Simple, Seedy Derivations of Generating Functions for Simple Polytopes and cd-indices

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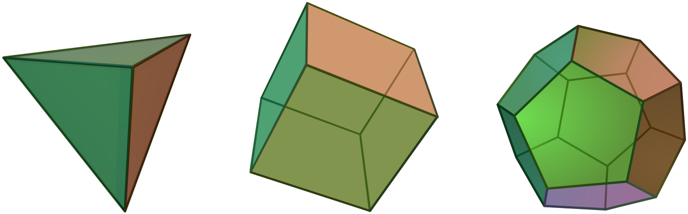
Section 1

Introduction to Polytopes

What are polytopes?

Definition (Polytope)

A *polytope* is the convex hull of a finite number of points in $\mathbb{R}^r$.



Examples of polytopes in $\mathbb{R}^3$

Faces of Polytopes

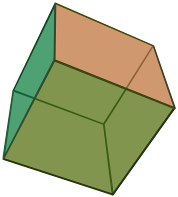
- Polytopes have *faces*.
- Faces are polytopes themselves.
- Faces have dimensions. It's the minimal integer d such that the face can live in $\mathbb{R}^d$.
- A j-dimensional face is called a j*-face*.
- A 0-face is usually called a *vertex*.
  A 1-face is usually called an *edge*.
  An r-face is the polytope itself.

f-vector and f-polynomial

Definition (f-vector and f-polynomial)

Define the *f-vector* of a r-dim Polytope P as $f(P) := (f_0, \ldots, f_r)$, where f_i is the number of i-dimensional faces of P.
Define its *f-polynomial* as $f_P(t) = \sum_{i=0}^{r} f_i t^i$.

Example:



A cube has 8 vertices, 12 edges and 6 faces.

$$f(P) = (8, 12, 6, 1)$$

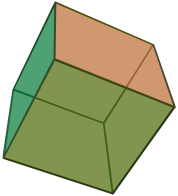
$$f_P(t) = 8 + 12t + 6t^2 + t^3$$

h-vector and h-polynomial

Definition (h-vector and h-polynomial)

Define the *h-polynomial* of a r-dim Polytope P as
$h_P(t) = f_P(t-1) = \sum_{i=0}^{r} f_i(t-1)^i$.
Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *$h$-vector* as
$h(P) := (h_0, h_1, \dots, h_r)$.

Example:



A cube has $f_P(t) = 8+12t+6t^2+t^3$.
Replace t with $t-1$.

$$h_P(t) = f_P(t-1) = 1+3t+3t^2+t^3$$

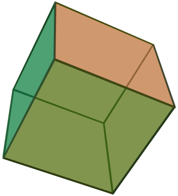
$$h(P) = (1,3,3,1)$$

h-vector and h-polynomial

Definition (h-vector and h-polynomial)

Define the *h-polynomial* of a r-dim Polytope P as
$h_P(t) = f_P(t-1) = \sum_{i=0}^{r} f_i(t-1)^i$.
Assume $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then define its *$h$-vector* as
$h(P) := (h_0, h_1, \ldots, h_r)$.

Example:



A cube has $f_P(t) = 8 + 12t + 6t^2 + t^3$.
Replace t with $t-1$.

$$h_P(t) = f_P(t-1) = 1 + 3t + 3t^2 + t^3$$

$$h(P) = (1, 3, 3, 1)$$

Is this always symmetric?

Dehn-Somerville Equation

Definition (Simple Polytope)

A r-dimensional polytope is called a *simple polytope* if and only if each vertex has exactly r incident edges.

For example, a cube is a simple polytope.

Theorem (Dehn-Sommerville equation)

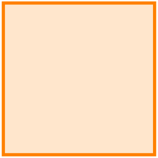
For any simple polytope P, its h-vector is symmetric.

Face Poset

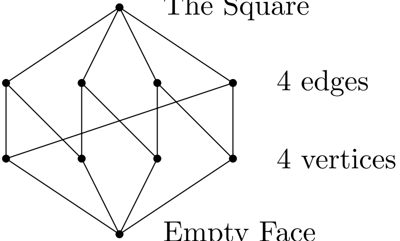
Definition (Face Poset)

The *face poset* of polytope P is the poset $\{\text{faces of } P\}$ ordered by inclusion of faces.

Example:



Polytope



Face Poset

*Note: A Face Poset is graded.

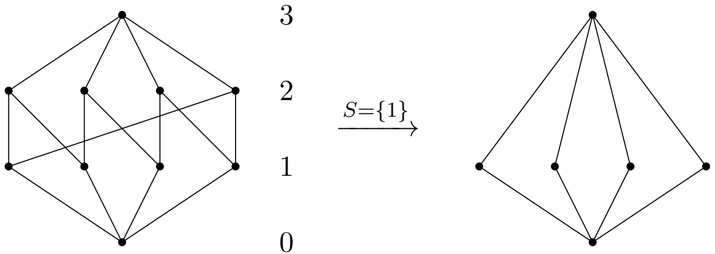
Rank Selected Poset

Definition (Rank Selected Poset)

Let $S \subseteq [r] = \{1, 2, \ldots, r\}$. The *rank-selected poset* P_S of P is

$$P_S = \{x \in P | \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\},$$

where ρ is the rank function.



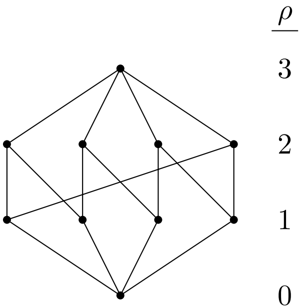
Flag f-vector and Flag h-vector

Definition (Flag f-vector and Flag h-vector)

Define the *flag f-vector* $\alpha(S)$ as the number of maximal chains in P_S. Based on that, define the *flag h-vector* $\beta(S)$ as:

$$\beta(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha(T) \quad \text{or,} \quad \alpha(S) = \sum_{T \subseteq S} \beta(T).$$

Example of Flag Vectors

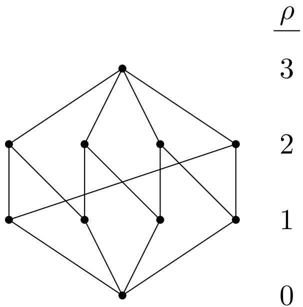


Face Poset

S	$\alpha(S)$	$\beta(S)$
$\emptyset$	1	1
$\{1\}$	4	3
$\{2\}$	4	3
$\{1,2\}$	8	1

Table of Flag Vectors

Example of Flag Vectors



Face Poset

S	$\alpha(S)$	$\beta(S)$
$\emptyset$	1	1
$\{1\}$	4	3
$\{2\}$	4	3
$\{1,2\}$	8	1

Table of Flag Vectors

$$\beta(S) = \beta(\bar{S})$$

ab-index

Definition (ab-index)

Define the *ab-index* of Polytope P as a polynomial over non-commutative variables a, b as

$$\Phi_P(a,b) = \sum_{S \subseteq [n]} \beta(S) u_S.$$

Here $u_S = u_n u_{n-1} \cdots u_1$, where

$$u_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

Example of ab-index

S	$\alpha(S)$	$\beta(S)$	u_S
$\emptyset$	1	1	a^2
$\{1\}$	4	3	ab
$\{2\}$	4	3	ba
$\{1,2\}$	8	1	b^2

Table of Flag Vectors

$$\Phi_P(a,b) = a^2 + 3ab + 3ba + b^2.$$

cd-index

Theorem (cd-index)

For any polytope P, there exists a polynomial $\Psi_P(c,d)$ in the non-commuting variables c and d such that

$$\Phi_P(a,b) = \Psi_P(a+b, ab+ba).$$

$\Psi_P(c,d)$ is also called the **cd-index** of polytope P.

Example of *cd*-index

S	$\alpha(S)$	$\beta(S)$	u_S
$\emptyset$	1	1	a^2
$\{1\}$	4	3	ab
$\{2\}$	4	3	ba
$\{1,2\}$	8	1	b^2

Table of Flag Vectors

$$\Phi_P(a,b) = a^2 + 3ab + 3ba + b^2$$

$$= (a+b)^2 + 2(ab+ba).$$

Replace $a+b \to c$, $ab+ba \to d$.

$$\Psi_P(c,d) = c^2 + 2d.$$

Summary

Methods to describe a polytope:

- f-polynomial/h-polynomial;
- face poset;
- cd-index.

Section 2

Coxeter Group and Weight Polytopes

Finite Reflection groups

Definition (Finite Reflection Group)

A *finite reflection group* is a finite subgroup $W \subset \mathrm{GL}_n(\mathbb{R})$ generated by reflections, i.e. elements w such that $w^2 = 1$ and they fix a hyperplane H and negate the line perpendicular to H

Example: One example of a finite reflection group is the Dihedral Group $I_n = \{s, t \mid s^2 = t^2 = e, (st)^n = e\}$.

Coxeter groups

Definition (Coxeter Group)

A *Coxeter Group* is a group W of the form

$$W \cong \langle s_1, \dots, s_n \mid s_i^2 = e, (s_i s_j)^{m_{ij}} = e \rangle$$

for some $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$.
If W is finite, then W is called a *Finite Coxeter Group*.
$S = \{s_1, s_2, \dots, s_n\}$ is called the *Generating Set* of W.

Finite Coxeter Groups = Finite Reflection Groups

Here is a BIG theorem of Coxeter:

Theorem (Coxeter)

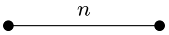
Finite Coxeter groups = Finite reflection groups.

Coxeter Diagram

Definition (Coxeter Diagram)

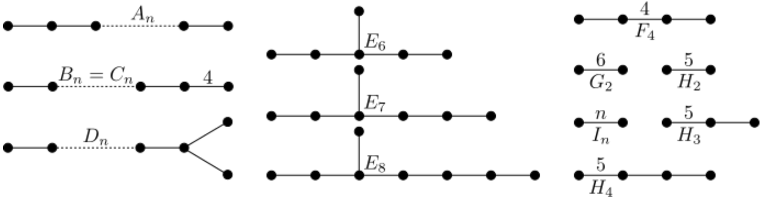
Given a Coxeter presentation (W, S), we can encapsulate it in the *Coxeter Diagram*, denoted $\Gamma(W)$, a graph with $V = S$ and if $m_{ij} = 3$, s_i and s_j are connected with no label and if $m_{ij} > 3$, s_i and s_j are connected with label m_{ij}.

Example: The dihedral group I_n has Coxeter diagram



Finite Coxeter Groups

Amazingly, finite Coxeter groups are classified! They come in four infinite families, A_n, B_n, D_n, and I_n, as well as a finite collection of exceptional cases. The Coxeter diagrams look as follows:



We will focus our energies on types A_n, B_n, D_n.

Weight Polytopes

Definition (Weight Polytope)

Given finite Coxeter group W, $\lambda \in \mathbb{R}^n$, we define the *Weight Polytope* P_λ to be the convex hull of $\{w \cdot \lambda \mid w \in W\}$.

Weight Polytopes

Definition (Stabilizer)

Let $J(\lambda) = \{s \in S \mid s(\lambda) = \lambda\}$ be the *stabilizer* of λ.

Theorem (Maxwell)

The f-vector and face lattice of a weight polytope P_λ is only dependent on W, S and $J(\lambda)$.

Weight Polytope Example 1

Coxeter Group

$W = A_n =$ symmetric group S_{n+1}

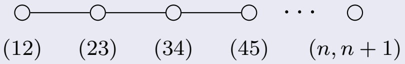
Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{n \text{ zeros}}, 1)$$

Weight Polytope Example 1

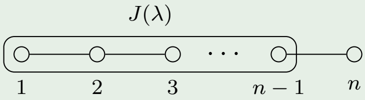
Coxeter Group

$W = A_n =$ symmetric group S_{n+1}



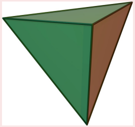
Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{n \text{ zeros}}, 1)$$



Polytope

Name: **Simplex**



Vertices: Set of vectors with n zeros and 1 one

Weight Polytope Example 2

Coxeter Group

$W = B_n =$ signed permutation group

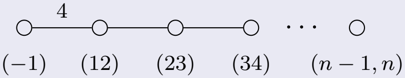
Vector λ

$$\lambda = (\underbrace{1, 1, \ldots, 1}_{n \text{ ones}})$$

Weight Polytope Example 2

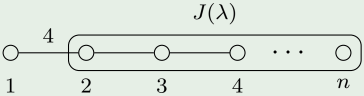
Coxeter Group

$W = B_n =$ signed permutation group



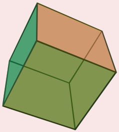
Vector λ

$$\lambda = (\underbrace{1, 1, \ldots, 1}_{n \text{ ones}})$$



Polytope

Name: **HyperCube**



Vertices: Set of vectors with 1 and -1

Weight Polytope Example 3

Coxeter Group

$W = B_n =$ signed permutation group

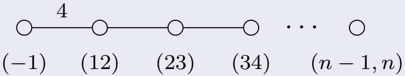
Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{n-1 \text{ zeros}}, 1)$$

Weight Polytope Example 3

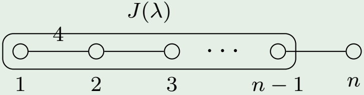
Coxeter Group

$W = B_n =$ signed permutation group



Vector λ

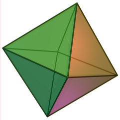
$$\lambda = (\underbrace{0, \ldots, 0}_{n-1 \text{ zeros}}, 1)$$



Polytope

Name:

HyperOctahedron



Vertices: Set of vectors with $n - 1$ zeros and one ± 1

Weight Polytope Example 4

Coxeter Group

$W = A_n =$ symmetric group S_{n+1}

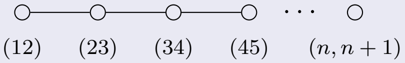
Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{k \text{ zeros}}, \underbrace{1, \ldots, 1}_{n-k+1 \text{ ones}})$$

Weight Polytope Example 4

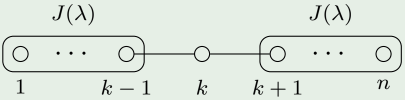
Coxeter Group

$W = A_n =$ symmetric group S_{n+1}



Vector λ

$$\lambda = (\underbrace{0, \ldots, 0}_{k \text{ zeros}}, \underbrace{1, \ldots, 1}_{n-k+1 \text{ ones}})$$



Polytope

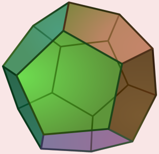
Name: **HyperSimplex**

Vertices: Set of vectors with k zeros and $n-k+1$ ones

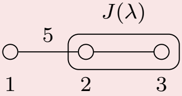
Other Examples

Example 5

Name: **Dodecahedron**

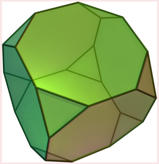


Coxeter Group: $W = H_3$

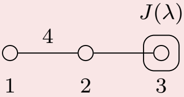


Example 6

Name: **Truncated Cube**



Coxeter Group: $W = B_3$



Recall Summary

Methods to describe a polytope:

- f-polynomial/h-polynomial;
- face poset;
- cd-index.

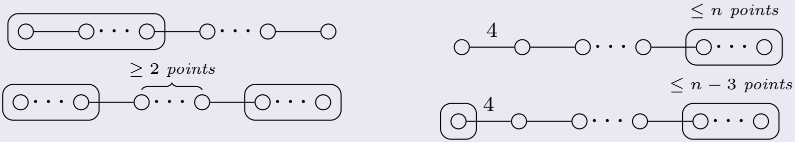
Section 3

f-polynomials of Simple Weight Polytopes

Renner's Classfication of Simple Polytopes

Theorem (Renner)

A type A_n or B_n weight polytope is simple iff its Coxeter diagram has one of the following structures.



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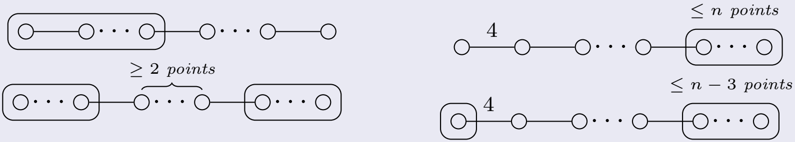
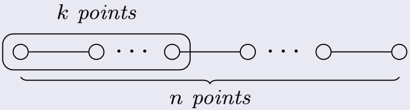


What are their f-polynomials?

Case 1

Theorem (Golubitsky)

Denote $F_{n,k}(t)$ as the f-polynomial for the f polytope of



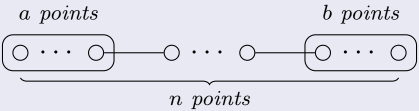
Then,

$$\sum_{n \geq k \geq 0} F_{n,k}(t) \cdot \frac{x^{n+1} y^k}{(n+1)!} = \frac{e^{xy}}{y-1} \cdot \left(y + \frac{e^{txy} - t - 1}{t + 1 - e^{tx}} \right) - 1.$$

Case 2

Theorem

Denote $F_{n,a,b}(t)$ as the f-polynomial for the f polytope of

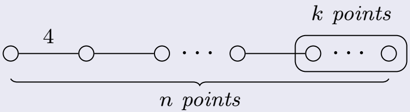


Then,
$$\sum_{a,b\ge 0}\sum_{n>a+b} F_{n,a,b}(t)\cdot \frac{x^{n+1}y^a z^b}{(n+1)!} = \frac{1}{y^2-y}\Bigg(x + \frac{(xy-e^{xy}+1)(xz-e^{xz})}{y} + \frac{\left(tz+(t+1)e^{xz}-t-e^{(t+1)xz}\right)\left(\frac{ty+(t+1)e^{(xy)}-t-e^{((t+1)xy)}}{\left(t-e^{(tx)}+1\right)y} - e^{(xy)}\right)}{t(y-1)z} + \frac{e^{(xy+xz)}}{ty} + \frac{\left(ze^{(txy)}-ye^{(txz)}\right)e^{(xy+xz)}}{t(y-z)y}\Bigg).$$

Case 3

Theorem

Denote $F_{n,k}(t)$ *as the* f*-polynomial for the* f *polytope of*



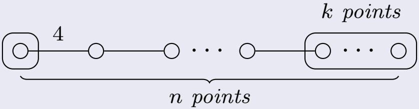
Then, $\displaystyle\sum_{n>k\geq 0} F_{n,k}(t)\cdot\frac{x^n y^k}{n!} =$

$$\frac{1}{y-1}\left(e^{(t+2)xy}+\frac{e^{tx}\cdot\left(e^{2(t+1)xy}-(t+1)\,e^{2xy}+t-ty\right)}{(t+1-e^{2tx})y}\right).$$

Case 4

Theorem

Denote $F_{n,k}(t)$ *as the* f*-polynomial for the* f *polytope of*



Then,

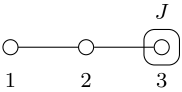
$$\sum_{n-2>k\geq 0} F_{n,k}(t)\frac{x^{n+1}y^k}{(n+1)!} = \frac{1}{y^2-y}\Bigg(xy + \left(y + \frac{(t+1)e^{(2\,xy)}}{t} - \frac{e^{(2\,(t+1)xy)}}{t} - 1\right)\left(\frac{(t+1)tx - te^{(tx)}}{t - e^{(2\,tx)} + 1} + 1\right) - x - \frac{\left((t+1)xy + \frac{1}{t} + 1\right)e^{(2\,xy)} - \frac{e^{(2\,(t+1)xy)}}{t} - e^{((t+2)xy)}}{y}\Bigg).$$

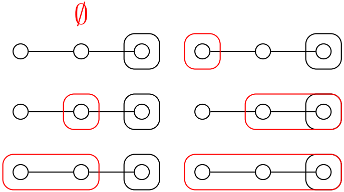
Ingredients of the Proof

Definition (J-minimal subset)

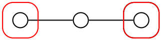
For a Coxeter diagram $\Gamma = (W, S)$ and subset $J \subseteq S$, a *J-minimal subset* is a subset $X \subseteq S$ such that no connected component of X on the Coxeter diagram lies entirely in J.

Example:





All six J-minimal subsets



Not J-minimal

Ingredients of the Proof

Theorem (Renner, Maxwell)

Consider the action of W on $\{$faces of $P_\lambda\}$, then there is a bijection

$$f : \{J(\lambda)\text{-minimal sets}\} \to \{\text{orbits of the action}\}.$$

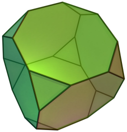
If X is $J(\lambda)$-minimal, then all faces in $f(X)$ are called X-type face. All X-type face has dimension $|X|$, and the number of X-type face is

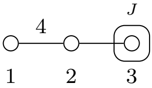
$$\frac{|W|}{|W_{X^*}|},$$

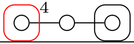
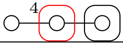
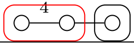
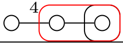
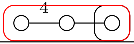
where $W_{X^*} \subseteq W$ is the subgroup generated by

$$\{s \in S | s \in X \text{ or } s \text{ and } X \text{ are not connected}\}.$$

Example of Renner/Maxwell





X	Face	W_{X^*}	$\|W\|/\|W_{X^*}\|$
$\emptyset$	Vertices	$\{3\}$	$48/2 = 24$
	Long Edges	$\{1,3\}$	$48/4 = 12$
	Triangle Edges	$\{2\}$	$48/2 = 24$
	Octagons	$\{1,2\}$	$48/8 = 6$
	Triangles	$\{2,3\}$	$48/6 = 8$
	Truncated Cube	$\{1,2,3\}$	$48/48 = 1$

However...

Renner only proved the case where W is a Weyl Group (a special type of Coxeter Group that forms a lattice).

Is this true for general finite Coxeter Group ?

However...

Renner only proved the case where W is a Weyl Group (a special type of Coxeter Group that forms a lattice).

Is this true for general finite Coxeter Group ?

Answer: Yes!

Section 4

Face Poset of General Weight Polytopes: Maxwell implies Renner

Maxwell

Theorem (Maxwell)

Given Coxeter System (W, S), and vector λ with stablizer J.

The face poset of polytope P_λ is isomorphic to the poset

$$L(W, J) = \{gW_X W_J | g \in W, X \subseteq S \text{ is } J\text{-minimal}\}$$

ordered by inclusion.

Here W_X is the subgroup generated by elements in X.

What does Maxwell Imply?

Corollary

- *All faces are labelled by some J-minimal set X;*
- *A X face lies inside a Y face if and only if $X \subseteq Y$;*
- *If $X \subseteq Y$, the number of X face inside a Y face is equal to*
$$\frac{|W_Y|}{|W_X| \cdot |W_{Y \cap (X^* \setminus X)}|}.$$

Take $Y = S$ the entire set, the number of X-face is

$$\frac{|W|}{|W_{X^*}|},$$

the same as Renner.

Section 5

A glimpse on the *cd*-index of Weight Polytopes

cd-index for simplices

Theorem (Stanley)

If Ψ_P denotes the cd-index for a poset P then

$$2\Psi_P = 2\Psi_{\hat{0}\hat{1}} = \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j-1}} (c^2-2d)^{j-1} c\Psi_{\hat{0}x} - \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j}} (c^2-2d)^{j}\Psi_{\hat{0}x} + \begin{cases} 2(c^2-2d)^{k-1} & \text{if } \rho(\hat{0},\hat{1}) = 2k-1 \\ 0 & \text{if } \rho(\hat{0},\hat{1}) = 2k. \end{cases}$$

Corollary (Stanley)

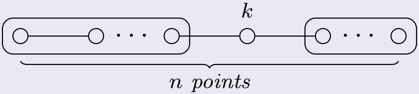
if $\Psi_n(c,d)$ is the cd-index for the n-simplex then

$$\sum_{n\geq 1} \frac{\Psi_n(c,d)x^n}{(n+1)!} = \frac{2\sinh((a-b)x)}{a-b} \cdot \left(1 - \frac{c\sinh((a-b)x)}{a-b} + \cosh((a-b)x)\right)^{-1}$$

cd-index for hypersimplices

Theorem

Denote $\Psi_{n,k}$ *as the cd-index for the hypersimplex*



$$\textit{Then, } \sum_{n \geq k \geq 1} \Psi_{n,k}(c,d) \frac{y^k}{(n+1)!} = (1 - s(y+1) \cdot c + c(y+1))^{-1}$$

$$\cdot \left(\frac{c(y+1) - c(y) - c(1) + 1}{c^2 - 2d} \cdot c - s(y+1) + \frac{y+1}{y-1} \cdot (s(y) - s(1)) \right)$$

where $c(x) = \cosh((a-b)x)$ *and* $s(x) = \sinh((a-b)x)/(a-b)$

Idea of Proof

Combine Stanley's Method with Renner/Maxwell's formula.

Section 6

Summary

What have we done?

	f-polynomial	Face Poset	cd-index
General Weight Polytopes	✓	Maxwell (we rewrote ✓)	
Weyl Group Weight Polytopes	✓ (some done by Golubitsky)	Renner	
Hypersimplex	✓	Renner	✓
Simplex	Known	Known	Stanley

Acknowledgement

We would like to thank Vic and Andy for their continual guidance and support!

The End!

Thank You!