Ice Models and Classical Groups

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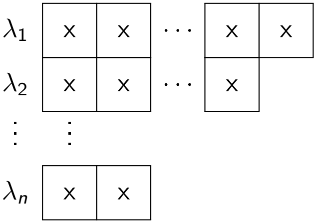
1. GL(n) Case

2. SO(2n+1) Case
   - Sundaram Tableaux
   - Koike-Terada Tableau
   - Proctor Tableaux

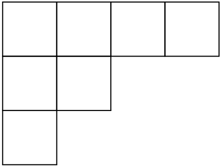
GL(n) Case: Tableaux

- Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$
- Our alphabet is $(1, 2, \cdots, n)$
- To form a Young tableau, fill the tableaux with elements of the alphabet such that:
  1. Weakly increasing along rows
  2. Strictly increasing along columns



GL(n) case: Tableaux Example

Let $\lambda = (4, 2, 1)$.
Our tableau will be of the shape:



A possible filling:

1	1	2	3
2	3		
3			

Gelfand-Tsetlin Patterns

$$GL(n) \downarrow GL(n-1)$$

Gelfand-Tsetlin pattern rules:

1. Rows weakly decreasing
2. Interleaving

1	1	1	3	3
2	2	2		
3	3			

$$\begin{array}{ccccc} 5 & & 3 & & 2 \\ & 3 & & 3 & \\ & & 3 & & \end{array}$$

Gelfand-Tsetlin Patterns

Gelfand-Tsetlin pattern rules:

1. Rows weakly decreasing
2. Interleaving

1	1	1

5		3		2
	3		3	
		3		

Gelfand-Tsetlin Patterns

Gelfand-Tsetlin pattern rules:

1. Rows weakly decreasing
2. Interleaving

1	1	1
2	2	2

$$\begin{array}{ccccc} 5 & & 3 & & 2 \\ & 3 & & 3 & \\ & & 3 & & \end{array}$$

Gelfand-Tsetlin Patterns

Gelfand-Tsetlin pattern rules:

1. Rows weakly decreasing
2. Interleaving

1	1	1	3	3
2	2	2		
3	3			

$$\begin{array}{ccccc} 5 & & 3 & & 2 \\ & 3 & & 3 & \\ & & 3 & & \end{array}$$

Strict Gelfand-Tsetlin Patterns

$\rho = (n-1, ..., 0)$

$$\begin{array}{ccccc} 5 & & 3 & & 2 \\ & 3 & & 3 & \\ & & 3 & & \end{array} \quad \begin{array}{l} +\rho = (2,1,0) \longrightarrow \\ +\rho = (1,0) \longrightarrow \\ +\rho = (0) \longrightarrow \end{array} \quad \begin{array}{ccccc} 7 & & 4 & & 2 \\ & 4 & & 3 & \\ & & 3 & & \end{array}$$

Tokuyama's Formula

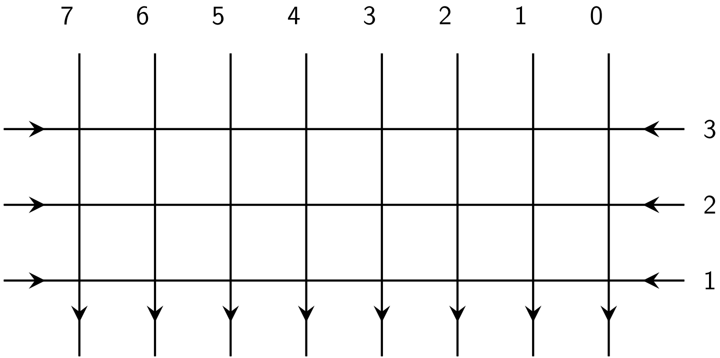
$$\sum_{T \in SGT(\lambda+\rho)} (1+t)^{S(T)} t^{L(T)} z^{wt(T)} = \prod_{i<j} (z_i + t z_j) s_\lambda(z_1, \ldots z_n)$$

$GL(n)$ Shifted Tableaux

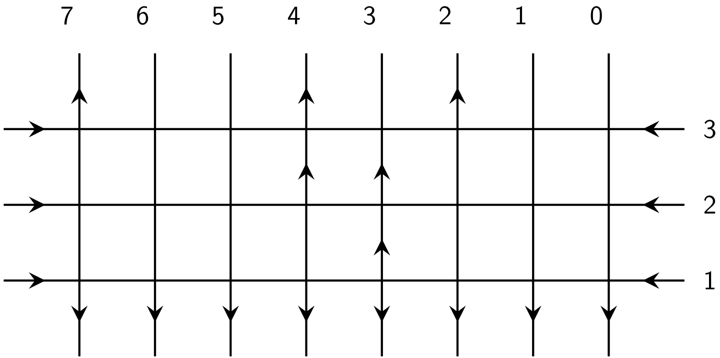
$$\begin{matrix} 7 & & 4 & & 2 \\ & 4 & & 3 & \\ & & 3 & & \end{matrix}$$

1	1	1	2	3	3	3
	2	2	2	3		
		3	3			

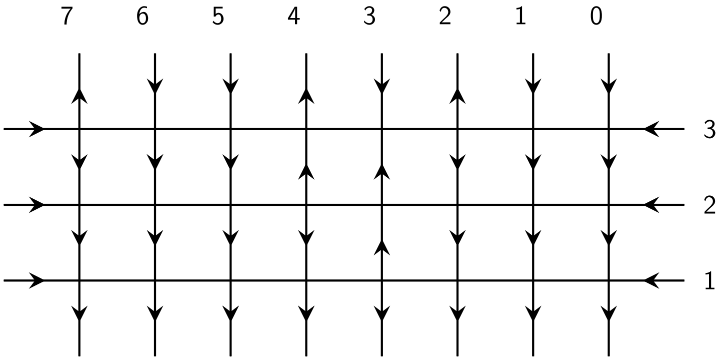
$GL(n)$ Ice Models: Boundary Conditions



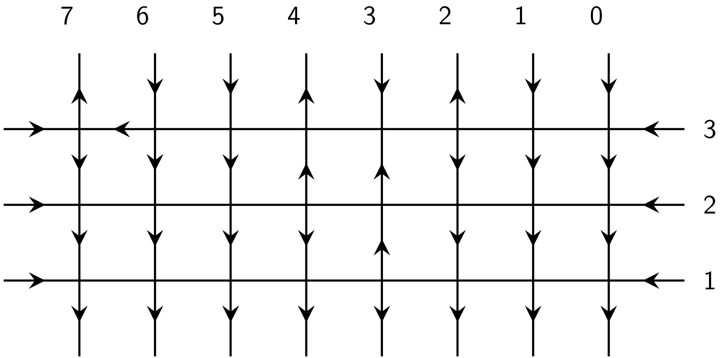
$GL(n)$ Ice Models: Gelfand-Tsetlin Pattern 1



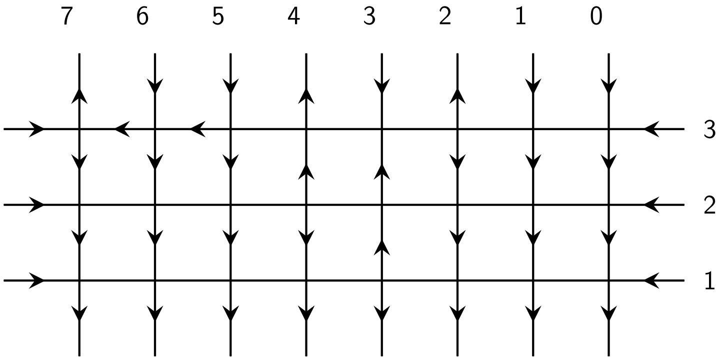
$GL(n)$ Ice Models: Gelfand-Tsetlin Pattern 2



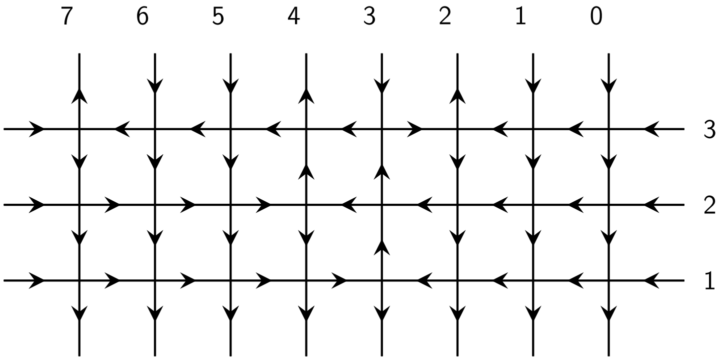
$GL(n)$ Ice Models: Gelfand-Tsetlin Pattern 3



$GL(n)$ Ice Models: Gelfand-Tsetlin Pattern 4



$GL(n)$ Ice Models: Gelfand-Tsetlin Pattern 5



Boltzmann Weights

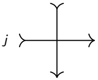
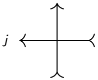
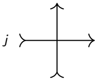
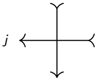
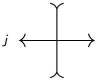
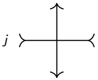
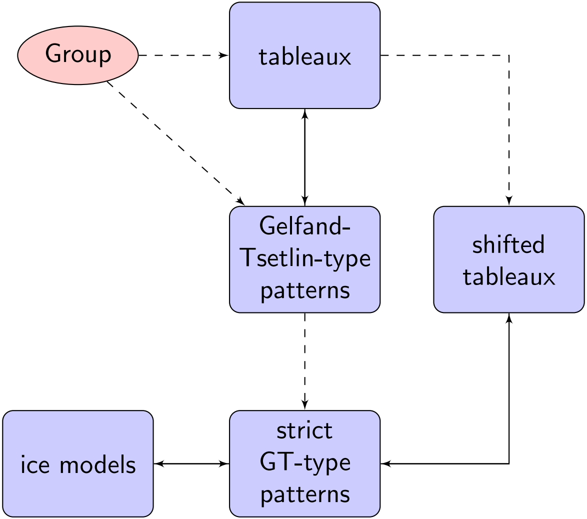
					
1	z_i	t_i	z_i	$z_i(t_i+1)$	1

Figure: Boltzmann weights for Gamma Ice

Introduction



Branching Rule for Sundaram Tableaux

$$s_\lambda^{so} = \sum_{\mu \subseteq \lambda} s_{(\mu)}^{sp}$$

$$Sp(2n) \downarrow Sp(2n-2) \otimes U(1)$$

Sundaram Tableaux

Partition: $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 1)$
Alphabet: $\{1 < \bar{1} < \cdots < n < \bar{n} < 0\}$

1. Rows are weakly increasing.
2. Columns are strictly increasing, but 0s do not violate this condition.
3. No row contains multiple 0s.
4. In row i, all entries are greater than or equal to i.

1	$\bar{1}$	0
2	0	

Sundaram Gelfand-Tsetlin-type patterns

Gelfand-Tsetlin-type pattern rules:

1. Rows weakly decreasing
2. Interleaving
3. Difference between top rows ≤ 1
4. Even rows cannot end in 0

1	$\bar{1}$	0
2	0	

$$\begin{array}{ccccc} 3 & & 2 & & 0 \\ & 3 & & 1 & \\ & & 2 & & 1 \\ & & & 2 & \\ & & & & 1 \end{array}$$

Sundaram Strict Gelfand-Tsetlin-Type Patterns

$$\begin{array}{ccccc} 3 & & 2 & & 0 \\ & 3 & & 1 & \\ & & 2 & & 1 \\ & & & 2 & \\ & & & & 1 \end{array} \quad \text{add } \rho \longrightarrow \quad \begin{array}{ccccc} 5 & & 3 & & 0 \\ & 5 & & 2 & \\ & & 4 & & 2 \\ & & & 3 & \\ & & & & 2 \end{array}$$

Sundaram Shifted Tableaux

$$\begin{array}{ccccc} 5 & & 3 & & 0 \\ & 4 & & 3 & \\ & & 3 & & 2 \\ & & & 2 & \\ & & & & 1 \end{array}$$

1	$\bar{1}$	2	$\bar{2}$	0
	2	2	$\bar{2}$	

Sundaram Ice Models: Boundary Conditions

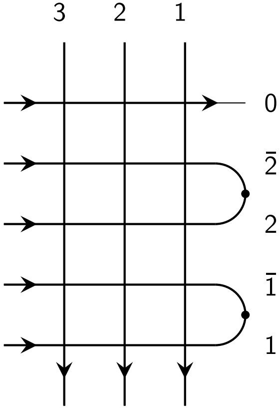
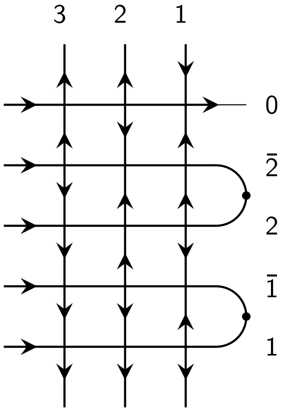


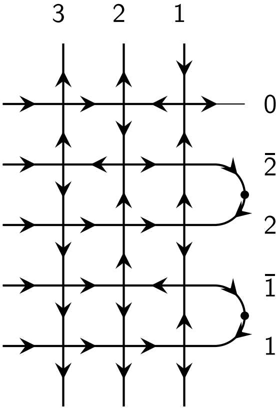
Figure:

Sundaram Ice Models: Modeling GT-Type Pattern



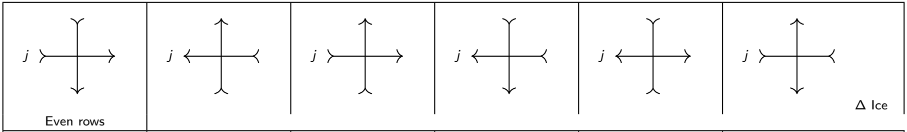
3		2		0
	3		1	
		2		1
			2	
				1

Sundaram Ice Models: Full Model

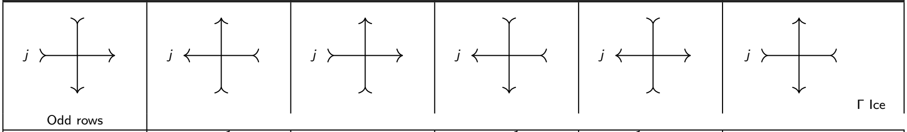


3 2 0
3 1
2 1
2
1

Sundaram Boltzmann Weights



1	tz_i	1	z_i	$z_i(t+1)$	1



1	z_i^{-1}	t	z_i^{-1}	$z_i^{-1}(t+1)$	1

Figure: Boltzmann weights for Δ and Γ Sundaram Ice

Figure: Boltzmann Weights for Sundaram Bends

Sundaram Botlzmann Weights

Alternate Bend Weights for B Deformation:

$$z_i^{-1} \cdot \left(z_i^{-n+i-1} \frac{(1+tz_i)}{(1+tz_i^2)} \right)$$

$$t \cdot \left(z_i^{-n+i-1} \frac{(1+tz_i)}{(1+tz_i^2)} \right)$$

Branching Rule for Koike-Terada Tableaux

$$SO(2n+1) \downarrow SO(2n-1) \otimes GL(1)$$

Koike-Terada Tableaux

Partition: $\lambda = (\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \geq 0)$
Alphabet: $\{1 < \overline{1} < \overline{\overline{1}} < 2 < \overline{2} < \overline{\overline{2}} \cdots n < \overline{n} < \overline{\overline{n}}\}$.
Let $T_{i,j}$ be the entry of the tableau in the i-th row and the j-th column. Then:

1. Rows are weakly increasing
2. Columns are strictly increasing
3. k can only appear in $T_{k,1}$
4. $T_{i,j} \geq i$

$\overline{1}$	$\overline{\overline{1}}$	$\overline{2}$	$\overline{2}$	$\overline{\overline{2}}$
2	$\overline{2}$	$\overline{\overline{2}}$		

Koike-Terada Gelfand-Tsetlin-type pattern

1. The pattern has $3n$ rows. Label these rows $1, \bar{1}, \bar{\bar{1}}, \cdots, n, \bar{n}, \bar{\bar{n}}$, starting from the bottom of the pattern. Rows $i, \bar{i}$, and $\bar{\bar{i}}$ must have i entries.
2. $a_{i,j-1} \geq a_{i,j} \geq a_{i,j+1} \geq 0$
3. $a_{i-1,j} \geq a_{i,j} \geq a_{i-1,j+1}$
4. Row i must end in a 1 or a 0 (for $i \in \{1, \cdots, n\}$)
5. Each entry in row $\bar{\bar{i}}$ (for $i \in \{1, \cdots, n\}$) must be left-leaning.

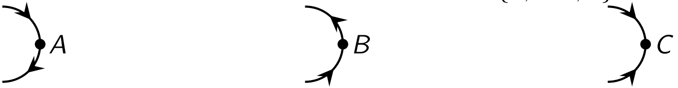
$$
\begin{array}{cccccccc}
5 & & 3 & & & & & \bar{\bar{2}} \\
& 4 & & 2 & & & & \bar{2} \\
& & 2 & & 1 & & & 2 \\
& & & 2 & & & & \bar{\bar{1}} \\
& & & & 1 & & & \bar{1} \\
& & & & & 0 & & 1
\end{array}
$$

Koike-Terada Shifted Tableaux

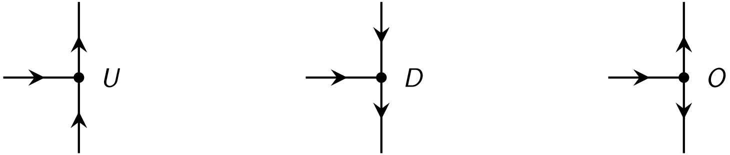
1. Rows are weakly increasing.
2. Columns are weakly increasing.
3. Diagonals are strictly increasing.
4. The first entry in row k is k, $\overline{k}$ or $\overline{\overline{k}}$.

Koike-Terada Ice: Bends and Ties

To connect rows $\overline{\overline{k}}$ and $\overline{k}$ for each $k \in \{1, \cdots, n\}$:



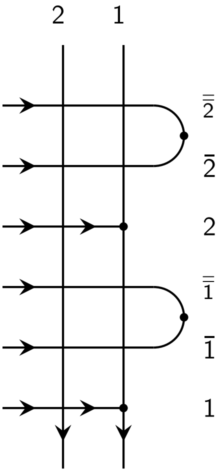
For rows k, where k $k \in \{1, \cdots, n\}$, there are 3 possible "ties":



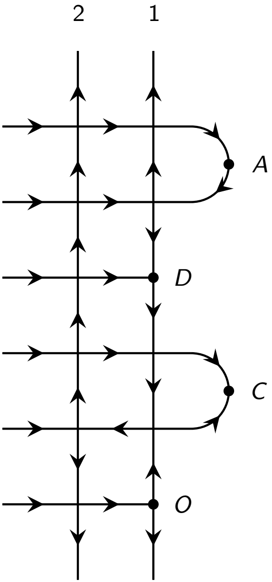
Note: Along with ties U, D and O, rows k $k \in \{1, \cdots, n\}$ are three-vertex models, the vertices being SW, NW, and NE.

Koike-Terada Ice: Boundary Conditions

The following depicts the boundary conditions for an ice model with top row $\lambda = (2, 1)$.



Koike-Terada Ice: Full Model



2 1
2 1
2 0
2
2
1

Theorem (1)

The following are equivalent:

1. *Koike-Terada Gelfand-Tsetlin-type pattern rules 4 and 5 are satisfied.*
2. *Each ice row labeled $k \in \{1, \cdots, n\}$ has no NS, SE, or EW configurations, and tie boundary conditions are satsified.*

Branching Rule for Proctor Tableaux

$$SO(2n+1) \downarrow SO(2n-1) \otimes SO(2)$$

Proctor Tableaux

1. Rows are weakly increasing
2. Columns are strictly increasing
3. Follows the 2c orthogonal condition
4. Follows the $2m$ protection condition

Proctor Tableaux

2c Orthogonal Condition: Less than or equal to 2c entries that are less than or equal to 2c in the first two columns.

1	1	3	5
x	3	4	
5			

Proctor Tableaux

2c Orthogonal Condition: Less than or equal to 2c entries that are less than or equal to 2c in the first two columns.

1	1	3	5
x	3	4	
5			

1	1	3	5
3	3	4	
5			

Proctor Tableaux

2m Protection Condition: For a 2m-1 entry in the first column, specified 2m entries must be "protected" by 2m-1 entries.

1	1	x	5
3	x	4	
5			

Proctor Tableaux

2m Protection Condition: For a 2m-1 entry in the first column, specified 2m entries must be "protected" by 2m-1 entries.

1	1	x	5
3	x	4	
5			

1	1	3	5
3	3	4	
5			

Proctor Gelfand-Tsetlin-type Patterns

n=4 shape

$$
\begin{array}{cccccccccccccc}
x & & x & & x & & x & & & & & & & (9) \\
 & x & & x & & x & & x & & & & & & (8) \\
 & & x & & x & & x & & x & & & & & (7) \\
 & & & x & & x & & x & & x & & & & (6) \\
 & & & & x & & x & & x & & x & & & (5) \\
 & & & & & x & & x & & x & & x & & (n=4) \\
 & & & & & & x & & x & & x & & & (3) \\
 & & & & & & & x & & x & & & & (2) \\
 & & & & & & & & x & & & & & (1)
\end{array}
$$

Proctor Gelfand-Tsetlin-type Patterns

2c Orthogonal Condition

$$\begin{array}{cccccccccccccc}
6 & & 5 & & 3 & & 1 & & & & & & & \\
& 6 & & 5 & & 3 & & 1 & & & & & & = 2+2+2+1\ (8) \\
& & 6 & & 4 & & 2 & & 1 & & & & & \\
& & & 4 & & 3 & & 1 & & 0 & & & & = 2+2+1 \qquad (6) \\
& & & & 4 & & 2 & & 1 & & 0 & & & \\
& & & & & 2 & & 1 & & 0 & & 0 & & = 2+1 \qquad (4) \\
& & & & & & 2 & & 1 & & 0 & & & \\
& & & & & & & 1 & & 0 & & & & = 1 \qquad (2) \\
& & & & & & & & 1 & & & & &
\end{array}$$

Proctor Gelfand-Tsetlin-type Patterns

2m Protection Condition:
Add in 0s to make all rows length n

6 5 3 2
6 5 3 2
6 4 2 1
4 3 1 0
4 2 1 0
2 1 0 0
2 1 0 0
1 0 0 0
1 0 0 0
0 0 0 0

Proctor Gelfand-Tsetlin-type Patterns

2m Protection Condition:

Identify non-left-leaning 0s in even rows

6		5		3		2									
	6		5		3		2								
		6		4		2		1							
			4		3		1		0						
				4		2		1		0					
					2		1		0		0				
						2		1		0		0			
							1		0		0		0		
								1		0		0		0	
									0		0		0		0

Proctor Gelfand-Tsetlin-type Patterns

2m Protection Condition

Check Example:

- For 0, if $2 > 1$
- Check:
- $2 \geq 2$
- $2 > 1$
- $0 \leq 1$

6		5		3		2									
	6		5		3		2								
		6		4		2		1							
			4		3		1		0						
				4		2		1		0					
					2		1		0		0				
						2		1		0		0			
							1		0		0		0		
								1		0		0		0	
									0		0		0		0

Proctor Strict Gelfand-Tsetlin-type Patterns

Change to be Strict and check Orthogonal Condition:

$$
\begin{array}{cccccccccccc}
6 & & 5 & & 3 & & 1 & & & & & \\
& 6 & & 5 & & 3 & & 1 & & & & \\
& & 6 & & 4 & & 2 & & 1 & & & \\
& & & 4 & & 3 & & 1 & & 0 & & \\
& & & & 4 & & 2 & & 1 & & 0 & \\
& & & & & 3 & & 2 & & 1 & & 0 \\
& & & & & & 2 & & 1 & & 0 & \\
& & & & & & & 1 & & 0 & & \\
& & & & & & & & 1 & & &
\end{array}
\qquad = 2 + 2 + 1 \quad (4)
$$

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Questions?