REU 2019 Day 1 6/3/2019

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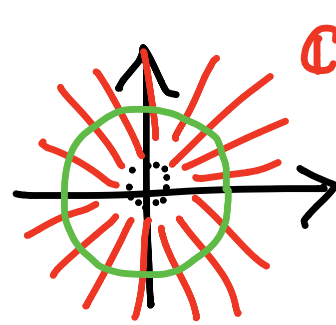
Virtual syzygies

$k = \mathbb{C}$

$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

algebraic torus



$\mathbb{C}^\times \subset \mathbb{C}$

Torus action:

$\bar{x} \in \mathbb{C}^{n+1}$

$$\mathbb{C}^\times \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$

$$(t, \bar{x}) \longmapsto (tx_0, tx_1, \ldots, tx_n) =: t \cdot \bar{x}$$

$$[\bar{x}] := \{\bar{y} \in \mathbb{C}^{n+1} : \exists\, t \in \mathbb{C}^\times \text{ s.t. } \bar{y} = t \cdot \bar{x}\}$$

= line through $\bar{0}$ containing $\bar{x}$, without $\bar{0}$

Projective space:

$$\mathbb{P}^n := \{[\bar{x}] : \bar{x} \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

$$\uparrow$$

$$\mathbb{C}^{n+1} \setminus \{\bar{0}\}$$

with $[\bar{x}] = [\bar{y}]$

$$\Updownarrow$$

$\exists t \in \mathbb{C}^\times$ s.t. $t \cdot \bar{x} = \bar{y}$

Why do this?

It helps to compactify $\mathbb{C}^n$.

For $\alpha \in \mathbb{N}^{n+1}$, $\mathbb{N} := \{0, 1, 2, \dots\}$

$$\bar{x}^\alpha := \prod_{i=0}^{n} x_i^{\alpha_i}, \quad |\alpha| = \sum_{i=0}^{n} \alpha_i$$

$$f(\bar{x}) = \sum_{\alpha} c_\alpha \bar{x}^\alpha \in \mathbb{C}[x_0, x_1, \ldots, x_n] =: S$$

(finite sum) $c_\alpha \in \mathbb{C}$

$t \in \mathbb{C}^\times$ acts on polynomials $f(\bar{x})$ via

$$f(t \cdot \bar{x}) = \sum_{\alpha} c_\alpha (tx_0, tx_1, \ldots, tx_n)^\alpha$$

$$= \sum_{\alpha} c_\alpha t^{|\alpha|} \bar{x}^\alpha$$

Def: f is **homogeneous** if $\forall c_\alpha \neq 0$, $|\alpha|$ is the same.

Then $f(\bar{y}) = 0$ $\forall y \in [\bar{x}]$ when $f(\bar{x}) = 0$ when f is homogeneous.

For $X \subset \mathbb{P}^n$

$$\mathcal{I}(X) := \langle f(\bar{x}) \in S : f(\bar{c}) = 0 \ \forall \bar{c} \in X \rangle$$

EXAMPLE

① $\mathbb{P}^2 \supset X = \{ [\overset{x_1}{1} : \overset{x_2}{0} : \overset{x_3}{0}], [0:1:0], [0:0:1] \}$

$[1:0:0] = [(1,0,0)]$

ideal generated by

has

$$\mathcal{I}(X) = \langle x_1, x_2 \rangle \cap \langle x_0, x_2 \rangle \cap \langle x_0, x_1 \rangle$$

$$= \langle x_0x_1, x_0x_2, x_1x_2 \rangle$$

② $\mathbb{P}^2 \supset Y = \{ [1:0:0], [1:1:0], [2:0:1] \}$

has $\mathcal{I}(X) = \langle x_2, x_1 \rangle \cap \langle x_0 - x_1, x_2 \rangle \cap \langle x_0 - 2x_2, x_1 \rangle$

$$= \langle x_1x_2, x_0x_2 - 2x_2^2, x_0x_1 - x_1^2 \rangle$$

DEF: $I \subseteq S$ is an ideal if

1) $I \neq \emptyset$ ($\Leftrightarrow 0 \in I$)
2) $a, b \in I \Rightarrow a+b \in I$
3) $a \in I, f \in S \Rightarrow fa \in I$

Claim: $\mathcal{I}(X) \subseteq S$ is an ideal!

The ideal generated by $f_1, \dots, f_r \in S$ is

(*) $$\langle f_1, \dots, f_r \rangle := \left\{ \sum_{i=1}^{r} h_i f_i : h_i \in S \right\}$$

Hilbert's Basis Theorem: In S, every ideal is finitely generated.

DEF. An ideal $I \subset S$ is homogeneous if it has a generating set containing only homogeneous polynomials.

CLAIM: For $X \subset \mathbb{P}^n$, $\mathbb{I}(X)$ is homogeneous.

DEF. Given a homog. ideal $I = \langle f_1, f_2, \ldots, f_r \rangle$ in S, let

$$\mathcal{V}(I) := \{ \bar{c} \in \mathbb{P}^n : f(\bar{c}) = 0 \ \forall f \in I \}$$
$$= \{ \bar{c} \in \mathbb{P}^n : f_1(\bar{c}) = \ldots = f_r(\bar{c}) = 0 \}$$

is a projective algebraic variety

EXAMPLE

$$\mathbb{P}^2 \supset X = \{[1:0:0],[0:1:0],[0:0:1]\}$$
$$= V(I)$$
where $I = \langle x_0x_1, x_0x_2, x_1x_2 \rangle$

Algebraic geometry:

Geometric properties of $V(I)$ $\overset{?}{\longleftrightarrow}$ Algebraic properties of S/I

e.g.

irreducible $\longleftrightarrow$ domain ($\Leftrightarrow$ I a prime ideal)

REU Exercise 1:

a) Prove (*) is an ideal.

b) Given $I \subset S$ a homog. ideal, set $\sqrt{I} := \{f \in S : \exists m \in \mathbb{Z}_{>0} \text{ s.t. } f^m \in I\}$ the radical of I.

- Compute $\sqrt{\langle x_0^2, x_1^3 \rangle}$
- Show $\sqrt{I}$ is a homog. ideal

c) Given $X \subseteq Y \subseteq \mathbb{P}^n$, show $I(X) \supseteq I(Y)$

Given $I \subseteq J \subseteq S$ homog. ideals, show $V(I) \supseteq V(J)$

d) · Find I s.t. $I \subsetneq \mathcal{I}(\mathcal{V}(I))$

· Given $I, J \subseteq S$ homog. ideals, show $\mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$

Not exercises, but related:

FACTS:

- $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$
- X projective variety $\Longrightarrow$ $\mathcal{V}(\mathcal{I}(X)) = X$

DEF: A graded minimal free resolution of S/I (I a homog. ideal) is

$$F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \leftarrow \cdots$$

such that

(i) $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\oplus \beta_{i,j}}$

where $S(-j)$ means S with 1 viewed as living in homogeneous degree j

(ii) $\varphi_i \circ \varphi_{i+1} = 0$ ($\Leftrightarrow F_*$ is a complex),
and in fact, more strongly,
$\ker \varphi_i = \operatorname{im} \varphi_{i+1}$

(iii) $\operatorname{coker} \varphi_1 := F_0 / \operatorname{im} \varphi_1 \cong S/I$

(iv) $\operatorname{im}\varphi_i \subseteq \langle x_0, \dots, x_n \rangle \cdot F_{i-1} \ \forall i$

(v) all φ_i have degree 0
(bearing in mind the degree shifts $S(-j)$!)

EXAMPLE:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in S(-2)^3$$

$$S = F_0 \xleftarrow[]{\varphi_1 = [x_0x_1 \ \ x_0x_2 \ \ x_1x_2]} S(-2)^3 \xleftarrow{\begin{bmatrix} x_2 & 0 \\ -x_1 & x_1 \\ 0 & -x_2 \end{bmatrix}} S(-3)^2 \leftarrow 0$$

<u>Hilbert's syzygy Theorem</u>: $X \subseteq \mathbb{P}^n \Rightarrow$
$S/\mathcal{I}(X)$ has a minimal graded free res'n
$$F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_n \leftarrow 0$$
of length $n = \dim \mathbb{P}^n$.

Computing minimal free resl'ns is not so easy in general, but there are algorithms, involving Gröbner bases, which allow computers to do it!

Virtual resolutions

Fix $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_N \in \mathbb{Z}^r$

$$(\mathbb{C}^\times)^r \times \mathbb{C}^N \longrightarrow \mathbb{C}^N$$

$$(\bar{t}\ ,\ \bar{x}) \longmapsto (\bar{t}^{\bar{\alpha}_1} x_1, \bar{t}^{\bar{\alpha}_2} x_2, \ldots, \bar{t}^{\bar{\alpha}_N} x_N)$$

EXAMPLE $r=2,\ N=5$

$$(\mathbb{C}^\times)^2 \times \mathbb{C}^5 \longrightarrow \mathbb{C}^5$$

$$(\bar{t},\ \bar{x}) \longmapsto (t_1x_1, t_1x_2, t_2x_3, t_2x_4, t_2x_5)$$

so $\bar{\alpha}_1 = \begin{pmatrix}1\\0\end{pmatrix} = \bar{\alpha}_2$

$\bar{\alpha}_3 = \bar{\alpha}_4 = \bar{\alpha}_5 = \begin{pmatrix}0\\1\end{pmatrix}$

$S = \mathbb{C}[x_1, \dots, x_N]$

$\cup$

$B =$ irrelevant ideal, determined by the fan of X

not defined yet!

In above example,

$S = \mathbb{C}[x_1, \dots, x_5]$

$\cup$

$B = \langle x_1, x_2 \rangle \cap \langle x_3, x_4, x_5 \rangle$

Then we define

$$X := \left(\mathbb{C}^N \setminus \mathcal{V}(B)\right) /\!\!/ (\mathbb{C}^\times)^r$$

In above example,

$$X = \mathbb{C}^5 \setminus \sqrt{\left(\langle x_1, x_2 \rangle \cap \langle x_3, x_4, x_5 \rangle\right)} /\!\!/ (\mathbb{C}^\times)^2$$

$$\cong \mathbb{P}^1 \times \mathbb{P}^2$$

... switched to projection of Macaulay2 software demo, showing a resolution of a curve in $\mathbb{P}^1 \times \mathbb{P}^2$ that took 4 steps.

DEF: $I \subseteq S$ an ideal

$$\operatorname{ann}(S/I) := \{ f \in S : f \cdot \bar{s} = 0 \ \forall \bar{s} \in S/I \}$$

DEF: For $Y \subseteq X$,

a virtual resolution of $S/I(Y)$ is

$$F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \leftarrow \cdots \quad \text{with}$$

- $F_i = \bigoplus_{\underline{j} \in \mathbb{Z}^r} S(-\underline{j})^{\beta_{i,\underline{j}}}$
- $V\left(\operatorname{ann}\left(\ker \partial_i / \operatorname{im} \partial_{i+1}\right)\right) = \emptyset \ \forall i \geq 1$
- $V(\operatorname{im} \partial_1) = Y$

Virtual Hilbert Syzygy Theorem:
[B.-Erman-Smith 2017]

$X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \dots \times \mathbb{P}^{n_r}$

or

$X =$ smooth toric of dimension 2

Then $\forall Y \subseteq X$,

$S/I(Y)$ has a virtual resolution

of length $\leq \dim X$

Macaulay2 showed a virtual resolution of same curve, now of length 2

Methods to make virtual resolutions (vres's)

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i) virtualOfPair (see [BES])

ii) res($I \cap B^{\bar{a}}$)

iii) take subsets of gens of I

Macaulay2 demo ...

REU PROBLEM 1.

X a smooth projective toric variety
$Z \subset X$ a (finite) set of points of X
Show that $S/I(Z)$ has a virtual resolution of length $\leq \dim X$

Macaulay2 has databases of such X,
e.g. smoothFanoToricVariety(n, m)

n: dim, m: #

kleinschmidt(n, L)

REU Exercise 2

(a) What is $I(([a_0:a_1],[b_0:b_1]))$?

$$\mathbb{P}^1 \times \mathbb{P}^1$$

(b) Find a vres of length 2 for $S/I(Y)$

where

$$Y = \{([0:1],[0:1]), ([1:0],[1:1]), ([1:2],[1:0])\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

Use $I(Y) \cap B^{\bar{a}}$ method.

Which $\bar{a}$'s work for length 2?

(c) Let X be a smoothFanoToricVariety(3,1).
Use the command toricPoints;
with

	(max X)_i ,	{a,b,c}
intersect the ideals of these three:	3	0, 0, 0
	0	0, 0, 0
	2	1, 1, 1

Use $(-) \cap B^{\bar{a}}$ method to make vres of Y = {3 points above}