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Resolutions of Stanley-Reisner rings and Alexander duality

John A. Eagon, Victor Reiner *

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract

Associated to any simplicial complex Δ on n vertices is a square-free monomial ideal I_Δ in the polynomial ring $A = k[x_1, \dots, x_n]$, and its quotient $k[\Delta] = A/I_\Delta$ known as the Stanley–Reisner ring. This note considers a simplicial complex Δ^* which is in a sense a canonical Alexander dual to Δ, previously considered in [1, 5]. Using Alexander duality and a result of Hochster computing the Betti numbers $\dim_k \mathrm{Tor}_i^A(k[\Delta], k)$, it is shown (Proposition 1) that these Betti numbers are computable from the homology of links of faces in Δ^*. As corollaries, we prove that I_Δ has a linear resolution as A-module if and only if Δ^* is Cohen–Macaulay over k, and show how to compute the Betti numbers $\dim_k \mathrm{Tor}_i^A(k[\Delta], k)$ in some cases where Δ^* is well-behaved (shellable, Cohen–Macaulay, or Buchsbaum). Some other applications of the notion of shellability are also discussed. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let Δ be an abstract simplicial complex on vertex set $[n] := \{1, 2, 3, \dots, n\}$, i.e. Δ is a collection of subsets $F \subseteq [n]$ called *faces* which is closed under inclusion. The *dimension* $\dim(F)$ of the face F is $|F| - 1$, and $\dim(\Delta)$ is the maximum dimension of its faces. We say that Δ is *pure* if all maximal faces of Δ have the same dimension, equal to $\dim(\Delta)$.

There is a well-known construction (see [14, Ch. 2]) of the *Stanley–Reisner* ring $k[\Delta]$ associated to Δ: one forms a certain square-free monomial ideal I_Δ in the polynomial ring $A := k[x_1, \dots, x_n]$, and then $k[\Delta]$ is the quotient ring A/I_Δ. The ideal I_Δ is generated by the monomials x^G as G runs over the inclusion-minimal subsets of $[n]$ which are *not* faces in Δ, where $x^G := \prod_{i \in G} x_i$.

* Corresponding author. E-mail: reiner@math.umn.edu.

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Every square-free monomial ideal I in A is of the form I_Δ for some simplicial complex Δ, and Δ plays a role in understanding the homological properties of I. Conversely, the rings $k[\Delta]$ have played a role in understanding combinatorial properties of simplicial complexes, and in particular the enumeration of their faces of various dimensions (see [14]).

One homological property of interest for $k[\Delta]$ are the *Betti numbers*

$$\beta_i(k[\Delta]) := \dim_k \mathrm{Tor}_i^A(k[\Delta], k),$$

where k is given the trivial A-module structure as the quotient A/A_+ by the irrelevant ideal $A_+ = (x_1, x_2, \ldots, x_n)$. The Betti numbers $\beta_i := \beta_i(k[\Delta])$ are of particular interest because they give the ranks of the ith resolvent in a minimal free resolution of $k[\Delta]$ as an A-module:

$$0 \to A^{\beta_h} \to \cdots \to A^{\beta_1} \to A \to k[\Delta] \to 0.$$

Since the monomial ideal I_Δ is homogeneous with respect to the $\mathbb{N}^n$-grading on A defined by letting the variable x_i have grade equal to the ith standard basis vector e_i, the Stanley–Reisner ring $k[\Delta]$ inherits this grading. The resolvents A^{β_i} may also be given this $\mathbb{N}^n$-grading so as to make the maps in the resolution homogeneous, and hence $\mathrm{Tor}_i^A(k[\Delta], k)$ inherits this grading. For a given grade $\alpha \in \mathbb{N}^n$, let $\mathrm{Tor}_i^A(k[\Delta], k)_\alpha$ denote the α-graded component of $\mathrm{Tor}_i^A(k[\Delta], k)$. One can then collate this finer information about the dimensions of these graded pieces into the *Betti polynomial*

$$T_i(k[\Delta], t) := \sum_\alpha \dim_k \mathrm{Tor}_i^A(k[\Delta], k)_\alpha \, t^\alpha,$$

where $t^\alpha = \prod_i t_i^{\alpha_i}$. Hochster gave the following formula for these Betti polynomials.

Theorem (Hochster [12]).

$$T_i(k[\Delta], t) = \sum_{V \subseteq [n]} \dim_k \tilde{H}_{|V|-i-1}(\Delta_V; k) \, t^V,$$

where Δ_V denotes the simplicial complex on vertex set V defined by

$$\Delta_V := \{V' \subseteq V : V' \in \Delta\}.$$

Here $\tilde{H}(\cdot; k)$ denotes reduced homology with coefficients in the field k, and $t^V := \prod_{i \in V} t_i$.

See [6, 7, 15–19] for some applications of Hochster's formula.

Our observation is that one may reinterpret the reduced homologies in Hochster's formula as the reduced (co-)homologies of links of faces in a certain simplicial complex Δ^* dual to Δ, defined by

$$\Delta^* := \{F \subseteq [n] \colon [n] - F \notin \Delta\}.$$

In other words, if one thinks of Δ as an *order ideal* in the Boolean algebra $2^{[n]}$, then Δ^* is obtained by taking the *order filter* $2^{[n]} - \Delta$, and applying the order-reversing map $F \mapsto [n] - F$ to each of these sets yielding another order ideal Δ^*. This same construction plays an important role in [5, Section 1].

Recall that the *link* of a face F in a simplical complex Δ on vertex set $[n]$ is the simplicial complex on vertex set $[n] - F$ defined by

$$\mathrm{link}_\Delta F := \{G \in \Delta\colon\ G \cup F \in \Delta, G \cap F = \emptyset\}.$$

We shall also need later the *deletion* of vertex v in a simplicial complex, defined by

$$\mathrm{del}_\Delta v := \{G \in \Delta\colon\ v \notin G\}.$$

Proposition 1. *For $i \geq 1$ we have*

$$T_i(k[\Delta], t) = \sum_{F \in \Delta^*} \dim_k \tilde{H}_{i-2}(\mathrm{link}_{\Delta^*} F; k)\, t^{[n]-F}$$

Proof. Given $V \subseteq [n]$ appearing as a term in Hochster's sum, let $F = [n] - V$. Note that if V is a face of Δ then Δ_V will be a simplex and hence have no reduced homology, therefore we may assume V is not a face of Δ. By definition of Δ^* then F *is* a face of Δ^*, so F appears in the sum on the right-hand side in the Proposition 1. Therefore it suffices to show

$$\dim_k \tilde{H}_{i-2}(\mathrm{link}_{\Delta^*} F; k) = \dim_k \tilde{H}_{|V|-i-1}(\Delta_V; k).$$

To see this, note that the complementation map

$$\{V' \subseteq [n]\colon\ V' \subseteq V\} \to \{F' \subseteq [n]\colon\ F \subseteq F'\}$$

given by $V' \mapsto [n] - F'$ identifies the Boolean algebra 2^V with the interval $[F, [n]]$ in the Boolean algebra $2^{[n]}$, and has the property that V' is a face of Δ if and only if $F' = [n] - V'$ is *not* a face of Δ^*. Thus this map gives an isomorphism between the complexes $\mathrm{link}_{\Delta^*} F$ and $(\Delta_V)^*$ if we think of both as having vertex set V. It only remains to apply the following lemma, and use the duality between reduced homology and cohomology over a field k [13, Theorem 53.5]:

Lemma 2 (see [5, Lemma 1.2; 9, Lemma 4; 1, Theorem 6.4.1]). *For any simplicial complex Δ on vertex set $[n]$, we have*

$$\tilde{H}_{i-2}(\Delta^*; k) \cong \tilde{H}^{n-i-1}(\Delta; k).$$

This concludes the proof of the proposition. □

We conclude this section with various remarks on Proposition 1.

Remark. The use of Alexander duality in connection with Hochster's formula is not new, although previously it has been most often used to relate $\tilde{H}_*(\Delta_V; k)$ and $\tilde{H}_*(\Delta_{[n]-V}; k)$ in the case where Δ is a k-homology sphere (e.g. [14, p. 76; 16–19]).

However, we found out that recently many others [1, 5, 15], have independently used this same Alexander duality to show, among other things, that the second Betti number $\beta_2(k[\Delta])$ depends only on Δ and not on the field k (as is clear from Proposition 1). In fact, the discussion in [5, p. 4 paragraph preceding Corollary 1.5] is almost the same as the assertion of Proposition 1, although the subcomplexes $\mathrm{link}_{\Delta^*}F$ which appear there implicitly are never identified as links.

Remark. We were led to this reformulation of Hochster's result by the results of [8], which give a procedure to construct the maps in a minimal free resolution of $k[\Delta]$, in the case where Δ^* is pure. In that paper, there is given more generally a procedure to construct maps in a minimal free resolution for all quotients of a polynomial ring by an ideal generated by monomials which all have the same degree.

2. Applications

Hochster's formula is clearly most useful when the homology of Δ and all of its subcomplexes Δ_V are comprehensible, a situation which is rare unless Δ is low-dimensional (although see [10, 17, 19] for some notable exceptions). On the other hand, the usefulness of Proposition 1 lies in situations where one has information about the links of faces in Δ^*, and there are several well-known hypotheses on a simplicial complex which state such information. We recall here the definitions for a simplicial complex to be Cohen–Macaulay, Buchsbaum, Gorenstein*, or a homology manifold over k, and refer the reader to [14] for equivalent definitions in terms of properties of the Stanley–Reisner ring $k[\Delta]$.

The simplicial complex Δ is said to be *Buchsbaum over the field* k if it is pure, and for every non-empty face F of Δ, we have $\tilde{H}_i(\mathrm{link}_\Delta F; k) = 0$ for $i < \dim(\mathrm{link}_\Delta F)$.

If in addition to Δ being Buchsbaum over k one has that $\tilde{H}_i(\Delta; k) = 0$ for $i < \dim(\Delta)$ then Δ is said to be *Cohen–Macaulay over* k.

If in addition to Δ being Cohen–Macaulay over k one has that

$$\tilde{H}_{\dim(\mathrm{link}_\Delta F)}(\mathrm{link}_\Delta F; k) = k$$

for every face F, then Δ is said to be a *homology sphere over* k or *Gorenstein* over* k.

If in addition to Δ being Buchsbaum over k one has that

$$\tilde{H}_{\dim(\mathrm{link}_\Delta F)}(\mathrm{link}_\Delta F; k) = k$$

for every non-empty face F, then Δ is said to be a *homology manifold over* k.

Examples. It is known [13, Section 63] that simplicial complexes Δ which triangulate a manifold without boundary are homology manifolds over any field k, and if Δ triangulates a sphere then it is a homology sphere over any field k.

All graphs (i.e. 1-dimensional simplicial complexes) are Buchsbaum over arbitrary fields k, and are furthermore Cohen–Macaulay when connected.

We say that an ideal I in A *has linear resolution* if there is a minimal free resolution for A/I in which all the non-zero entries in the matrices $\partial_i : A^{\beta_i} \to A^{\beta_{i-1}}$ for $i \geq 2$ are of degree 1 in the standard grading on A where $deg(x_i) = 1$. Fröberg [9] gave a characterization of the ideals I generated by monomials which have linear resolution, by first reducing to the case of square-free monomial ideals I_Δ, and then using Hochster's formula. Using Proposition 1 we obtain an elegant dual formulation of this result.

Theorem 3. *I_Δ has linear resolution if and only if Δ^* is Cohen–Macaulay over k.*

Proof. It is easy to see that I_Δ has linear resolution if and only if
- its minimal generators all have the same degree t, and
- for each i we have that $\mathrm{Tor}_i^A(k[\Delta], k)$ is homogeneous of degree $t+i$ in the standard grading

(in fact, this is the definition of having *t-linear resolution* used in [9]). The first of these conditions is equivalent to Δ^* being pure. Using Proposition 1, the second condition is equivalent to $\mathrm{link}_{\Delta^*} F$ having no homology over k except in its top dimension for all faces F of Δ^*. Thus these two conditions are exactly equivalent to Δ^* being Cohen–Macaulay over k. □

Remark. Theorem 3 explains some of the "bad" behavior of resolutions of $k[\Delta]$ with respect to the topology of Δ, as discussed in [9]. In [9, Remark 9] it is noted that having linear resolution is not a topological invariant of Δ. However, it *is* a topological invariant of Δ^*. Also, [9, Example 3] points out that when Δ is the well-known 6-point triangulation of $\mathbb{RP}^2$, the resolution is linear when k has characteristic 0 but not when it has characteristic 2. This is because in this case Δ^* is isomorphic to Δ, and hence triangulates $\mathbb{RP}^2$ which is Cohen–Macaulay over k exactly when k has characteristic not equal to 2.

In the case where Δ^* is at least Buchsbaum, Proposition 1 gives an easy computation of the Betti numbers $\beta(k[\Delta])$, in terms of the number of faces of various dimensions and (topological) Betti numbers of Δ^*. Recall (see [14, Appendix 2]), the definition of the *f-vector* of a $(d-1)$-dimensional simplicial complex

$$f(\Delta) := (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

where f_i is the number of i-dimensional faces of Δ. Also recall that the same information may be encoded in the *h-vector* defined by

$$h(\Delta) := (h_0, h_1, \ldots, h_d), \qquad \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}. \tag{1}$$

Also define the (reduced) *Poincaré polynomial* $\mathrm{Poin}(\Delta, t)$ by

$$\mathrm{Poin}(\Delta, t) = \sum_{i \geq -1} \dim_k \tilde{H}_i(\Delta; k) t^i$$

and the (reduced) *Euler characteristic* $\tilde{\chi}(\Delta) = \mathrm{Poin}(\Delta, -1)$.

Theorem 4. *Let Δ be a simplicial complex, and Δ^* its Alexander dual as defined earlier, with* $\dim(\Delta^*) = d-1$.

- *If Δ^* is Buchsbaum, then*

$$\sum_{i\geq 1} \beta_i(k[\Delta])\, t^{i-1} = t\,\mathrm{Poin}(\Delta^*, t) + (-t)^d \tilde{\chi}(\Delta^*) + \sum_{i=0}^{d} h_i(\Delta^*)(t+1)^i. \tag{2}$$

- *If Δ^* is Cohen–Macaulay, then Eq.* (2) *collapses to*

$$\sum_{i\geq 1} \beta_i(k[\Delta])\, t^{i-1} = \sum_{i=0}^{d} h_i(\Delta^*)(t+1)^i.$$

- *If Δ^* is a homology manifold over k, then*

$$\sum_{i\geq 1} \beta_i(k[\Delta])\, t^{i-1} = t\,\mathrm{Poin}(\Delta^*, t) + \sum_{i=0}^{d-1} f_{d-i-1}(\Delta^*) t^i.$$

- *If Δ^* is a homology sphere* (*Gorenstein**) *over k, then*

$$\sum_{i\geq 1} \beta_i(k[\Delta])\, t^{i-1} = \sum_{i=0}^{d} f_{d-i-1}(\Delta^*) t^i.$$

Proof. To prove Eq. (2), assume Δ^* is Buchsbaum, and use Proposition 1 to conclude that

$$\begin{aligned}\sum_{i\geq 1} \beta_i(k[\Delta])\, t^{i-1} &= \sum_{F\in\Delta^*}\sum_j \dim_k \tilde{H}_j(\mathrm{link}_{\Delta^*} F) t^{j+1}\\ &= \sum_j \dim_k \tilde{H}_j(\Delta^*) t^{j+1} + \sum_{\emptyset\neq F\in\Delta^*}\sum_j \dim_k \tilde{H}_j(\mathrm{link}_{\Delta^*} F) t^{j+1}\\ &= t\,\mathrm{Poin}(\Delta^*, t) + \sum_{\emptyset\neq F\in\Delta^*} (-1)^{d-\dim(F)} \tilde{\chi}(\mathrm{link}_{\Delta^*} F) t^{d-|F|}.\end{aligned} \tag{3}$$

Combining equations from [14, Sections II.7 and II.2] gives the equation

$$\sum_{F\in\Delta^*} (-1)^{d-\dim(F)} \tilde{\chi}(\mathrm{link}_{\Delta^*} F) \left(\frac{t}{1-t}\right)^{|F|} = \frac{\sum_{i=0}^{d} h_i(\Delta^*) t^{d-i}}{(1-t)^d}.$$

If we replace t by $1/(1+t)$ and then multiply by t^d, we obtain

$$\sum_{F\in\Delta^*} (-1)^{d-\dim(F)} \tilde{\chi}(\mathrm{link}_{\Delta^*} F)\, t^{d-|F|} = \sum_{i=0}^{d} h_i(\Delta^*)(1+t)^i$$

which combined with the last equation in (3) yields Eq. (2).

The formula in the case Δ^* is Cohen–Macaulay follows from Eq. (2) upon observing that

$$(-t)^{d-1}\tilde{\chi}(\Delta^*) = t\,\mathrm{Poin}(\Delta^*, t)$$

since Δ^* has only top dimensional reduced homology.

The formula in the case of Δ^* is a homology manifold over k, follows directly from Eq. (3) and the definition of the f-vector, using the fact that

$$(-1)^{d-\dim(F)}\tilde{\chi}(\mathrm{link}_{\Delta^*}F) = 1$$

for all non-empty faces F. The case where Δ^* is a homology sphere over k is then a trivial specialization of this. □

Remark. Note that the definition of Δ^* gives an obvious relation between the f-vectors of Δ and Δ^*, namely

$$f_i(\Delta^*) = \binom{n}{i+1} - f_{n-i-2}(\Delta).$$

Similarly, Lemma 2 gives a simple relation between the topological Poincaré polynomials of Δ and Δ^*. Therefore one has a choice in the previous theorem to express the formulas in terms of the f-vector and Poincaré polynomial of Δ^*, or in terms of Δ itself.

The next result provides a large class of examples where the Betti numbers of $k[\Delta]$ do not depend upon the field (see [15–19] other such results). It is pointed out in [16, Section 3] that this is equivalent to the existence of a minimal free resolution of $\mathbb{Z}[\Delta]$ over $\mathbb{Z}[x_1, \ldots, x_n]$.

The field independence comes from the condition of *shellability* [3, 4]. Say that a simplicial complex Δ is *shellable* if one can order its maximal faces $F_1, F_2, \ldots, F_m$ in such a way that for each $i \geq 2$ the intersection

$$F_i \cap \left(\bigcup_{j<i} \overline{F_j} \right) \tag{4}$$

between F_i and the subcomplex generated by the previous maximal faces is a subcomplex of codimension 1 inside F_i. When Δ is shellable and pure of dimension $d-1$, the h-vector has the following interpretation: h_r is the number of maximal faces F_i for which the intersection in (4) consists of exactly $d-r$ of the $(d-2)$-faces of F_i.

Corollary 5. *If Δ^* is shellable then the Betti numbers β_i of $k[\Delta]$ are independent of the field k. If furthermore Δ^* is pure and shellable, then regardless of the field k we have that the resolution of I_Δ is linear and*

$$\sum_{i \geq 1} \beta_i(k[\Delta])\, t^{i-1} = \sum_{i \geq 0} h_i(\Delta^*)(t+1)^i.$$

Proof. When Δ^* is shellable, its homology is independent of the field [3, Corollary 4.2], and all of its links $\text{link}_{\Delta^*}F$ inherit the property of shellability [4, Proposition 10.14] so their homology is also independent of the field. The first assertion of the theorem then follows from Proposition 1.

Since Δ^* being pure and shellable implies it is Cohen–Macaulay over any field (see e.g. [3, 4, Corollary 4.1, Proposition 10.14]), the rest of the assertions follow from Theorems 3 and 4. □

Directly translating the definition of pure shellability of Δ^* produces the following condition on the generators of the monomial ideal $I = I_\Delta$: one can linearly order the monomial generators $m_1, m_2, \ldots, m_r$ of I in such a way that for each $i < k$ there exists a $j < k$ satisfying

- m_j divides the least common multiple $\text{lcm}(m_i, m_k)$,
- m_j, m_k differ in at exactly 2 variables, i.e. $m_j = (x_p/x_q)m_k$ for some p, q.

It follows immediately from the preceding corollary that any square-free monomial ideal satisfying this condition will have a linear resolution regardless of the field k. On the other hand, this same definition also makes sense for monomial ideals I which are not necessarily square-free. Say that such a monomial ideal (not necessarily square-free) is *dually shellable.*

Theorem 6. *Let I be a dually shellable monomial ideal in $k[x_1, \ldots, x_n]$. Then I has linear resolution regardless of the field k.*

Proof. Assume I is dually shellable, with linear order $m_1, m_2, \ldots, m_r$ on its monomial generators as in the definition. If all the monomials m_i are square-free, then we are done by the previous corollary. Otherwise there is some variable, say x, for which the maximum x-degree appearing among all the m_i's is $d > 1$. In this case we introduce a new ideal I' which is "closer" to being square-free, by defining m_i' to be

$$m_i = \begin{cases} \frac{x_0}{x} m_i & \text{if } x^d \text{ divides } m_i, \\ m_i & \text{otherwise,} \end{cases}$$

and letting I' be the ideal generated by $m_1', m_2', \ldots, m_r'$. Since A/I is the quotient of $A[x_0]/I'$ by the linear non-zero divisor $x_0 - x$, it follows from [9, Lemma 1] that I will have linear resolution if and only if I' does.

Therefore it suffices (by induction on d) to show that I' inherits dual shellability from I, with respect to the ordering $m_1', m_2', \ldots, m_r'$ of its generators. So let $i < k$, and let $j < k$ be the index satisfing m_j divides $\text{lcm}(m_i, m_k)$ with m_j, m_k differing in exactly 2 variables. We claim both that m_j' will divide $\text{lcm}(m_i', m_k')$ and that m_j', m_k' differ in exactly 2 variables. To see the first claim, note that the power of any variable x_t other than x or x_0 is the same in m_i', m_j', m_k' as it was in m_i, m_j, m_k, so we only need to check that the x-degree and x_0-degree of m_j' are no bigger than their minimum values for m_i', m_k'. This is true for the x_0-degrees because m_j' has a factor of x_0 exactly when x^d divides m_j, which implies that x^d divides at least one of the two monomials m_i, m_j, and so one of m'_i, m'_k will be divisible by x_0. Similar reasoning shows that it is true for the x-degrees. To show the second claim about m'_j, m'_k differing in exactly 2 variables, consider the four cases

- $m'_j = m_j, m'_k = m_k$. Trivial.
- $m'_j \neq m_j, m'_k \neq m_k$. Here it must be that m_j, m_k were both divisible by x^d, so we must have $m_j = (x_p/x_q)m_k$ for two variables $x_p, x_q \neq x$. But then $m'_j = (x_p/x_q)m'_k$.
- $m'_j = m_j, m'_k \neq m_k$. In this case it must be that m_k is divisible by x^d while m_j is not, so $m_j = (x_p/x)m_k$ for some variable x_p. But then

$$m'_j = m_j = \frac{x_p}{x} m_k = \frac{x_p}{x_0} \frac{x_0}{x} m_k = \frac{x_p}{x_0} m'_k$$

  as desired.
- $m'_j \neq m_j, m'_k = m_k$. Symmetric to the previous case.

This completes the proof of the claim, and hence the theorem follows. □

The remainder of this section discusses two situations where the conclusion of Corollary 5 applies because the dual complex Δ^* is not only shellable, but satisfies the stronger condition of vertex-decomposability. A simplicial complex Δ is said to be *vertex-decomposable* if it satisfies the following recursive definition: either $\Delta = \{\emptyset\}$ or there exists some vertex v of Δ for which both subcomplexes $\mathrm{del}_\Delta v$ and $\mathrm{link}_\Delta v$ are vertex-decomposable. This concept was introduced by Provan and Billera, who showed that vertex-decomposable complexes are shellable (see [2, Lemma 4.14]).

Say that I_Δ is *matroidal* if its set of minimal generators $\{x^{G_\alpha}\}$ satisfy the *MacLane–Steinitz exchange axiom*: For any α, β, i, if x_i divides x^{G_α} then there exists a j such that x_j divides x^{G_β} and $(x_j/x_i)G_\alpha$ is also a minimal generator. Equivalently, I_Δ is matroidal if the set of exponents $\mathscr{B} := \{G_\alpha\}$ of its minimal generators form the set of bases for a *matroid* $\mathscr{M}$ on the ground set $[n]$ (see [2]).

Proposition 7. *If I_Δ is matroidal, then Δ^* is vertex-decomposable, and hence shellable. Therefore I_Δ has linear resolution over any field k.*

Proof. In this situation, Δ^* will be the *dual complex* for the matroid $\mathscr{M}$, i.e. the complex of *independent sets* in the *dual matroid* $\mathscr{M}^*$. As a consequence it is vertex-decomposable (see [2, Section 5]). □

Note that [9, Example 4] is an instance of a matroidal ideal I_Δ, in which $\mathscr{M}$ is the *uniform* matroid of rank $k+1$ on ground set $[n]$. We also remark that the hypotheses in the previous proposition may be weakened somewhat to assume only that the generators of I_Δ correspond to the set of *bases* in a *greedoid* $\mathscr{G}$ on the ground set $[n]$ (see [2] for definitions). In this situation Δ^* again forms the *dual complex* of $\mathscr{G}$, which is known to be vertex-decomposable [2, Theorem 5.1]. Unfortunately we are not aware of any simple characterization for when a family of subsets $\mathscr{B}$ form the bases of some greedoid (and there may be many such greedoids), so it is not easy to check these weaker hypotheses.

Lastly we re-interpret a result of Fröberg [10] which characterizes the ideals I_Δ generated by *quadratic* square-free monomials having linear resolutions. Note that I_Δ is generated by quadratic square-free monomials exactly when Δ is the *flag complex* $\Delta(G)$ associated to some graph G on the vertex set $[n]$, i.e. the simplices of $\Delta(G)$ are exactly the subsets F of $[n]$ for which every pair in F is an edge of G. Fröberg's characterization involves chordal graphs, which we now discuss. Say that a graph G is *chordal* if for every cycle $v_1, v_2, \dots, v_{m_1}, v_m, v_1$ in G with $m \geq 4$, there exists some *chord* i.e. an edge in G between two vertices which are not adjacent in the cycle. It is well-known (see [11]) that chordal graphs may also be characterized by the existence of an *elimination ordering* $v_1, v_2, \dots, v_n$ on the vertices, meaning that for all i there are edges between all pairs of v_i's neighbors in $G - \{v_1, v_2, \dots, v_{i-1}\}$ (v_i is said to be a *simplicial vertex* of $G - v_1, v_2, \dots, v_{i-1}\}$ in this situation).

Theorem (Fröberg [10, Theorem 1]). *A Stanley–Reisner ideal I_Δ generated by quadratics has linear resolution if and only if $\Delta = \Delta(G)$ for some chordal graph G.*

In light of Theorem 3 and Corollary 5, the following proposition gives a "dual" explanation of this result:

Proposition 8. *The following are equivalent for a graph G:*
(i) *$\Delta(G)^*$ is vertex-decomposable.*
(ii) *$\Delta(G)^*$ is Cohen–Macaulay over any field k.*
(iii) *$\Delta(G)^*$ is Cohen–Macaulay over some field k.*
(iv) *G is chordal.*

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are all trivial.

(iii)$\Rightarrow$(iv): If G is not chordal then there exist some subset V of the vertices which supports a cycle in G having no chord. Borrowing from the argument of [10], note that $\Delta(G)_V$ is homeomorphic to a circle. Lemma 2 then implies that

$$\tilde{H}_{|V|-4}(\mathrm{link}_{\Delta(G)^*} F; k) = \tilde{H}_1(\Delta(G)_V; k) \neq 0$$

so that $\Delta(G)^*$ is not Cohen–Macaulay over any field k.

(iv) $\Rightarrow$ (i): If G is chordal, let $v_1, v_2, \dots, v_n$ be an elimination ordering for its vertices. A vertex decomposition for $\Delta(G)^*$ starting with v_1 will then follow from the following lemma, whose proof is straightforward.

Lemma 9. 1. *For any vertex v in a graph G, we have $\mathrm{link}_{\Delta(G)^*} v = (\Delta(G - v))^*$ as complexes on the vertex set $[n] - \{v\}$.*

2. *For any simplicial vertex v in a graph G, the deletion $\mathrm{del}_{\Delta(G)^*} v$ is the simplicial complex on vertex set $[n] - \{v\}$ having maximal faces $\{[n] - \{v, v'\}\}$ as v' runs over all non-neighbors of v in G.*

We must show that the lemma implies both subcomplexes $\text{link}_{\Delta(G)^*} v_1$ and $\text{del}_{\Delta(G)^*} v_1$ are vertex decomposable. By induction and part 1 of the lemma we have that $\text{link}_{\Delta(G)^*} v_1 = (\Delta(G - v_1))^*$ is vertex-decomposable, since $G - v_1$ is chordal whenever G is chordal. By part 2 of the lemma, since v_1 is simplicial, $\text{del}_{\Delta(G)^*} v_1$ is the complex generated by a collection of codimension 1 faces of a simplex, and all such complexes are easily seen to be vertex-decomposable. □

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