REU 2019 Day 2 6/4/2019

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Lattice Models and Combinatorial Representation Theory

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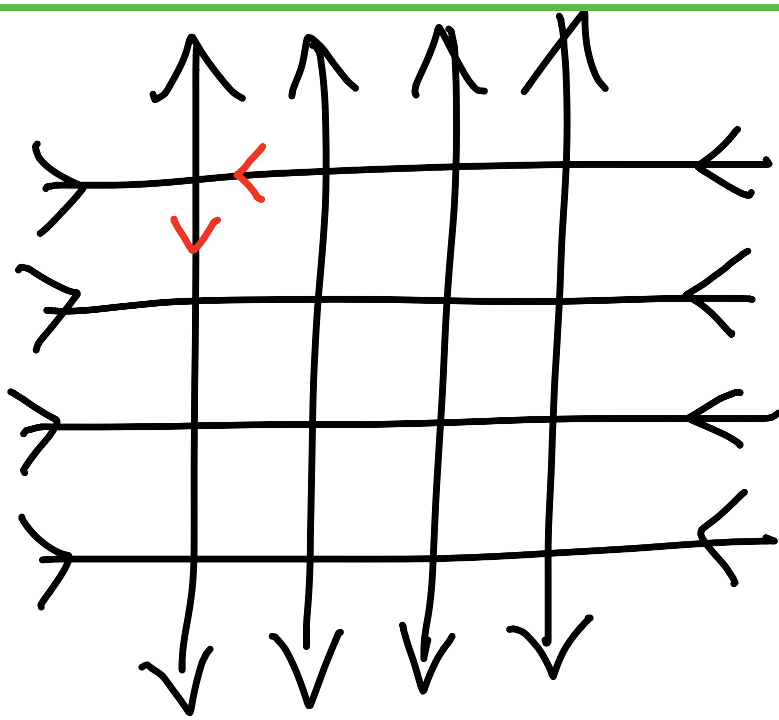
Katy Weber

Lattice models are inspired by statistical mechanics

Slogan: use local behavior to infer global properties

Example: Square lattice

Assign arrows to every edge in lattice, having pre-assigned boundary edges



Game: How many ways can I fill this lattice with distinct edge assignments with every vertex having 2 edges in, 2 edges out.

This means every vertex has adjacent edges in one of $\binom{4}{2}=6$ configurations ("6-vertex model")

How many fillings?

A hard question; not hard to see it's equivalent to the alternating sign matrix conjecture proven by Zeilberger/Kuperberg

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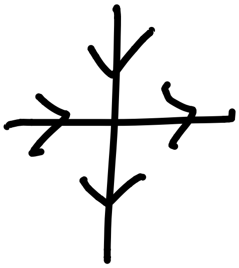
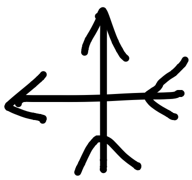
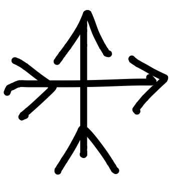
Let's describe a method for attaching a weight to each filling...

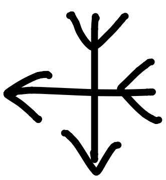
Example:

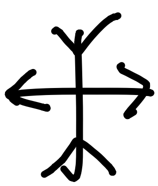
weight

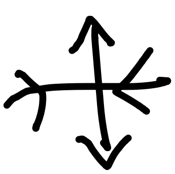
 NW : 1

 SE : z_i (if in row i)

 SW : 0 ← so it's really a five-vertex model!

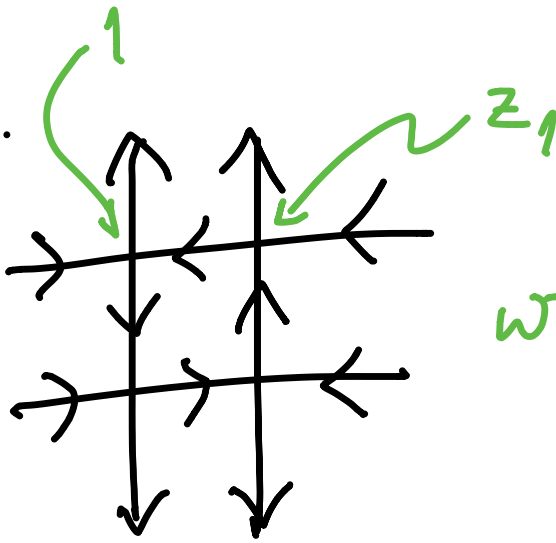
 NE : z_i

 NS : z_i

 EW : 1

Then the weight of a ~~filling~~ admissible state $S = \prod_{v \in S} wt(v)$

e.g.

$S =$ 

$wt(S) = 1 \cdot z_1 \cdot 1 \cdot 1 = z_1$

Finally define the lattice model partition function (LMPF)

$$P(\underline{z}) := \sum_{\substack{\text{admissible} \\ \text{states } S}} wt(S)$$

$\underline{z} = (z_1, z_2, \dots, z_r)$

New Question:

How to find closed form expressions for $P(\underline{z})$ given various weight schemes?

$\underline{\underline{A}}$: In general, impossible.

REU Exercise 3

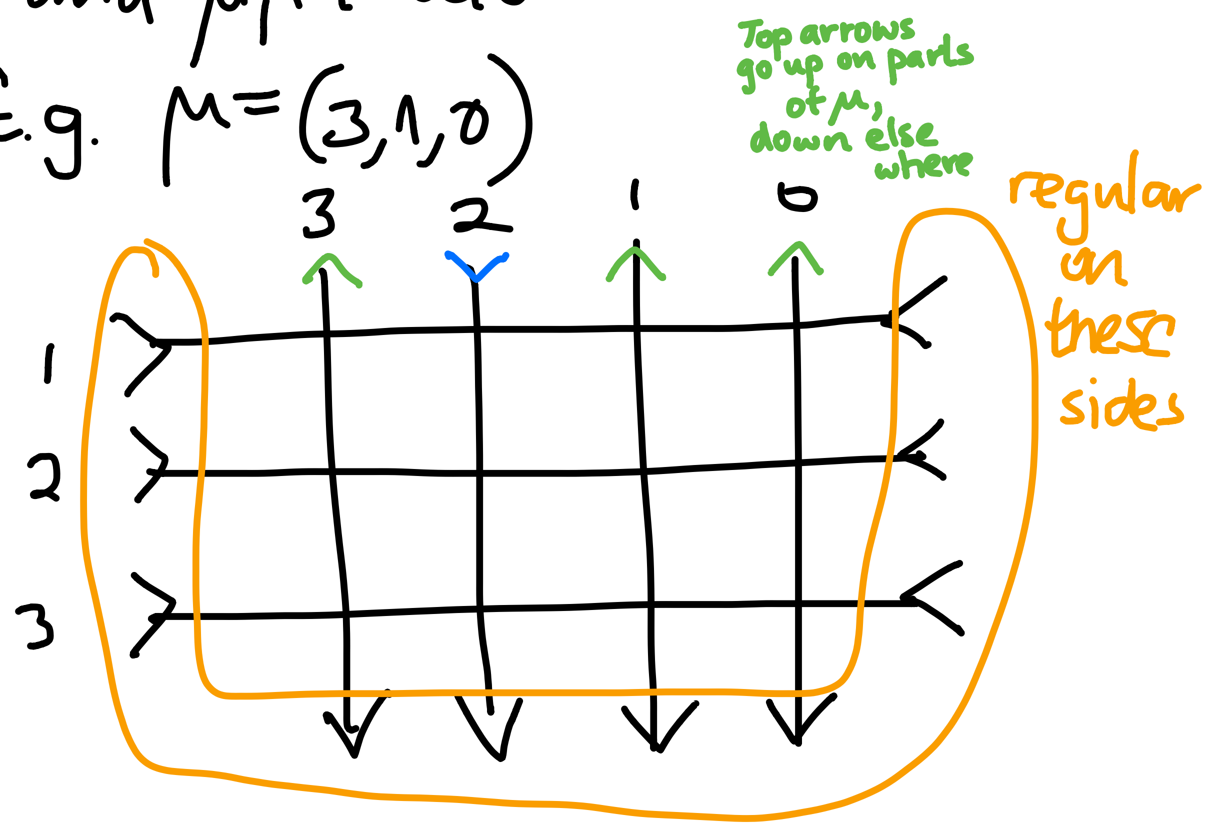
Compute $P(z_1, \dots, z_n)$ for $n \times n$ square lattice with the above weights.

(i.e. for all n; it's actually do-able!)

Let's generalize the lattice, depending on $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{Z}^r$ with $\mu_i > \mu_{i+1}$ (and we often take $\mu_r = 0$ for simplicity).

The lattice will have r rows and $\mu_1 + 1$ columns.

E.g. $\mu = (3, 1, 0)$



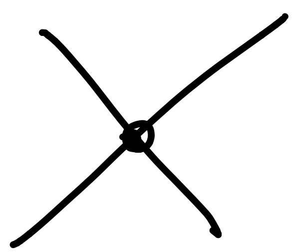
Q: What is $P_\mu(\underline{z})$?

A: The most famous symmetric functions of all!

Q: What tools do we use to evaluate $P(\underline{z})$?

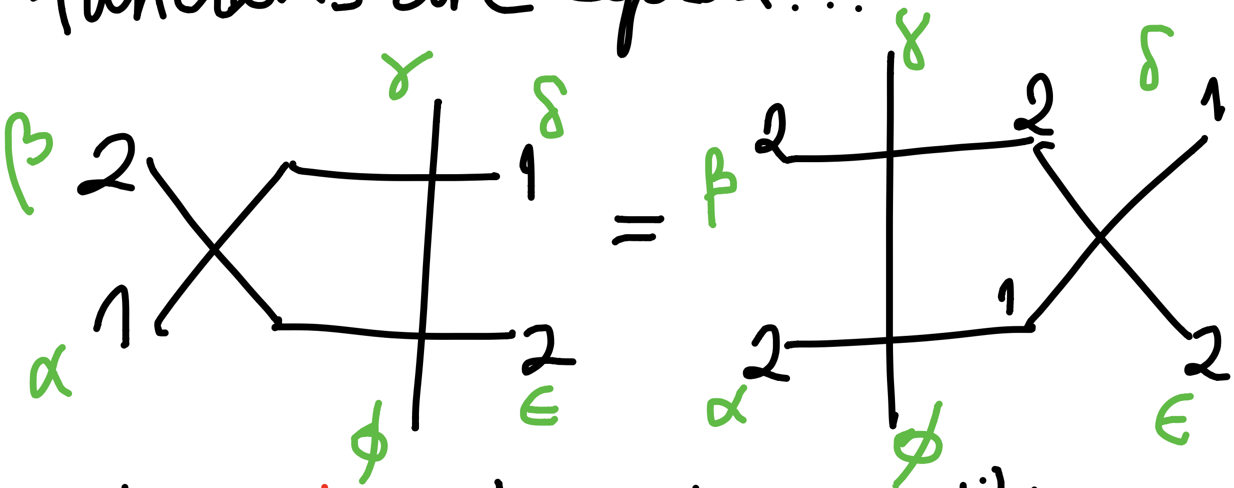
A: The Yang-Baxter Equation

We want a new set of 6 vertices, i.e. weights for rotated vertices



so that . . .

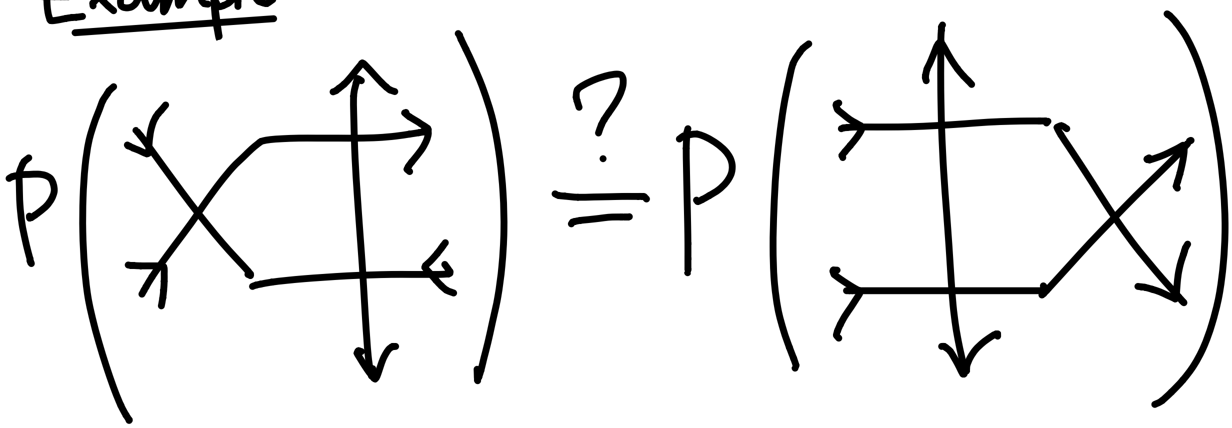
. . . the following mini partition functions are equal . . .

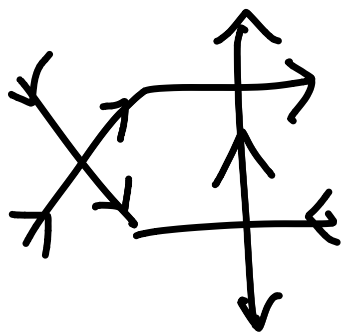


with arbitrary boundary conditions

$\alpha, \beta, \gamma, \delta, \epsilon, \phi \in \{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

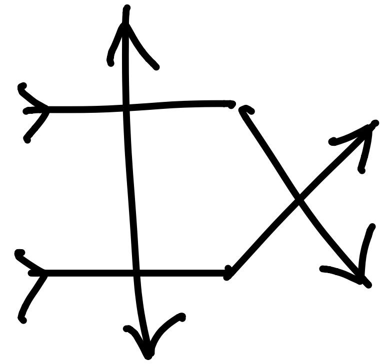
<u>Example:</u>





unique admissible configuration on left

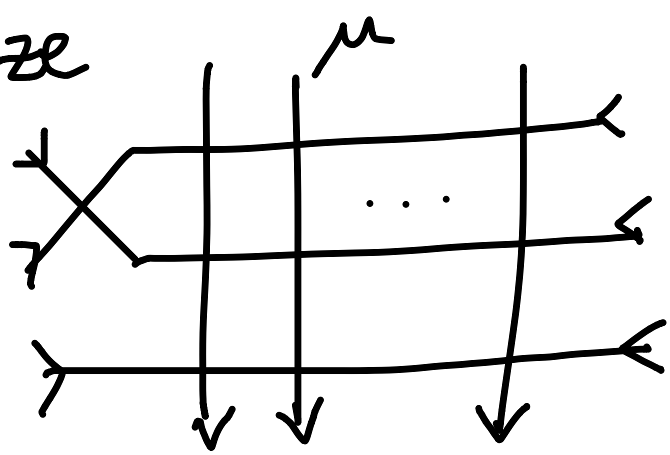
vs.

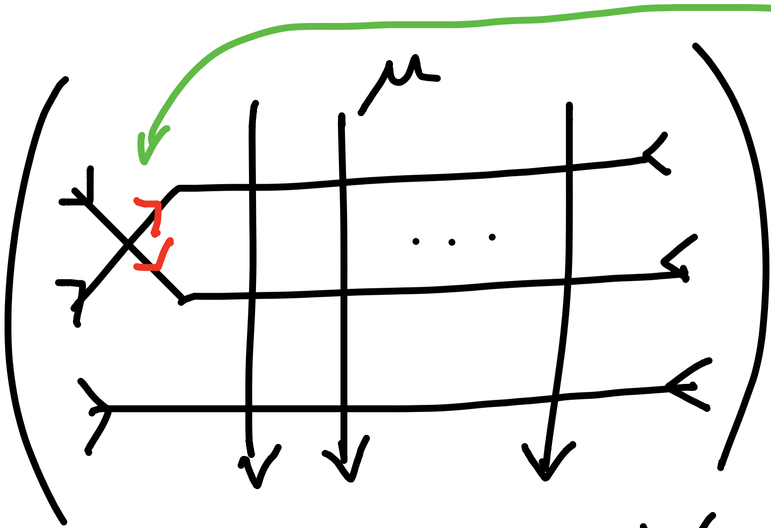


two admissible configurations on right (check this!)

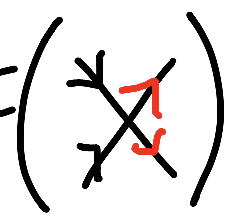
Amazing idea to prove $P_\mu(z_1,\ldots,z_r)$ symmetric in the variables $z_1, z_2, \ldots, z_r$

Analyze

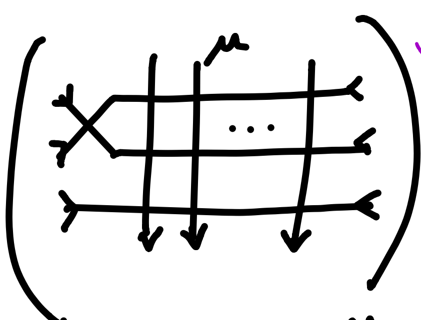


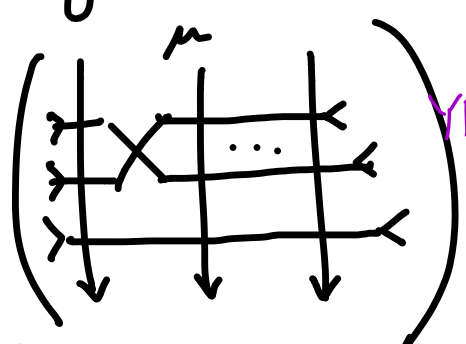
P 

forced by 2 in 2 out rule

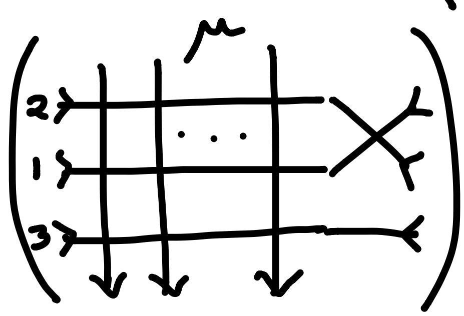
$= \mathrm{wt}$  $P_\mu(\underline{z})$

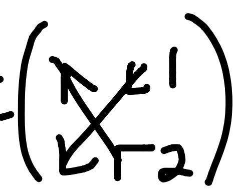
On the other hand, by YBE

P 

$\overset{\text{YBE}}{=} P$ 

$\overset{\text{YBE}}{=} \cdots \overset{\text{YBE}}{=}$

$\overset{\text{YBE}}{=} P$ 

$= \mathrm{wt}$ 

$\cdot P_\mu(z_2, z_1, z_3)$

REU Exercise 4

(1) Find a solution to the Yang-Baxter equation with our weights as above.

(2) Read about how to encode it as a matrix in B.-Bump-Friedberg, and do the translation into matrix multiplication.

$\underline{\underline{Q}}$: When is a solution to the YBE possible?

Baxter and B-B-F give sufficient conditions on the weights to satisfy YBE

(2 degree two equations in weights)

Let $\underline{z} = (z_1, \ldots, z_r)$ in $P_\mu(\underline{z})$

and let $\rho := (r-1, r-2, \ldots, 2, 1, 0)$

Then $\lambda := \mu - \rho$ has $\lambda_i \geq \lambda_{i+1}$

i.e. λ is a (combinatorialist's) partition of $n = \lambda_1 + \lambda_2 + \ldots + \lambda_r$

Then, letting $\underline{\underline{z}}^{\rho} = z_1^{r-1} z_2^{r-2} \cdots z_{r-2}^{2} z_{r-1}$

one has

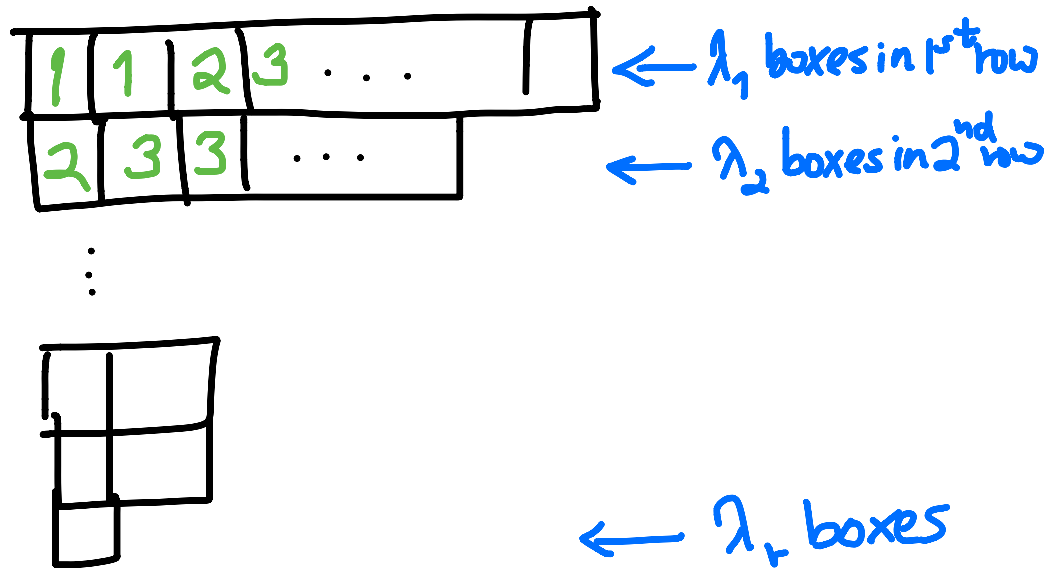
$$P_\mu(\underline{\underline{z}}) \overset{(*)}{=} \underline{\underline{z}}^{\rho} s_\lambda(\underline{\underline{z}})$$

where $s_\lambda(\underline{\underline{z}}) :=$ Schur polynomial for λ

$$s_\lambda(\underline{\underline{z}}) = \sum_{T \in SSYT(\lambda)} \underline{\underline{z}}^{wt(T)}$$

semistandard Young tableaux T of shape λ and entries $\leq r$

Young diagram for $\lambda = (\lambda_1, \dots, \lambda_r)$



SSYT = semistandard Young tableaux
is a filling T of the Young diagram
with alphabet $\{1, 2, \dots, r\}$ so that

- weakly increasing along rows
- strictly increasing down columns

$$\underline{z}^{wt(T)} := z_1^{\#1\text{'s in } T} \cdots z_r^{\#r\text{'s in } T}$$

REU Exercise 5:

ⓐ Compute $s_{(4,2,1)}(z_1,z_2,z_3)$

ⓑ Show $(*)$ by establishing a bijection between $SSYT(\lambda)$ and $\underset{\lambda+\rho}{\underset{\shortparallel}{\mu}}$-lattice states

ⓒ Use ⓑ to give a rule for how to expand

$$s_\lambda(\underline{z}) = \sum_{\nu} \underbrace{c_{\lambda,\nu}(z_1)} s_\nu(z_2,\dots,z_r)$$

What to sum over?

Come up with this!

REU Problem #2

Find conditions under which partition functions of μ-lattice models satisfy nice identities like the Schur polynomials.

branching rules
Cauchy identities
Littlewood-Richardson rules
(-insert your favorite rule here-)

Which of these identities (for Schur polynomials) can be expressed as lattice model partition functions computed in two ways?