2^{\lfloor (n-2)/2 \rfloor}$.

Theorem (Anzis-Prasad)

The size of the largest factor in $\mathrm{Syl}_2(K(Q_n))$ is $\leq 2^{n+\lfloor \log_2 n \rfloor}$.

We will generalize Bai's first result and Anzis-Prasad, but not Bai's second result.

Section 2

Results on the Number of Even Invariant Factors

Invariant factors

Definition

We define $d(M)$ to be the number of Sylow-2 cyclic factors in $K(G)$.

Proposition (Parity Invariance)

Let our matrix of generators M have multiplicities $(a_{v_1}, \dots a_{v_{2^r-1}})$ for each nonzero vector in $\mathbb{F}_2^r$. Then $d(M)$ only depends on the parity of the multiplicities of generators.

Example

If $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ then

$$K(G(\mathbb{F}_2^3, M)) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})$$
$$K(G(\mathbb{F}_2^3, M')) = (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z}) \oplus (\mathbb{Z}/120\mathbb{Z})$$

Computing $d(M)$

Definition

A Cayley graph $G(\mathbb{F}_2^r, M)$ with $M = \{v_1, \ldots, v_n\}$ is called **generic** if $\sum_{i=1}^{n} v_i \neq \vec{0}$. For example, Q_n is generic for all $n \geq 1$.

Theorem

If $G(\mathbb{F}_2^r, M)$ is generic, then $d(M) = 2^{r-1} - 1$.

Proof Sketch.

Consider $(\mathbb{Z} \oplus K(G)) \otimes (\mathbb{Z}/2\mathbb{Z})$. $d(M)$ is equal to the dimension of $K(G) \otimes (\mathbb{Z}/2\mathbb{Z})$ as a vector space. Theorem's condition gives us a nonzero degree 1 term of the form u_i which allows us to construct an explicit isomorphism $(\mathbb{Z} \oplus K(G)) \otimes (\mathbb{Z}/2\mathbb{Z}) \overset{(\mathbb{Z}/2\mathbb{Z})-\text{mod}}{\cong} (\mathbb{Z}/2\mathbb{Z})^{2^{r-1}}$. □

Conjectures

Conjecture

For a collection of generators, M, yielding a connected Cayley graph on $\mathbb{F}_2^r$, $d(M) \geq 2^{r-1} - 1$ with equality occurring iff M is generic.

Conjecture

$d(M)$ is odd unless all of the eigenvalues have the same power of 2, in which case $d(M) = 2^n - 2$.

Section 3

Reducing the Sandpile Group and Results for Small Cases

Reducing multiplicities in M

Given generators for $V = \mathbb{F}_2^r$, we can express these generators in terms of their multiplicities $\vec{a} = (a_{v_1}, \dots, a_{v_{2^r-1}})$, where the multiplicity, a_{v_i}, denotes the number of times the vector v_i occurs. Here, we will use the binary naming convention for vectors, so $v_3 = (1,1,0)$.

Lemma

Let $G_1 = G(\mathbb{F}_2^r, M_1)$ and $G_2 = G(\mathbb{F}_2^r, M_2)$ such that $\vec{a}_2 = \lambda \vec{a}_2$ for $\lambda \in \mathbb{N}$ and let $\{\alpha_i\}$ be the invariant factors in the Smith Normal Form of $L(G_1)$. Then

$$K(G_1) = \prod_{i=1}^{2^r} \mathbb{Z}/\alpha_i\mathbb{Z} \implies K(G_2) = \prod_{i=1}^{2^r} \mathbb{Z}/(\lambda\alpha_i)\mathbb{Z}$$

Proof.

$\vec{a}_2 = \lambda \vec{a}_2 \implies L(G_2) = (\lambda Id) \cdot L(G_1)$. Now consider SNF of $L(G_2)$. (Note: reduces analysis to $gcd(\vec{a}) = 1$ case.) □

Example

Consider the two matrices of generators:

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\implies K(G(\mathbb{F}_2^2, M_1)) = (\mathbb{Z}/1\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$
$$\implies K(G(\mathbb{F}_2^2, M_2)) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$$

Invariance Under GL Action

Theorem

Given a matrix of generators on $\mathbb{F}_2^r$

$$M = \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_n \\ | & | & \dots & | \end{pmatrix}$$

and an element $g \in \mathrm{GL}_r(\mathbb{F}_2)$, *define* $M' := g \cdot M$, *then* $G(\mathbb{F}_2^r, M)$ *and* $G(\mathbb{F}_2^r, M')$ *have the same sandpile group.*

Proof.

An element of GL_r permutes the nonzero vertices of the graphs and the edges in a consistent manner. This induces a graph isomorphism, and thus a sandpile group isomorphism. □

Example

Assume $a_{v_1} = a_{v_2} = a_{v_4} = a_{v_8} = 2$. Let $\{i_1, \ldots, i_k\}$ denote $a_{v_{i_1}} = \cdots = a_{v_{i_k}} = 2$ and $a_{v_j} = 1$ for all $j \notin \{1,2,4,8\} \cup \{i_1, \ldots, i_k\}$:

$$\{6,10,12\},\{5,9,12\},\{3,5,6\},\{3,9,10\},\{10,12,14\},$$
$$\{9,12,13\},\{5,6,7\},\{3,10,11\},\{6,12,14\},\{5,9,13\},\{5,12,13\},$$
$$\{3,6,7\},\{3,9,11\},\{6,10,14\},\{3,5,7\},\{9,10,11\}$$

All of the $\underline{16}$ previous cases yield

$$K(G) = (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/48\mathbb{Z}) \oplus (\mathbb{Z}/48\mathbb{Z})$$
$$\oplus (\mathbb{Z}/528\mathbb{Z}) \oplus (\mathbb{Z}/6864\mathbb{Z}) \oplus (\mathbb{Z}/6864\mathbb{Z})$$

Size of 2-Sylow component

By Kirchoff's Matrix Tree Theorem

$$|K(G)| = \det \overline{L(G)}^{i,i} = \frac{\lambda_2 \cdots \lambda_m}{m}$$

where $\lambda_1 = 0$ is only 0 eigenvalue by convention. Here, $m = 2^r$, so

$$|Syl_2(K(G))| = \frac{1}{2^r} \text{Pow}_2 \left(\prod_{u \in \mathbb{F}_2^r - \{0\}} \lambda_{u,M} \right)$$

where for $n = 2^k \cdot b$ with k-maximal, we define $\text{Pow}_2(n) := 2^k$ and $v_2(n) := k$.

Application: Determining Sandpile Group for $r = 2$

In the generic case for $r = 2$ with multiplicities $\vec{a} = (a_1, a_2, a_3)$, we have $Syl_2(K(G)) \cong \mathbb{Z}/2^e\mathbb{Z}$ with

$$\lambda_2 = 2(a_1 + a_3), \quad \lambda_3 = 2(a_2 + a_3), \quad \lambda_4 = 2(a_1 + a_2)$$

$$\vec{a} \equiv (1,0,0) \implies e = v_2(\lambda_2\lambda_3\lambda_4) - 2 = v_2(a_2 + a_3) + 1$$
$$\vec{a} \equiv (1,1,0) \implies e = v_2(a_2 + a_3) + 1$$

by GL equivalence of generators, these handle all generic cases. Note the symmetry of the 2 Sylow w.r.t. the eigenvalues

Non-generic case for $r = 2$

Only case left is all odd. By parity invariance, suffices to check $\vec{a} \equiv (1, 1, 1)$ to find $d(M)$. $d(M) = 2$, so we need only determine largest 2-factor. WLOG $a_1 + a_2 \equiv 2 \mod 4$. Through algebraic manipulation we get that

$$Syl_2(K(G)) \cong \mathbb{Z}/2^e\mathbb{Z} \oplus \mathbb{Z}/2^f\mathbb{Z}$$

$$e = v_2(a_2 + a_3) + 1, \qquad f = v_2(a_1 + a_3) + 1$$

Results for $r = 3$

Proposition

For $r = 3$, let $d_1 \leq d_2 \leq \cdots \leq d_7$ be all the powers of 2 in the nonzero eigenvalues of $L(G)$ for M reduced. Let c_{top} be the top Sylow-2 cyclic factor. Then

$$c_{top} = \begin{cases} 2^{d_7+1} & \text{not all } d_i \text{ equal} \\ 2^{d_7} & d_i = d_j \text{ for all } i, j \in \{1, \ldots, 7\} \end{cases}$$

.

Theorem

Let $G = G(\mathbb{F}_2^3, M)$ be **generic**, *and with d_i as above. Then*

$$\mathrm{Syl}_2(K(G)) = \begin{cases} \mathbb{Z}/2^{d_5-1}\mathbb{Z} \times (\mathbb{Z}/2^{d_7+1}\mathbb{Z})^2 & d_6 = d_7 \\ \mathbb{Z}/2^{d_5}\mathbb{Z} \times \mathbb{Z}/2^{d_6}\mathbb{Z} \times \mathbb{Z}/2^{d_7+1}\mathbb{Z} & d_6 < d_7 \end{cases}$$

Section 4

Investigating Largest Cyclic Factors of the Sandpile Group

Background Theory

- In quotient ring of the hypercube sandpile group, Anzis and Prasad showed that $x_j - 1$ has maximal, finite, additive order for any $j \in \{1, \ldots, r\}$
- We adapted proof to show that for any generating set $(v_1, \ldots, v_m)$, the maximal order element of the form $x_{v_k} - 1$ has maximal finite order. By changing variables, we can assume that $x_1 - 1$ has maximal finite order.
- From definition of cokernel, $\mathrm{ord}(x_1 - 1)$ is smallest C s.t.

$$\exists v \in \mathbb{Z}^{2^r} \text{ s.t. } L(G)v = C(1, -1, 0, \ldots, 0) = Cw$$

here we use the isomorphism:

$$\mathbb{Z}^{2^r} \cong \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1)$$

Top Cyclic Factor

Theorem

Let d be the size of the largest cyclic factor in $K(G)$. Then $d \mid 2^{r-2} lcm(\lambda_i : i \geq 2)$.

Proof.

An adaptation of the argument from Anzis and Prasad. □

Corollary

The largest 2-cyclic factor, $\mathbb{Z}/2^e\mathbb{Z}$ has bound

$$e \leq \lfloor \log_2(n) \rfloor + r - 1$$

which is sharp when $G = Q_{2^k}, Q_{2^k+1}$.

Proof of Corollary.

Apply theorem while noting that the largest eigenvalue is bounded by $2n$, so that

$$\begin{aligned} v_2(d) &\leq v_2\left[2^{r-2}\text{lcm}\left(\lambda_i : i \geq 2\right)\right] \\ &\leq r - 2 + \lfloor \log_2(2n) \rfloor = \lfloor \log_2(n) \rfloor + r - 1 \end{aligned}$$

when $G = Q_{2^k}$, we use the fact that each eigenvalue is distinct with largest value being 2^{k+1} and that $\lfloor \log_2(2^{k+1}) \rfloor = k + 1$. □

Main Result of Interest

We can actually improve the previous result:

Corollary

The order of $x_r - 1$ in $K(G)$ is equal to minimum integer C, such that for any $S \subseteq [n], |S| \geq 2, d \in \mathbb{F}_2^{|S|} \setminus \{\mathbf{0}\}$,

$$\frac{C}{2^{r-|S|}} \sum_{u_S = d} \frac{1}{\lambda_u} \in \mathbb{Z}$$

Specialization to $G = Q_n$

When $G = Q_n$ we know the eigenvalues and their multiplicities explicitly from Bai's paper, so searching for $v \in \mathbb{F}_2^n$ and C minimal such that $L(Q_n)v = Cw$ can be solved explicitly.

Theorem

For $n \geq 2$, let c_n be the size of the largest cyclic factor in $K(Q_n)$. Then,

$$v_2(c_n) = \max\{\max_{x<n}\{v_2(x) + x\}, v_2(n) + n - 1\}.$$

Theorem

For $n \geq 3$, the 2^{nd} to the $(n-1)^{th}$ largest cyclic factor in $K(Q_n)$ all have the same size d_n. Moreover,

$$v_2(d_n) = \max_{x<n}\{v_2(x) + x\}.$$

Remaining Conjectures

Conjecture

For $n \geq 3$, let e_n be the size of the n^{th} largest cyclic factor in $K(Q_n)$. Then,

$$v_2(e_n) = \max\{\max_{x<n-1}\{v_2(x)+x\}, v_2(n-1)+n-3\}.$$

Similarly, for $n \geq 4$, let f_n be the size of the $(n+1)^{th}$ largest cyclic factor in $K(Q_n)$. Then,

$$v_2(f_n) = \max_{x<n-1}\{v_2(x)+x\}.$$

Section 5

Future Areas of Research

Groebner Bases

- Very difficult! Groebner bases must be redefined over $\mathbb{Z}$, or in general PIDs vs. fields
- Recall we can order monomials $x_I = \prod_{i \in I} x_i$ by the multi-indices they are indexed by
- For $f = \sum_I a_I x_I = a_{I_0} x_{I_0} + \sum_{I \neq I_0} a_I x_I$ with x_{I_0} largest present, $LT(f) = a_{I_0} x_{I_0}$, $lm(f) := x_{I_0}$, and $lc(f) = a_{I_0}$
- Assuming a novel (unstated) definition of groebner basis, we have...

Groebner Isomorphism

Theorem

For A a PID, and ideal $s \subseteq A[x_1, \ldots, x_n]$. Let $G = \{g_i\}_{i=1}^{t}$ be a groebner basis for s. Let

$$J_{x_\alpha} := \{i : \ lm(g_i) \mid x_\alpha, \ g_i \in G\}, \qquad I_{J_{x_\alpha}} := \langle \{lc(g_i) : \ i \in J_{x_\alpha}\} \rangle$$

Call $I_{J_{x_\alpha}}$ the leading coefficient ideal. Under a few other conditions (which hold for $A = \mathbb{Z}$), there exists an isomorphism

$$\phi : A[x_1, \ldots, x_n]/\langle G \rangle \to A/I_{J_{x_{\alpha,1}}} \oplus \cdots \oplus A/I_{J_{x_{\alpha,m}}}$$

Example

Consider

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} \{LT(g_i) \mid g_i \in G\} = &\{x_1^2, x_1x_2, x_2^2, \\ &x_1x_3, x_3^2, 2x_1x_4, x_4^2, 6x_1, 24x_2, 24x_3, 480x_4\} \end{aligned}$$

$$K(G(\mathbb{F}_2^4, M)) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})^4 \oplus (\mathbb{Z}/480\mathbb{Z})$$

Note

$$J_{x_2x_3} = \{9, 10\}, \qquad I_{J_{x_2x_3}} = 24$$

$$J_{x_3x_4} = \{10, 11\}, \qquad I_{J_{x_3x_4}} = 24$$

Flaws with Groebner Basis Method

Sage's implemented version of groebner basis is not general enough for this isomorphism to always hold.

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

should yield same sandpile, but

$$\begin{aligned} \{LT(g_i) \mid g_i \in G\} = &\{x_1^2, x_1x_2, x_2^2, x_1x_3, x_3^2, x_1x_4, \\ &x_2x_4, 2x_3x_4, x_4^2, 24x_1, 24x_2, 48x_3, 60x_4\} \end{aligned}$$

which does not match the sandpile group (no order 480 term!). Sage is not to be trusted, but groebner bases could be useful in the future.

Matroid Contraction

For

$$M = \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{pmatrix}$$

where each $v_i \in \mathbb{F}_2^r$, consider

$$M' = \pi_{r-1}(M) = \begin{pmatrix} | & \dots & | \\ \pi_{r-1}(v_1) & \dots & \pi_{r-1}(v_n) \\ | & \dots & | \end{pmatrix} = \begin{pmatrix} | & \dots & | \\ v_1' & \dots & v_n' \\ | & \dots & | \end{pmatrix}$$

Continued

Gives rise to surjection

$$\mathbb{Z}[x_1, \ldots, x_r] / \left(x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j} \right)$$

$$\overset{"x_r = 1"}{\twoheadrightarrow} \mathbb{Z}[x_1, \ldots, x_{r-1}] / \left(x_1^2 - 1, \ldots, x_{r-1}^2 - 1, n' - \sum_{i=1}^{n} \prod_{j=1}^{r-1} x_j^{(v_i)_j} \right)$$

- Comparing torsion components: the cyclic factors in image can be viewed as subgroups of a larger cyclic factor in the domain sandpile group.
- Process of evaluating at $x_r = 1$ is *matroid contraction*.

Example

Consider

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \mapsto M' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$[K(G(\mathbb{F}_2^3, M)) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/48\mathbb{Z} \oplus \mathbb{Z}/240\mathbb{Z}] \mapsto [K(G(\mathbb{F}_2^2, M')) = \mathbb{Z}/24\mathbb{Z}]$$

From Groebner basis approach, we can think of each invariant factor being generated by a monomial $\overline{x_I}$. In fact...

Continued

	M	M'
$\overline{1}$	0	0
$\overline{x_1 x_2 x_3}$	1	NA
$\overline{x_1 x_3}$	1	NA
$\overline{x_1 x_2}$	1	NA
$\overline{x_1}$	12	3
$\overline{x_3}$	240	NA
$\overline{x_2}$	16	8
$\overline{x_2 x_3}$	1	NA

- Notice that $\mathrm{ord}(\overline{x}_I)_{M'} \mid \mathrm{ord}(\overline{x}_I)_M$. Consistent with map of quotients
- Indicates "growth" of sandpile group

Future Work

Our data and preliminary results raise other questions:

- Can we find a bound from below the top cyclic factor of the sandpile group in terms of r, n? We have one for the cube, but not in general.
- Can we implement the novel definition of groebner bases for PIDs as described by Franz Pauer in his work "Groebner basis with coefficients in rings"?
- Can we show the sandpile group of a Cayley graph only depends on the *set* of eigenvalues, and not by their indexing set?
- Is there a larger pattern to the number of even invariant factors?
- Can we describe $r = 3$ in full generality? We have conjecture for all the cases except all odd parities
- Maybe $r = 4$ as well?

Unfortunately our funding has run out, so the world may never know. . .

Acknowledgements

We'd like to thank our mentor, Victor Reiner, as well as the teaching assistant, Eric Stucky, for providing us with guidance and independence during the course of our research efforts.

The End!

Questions?