Digraph associahedra

6/11

1. Associahedra, building and nested sets
2. Extended version, REU Problem 7 (a,b,c)
3. f, h, γ - vectors
4. REU Problem 7 (d)

My own TA: (Vic)
2:30 Vin.H. 2.

1. Associahedra

Recall (Day 4) that

Kirkman-Cayley #s

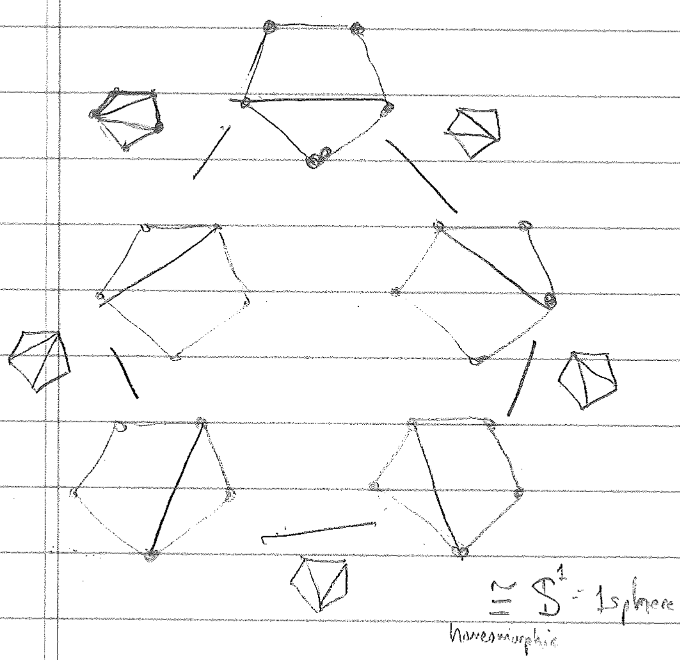
$$\#\left\{\begin{array}{c}\text{dissections of an } (n+2)\text{-gon} \\ \text{with } k \text{ diagonals}\end{array}\right\} = \frac{1}{n}\binom{n}{k+1}\binom{n+k+1}{k},$$

but they're also the f-vector $f = (f_{-1}, f_0, f_1, \ldots, f_{n-2})$ in an interesting simplicial complex $Assoc_n = \Delta$

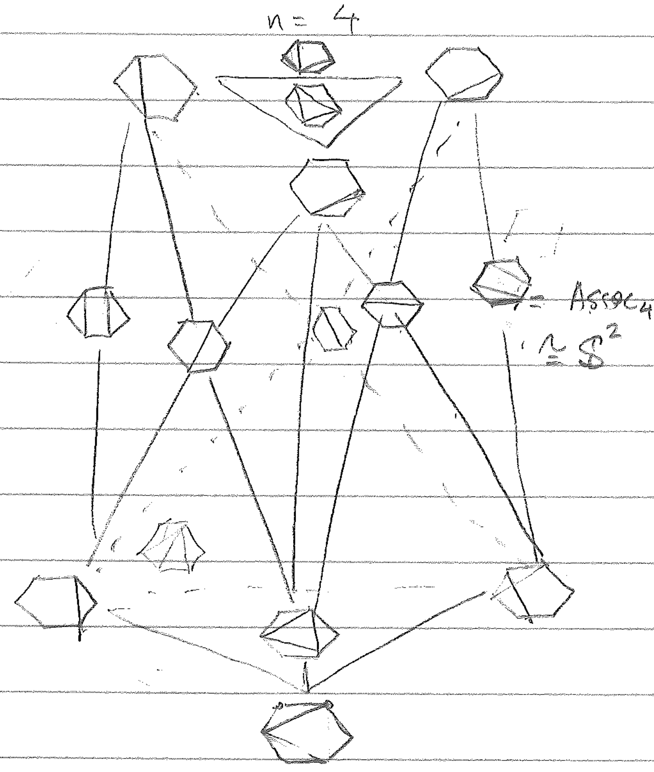
$Assoc_n$ has vertices $= \left\{\begin{array}{c}\text{diagonals of} \\ (n+2)\text{-gon}\end{array}\right\}$

simplices $= \left\{\begin{array}{c}\text{non crossing} \\ \text{subsets of} \\ \text{diagonals}\end{array}\right\}$

e.g. n=3.

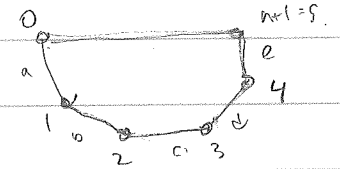


n = 4



Thm (Stasheff 1963) $Assoc_n \cong S^{n-2}$

(even polytopal spheres; Milnor, Lee 1989, Haiman 1989).



Associative parenthesis

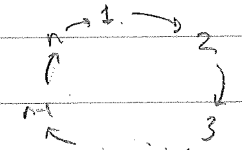
Generalize vastly:

def/ A (connected) building set $\mathcal{B}$ on $[n] = \{1, \dots n\}$ is a collection of subsets $\mathcal{B} \subseteq 2^{[n]}$ = all subsets of $[n]$, with $\mathcal{B}$ containing $[n]$ (i.e. $\mathcal{B}$ is connected) and all singletons $\{1\}, \{2\}, \dots \{n\}$, and $I, J \in \mathcal{B}$, and $I \cap J = \emptyset \Rightarrow I \cup J \in \mathcal{B}$.

e.g. $\mathcal{B}_{min} = \{[n], \{1\}, \dots \{n\}\}$

eg. Γ a strongly connected digraph on vertex set $[n]$.

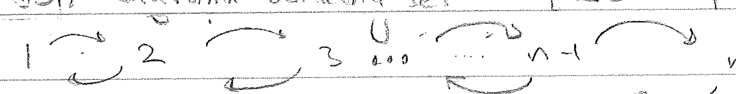
$\Gamma =$



$\rightsquigarrow \mathcal{B}_\Gamma = \mathcal{B}_{min}$ in this case

$\rightsquigarrow \mathcal{B}_\Gamma$ = digraph building set = $\{ I \subset [n] \mid \Gamma|_I$ is strongly connected $\}$

e.g. $\Gamma =$



$= 1 - 2 - 3 - \cdots - n-1 - n.$

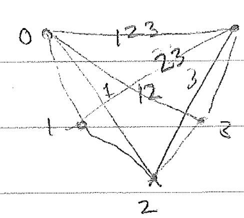
$\mathcal{B}_\Gamma = \{$ contiguous segments $I = \{a, a+1, \dots b-1, b\}\}$

$= \{$ intervals in $[n]$ $\}$

I = vertices separated from long edge by diagonal $\longleftrightarrow$ diagonal

bijection $\updownarrow$

$\{$diagonals in $(n+2)$-gon$\}$.



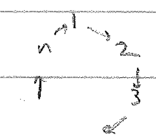
(labels on diagonals are contiguous segments in $\mathcal{B}_\Gamma$)

def/ A $\mathcal{B}$-nested collection $\sigma = \{I_1, I_2, \ldots, I_m\}$ is a collection of subsets I_j in $\mathcal{B} - \{[n]\}$ which are pairwise nested ($I_i \subseteq I_j$ or $I_j \subseteq I_i$) or disjoint ($I_i \cap I_j = \emptyset$) and whenever $I_{i_1}, \ldots I_{i_k}$ are pairwise disjoint, then $I_{i_1} \cup \ldots I_{i_k} \notin \mathcal{B}$.

The complex of nested sets $\mathcal{N}(\mathcal{B})$ $(=\Delta)$ is the simplicial complex vertices $= \mathcal{B} - \{[n]\} = \{y_I\}_{I \in \mathcal{B} - \{[n]\}}$

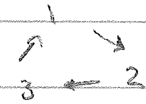
simplices $=$ nested collections σ

e.g. when $\mathcal{B} = \mathcal{B}_\Gamma$ for $\Gamma =$



, then $\mathcal{N}(\mathcal{B})$ is the boundary of a simplex on $y_1, y_2, \ldots, y_n$.

e.g. $n=3$



$\mathcal{B} = \{1, 2, 3, 123\}$

$\mathcal{N}(\mathcal{B}_\Gamma) =$

$\cong S^1$

$= \{\emptyset, y_1, y_2, y_3, y_1y_2, y_2y_3, y_1y_3\}$

(but not y_{123})

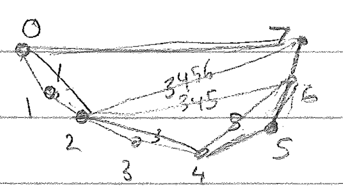
e.g. $\mathcal{B} = \mathcal{B}_\Gamma$ for $\Gamma = 1 - 2 - \cdots - n{-}1 - n$

$\mathcal{N}(\mathcal{B}_\Gamma) \cong \mathrm{Assoc}_n$.

e.g.

$n = 6$

$\{y_1, y_{3456}, y_{345}, y_3, y_5\} \longleftrightarrow$ $\sigma =$



Thm (DiConcini - Procesi 1995, Feichtner - Yuzvinsky 2004, Postnikov 2005)

For any $\mathcal{B}$ on $[n]$, $\mathcal{N}(\mathcal{B}) \cong S^{n-2}$

(and it's the boundary of an $(n-1)$-dim'l convex polytope.)

2. Extended Version (Lam and Pylyavskyy, 2012, for digraphs)

def/ $\tilde{N}(\mathcal{B})$ = extended nested set complex (for $\mathcal{B}$)

= simplicial complex w/ vertices $\{x_1, \dots x_n\} \cup \{y_I\}_{I \in \mathcal{B}}$ (includes $y_{[n]} = y_{12\dots n}$!)

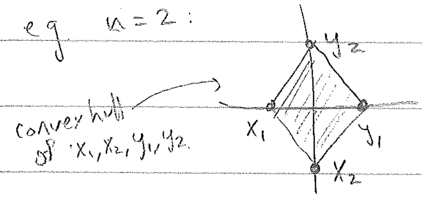
simplices $\{x_{i_1}, \dots x_{i_k}, y_{I_1}, \dots y_{I_r}\}$

where $\{I_1, \dots I_r\}$ is a nested family in $\mathcal{B}$

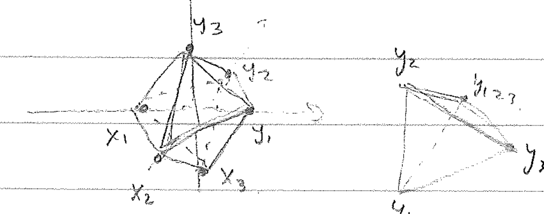
and $x_{i_k} \notin I_j \ \forall k, j$ i.e. $\{x_{i_1} \dots x_{i_k}\} \cap \bigcup_{j=1}^{r} I_j = \phi$.

eg. $\tilde{N}(\mathcal{B}_{n \to 1 \to 2 \to 3 \dots})$ = boundary of n-dim'l cross polytope / hyperoctahedron with face $\{y_1, \dots y_n\}$ stellarly subdivided

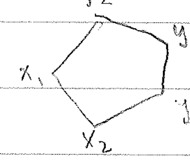
eg $n=2$: $\mathbb{R}^2$

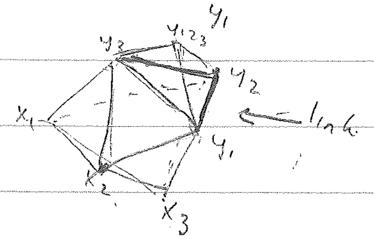


e.g. $n=3$ $\mathbb{R}^3$.

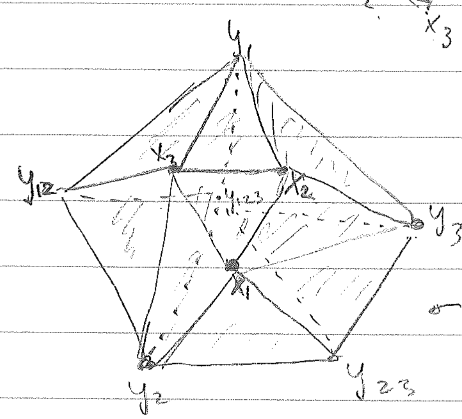


stellarly subdivide $y_1, y_2, \dots y_n$.





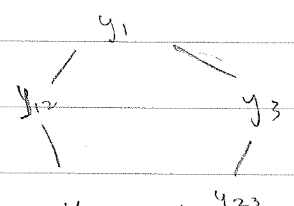
e.g. $\tilde{N}(\mathcal{B}_{1 \leftrightarrows 2 \leftrightarrows 3})$ =
$1 - 2 - 3$



$f = (f_{-1}, \dots f_2) = (1, 9, 21, 14)$

Note! same f-vector as $Assoc_4 = N(\mathcal{B}_{1-2-3-4})$!

$N(\mathcal{B})$ =

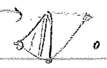


$Assoc_3$ $f = (1, 5, 5)$

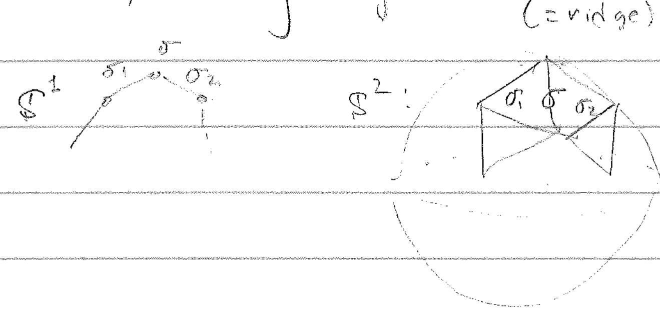
REU Exercise 16 Show $\tilde{N}(\mathcal{B})$, $N(\mathcal{B})$ are both...

(a) pure of dimensions $n-1$, $n-2$ respectively

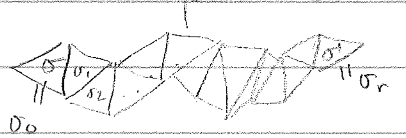
i.e. all facets have same dimension $n-1$ for $\tilde{N}(\mathcal{B})$, $n-2$ for $N(\mathcal{B})$)



(b) thin, meaning every codimension one simplex σ (= ridge) lies in exactly 2 facets, σ_1, σ_2



(c) pseudomanifolds, meaning thin and every pair σ, σ' of facets have a path $\sigma = \sigma_0, \sigma_1, \ldots \sigma_{r-1}, \sigma_r = \sigma'$ with



$\sigma_i \cap \sigma_{i+1}$ a ridge $\forall i$.

And (d) they're both flag complexes for any $\mathcal{B} = \mathcal{B}_\Gamma$ w/ Γ undirected, meaning the minimal nonfaces all have size 2

$\Updownarrow$

Stanley-Reisner ideal $I_{\tilde{N}(\mathcal{B}_\Gamma)}$, $I_{N(\mathcal{B}_\Gamma)}$ are quadratically generated
$\langle x_{i_1} x_{j_1}, \ldots x_{i_\ell} x_{j_\ell} \rangle$

REU Problem 7(a,b,c)

(a) Prove Conj (Lam-Pylyavskyy 2012 for $\mathcal{B} = \mathcal{B}_\Gamma$)

$\tilde{N}(\mathcal{B}) \cong \mathbb{S}^{n-1}$ and even is the boundary of a (simplicial) convex polytope

(b) Characterize the digraphs Γ for which $\tilde{N}(\mathcal{B}_\Gamma)$, $N(\mathcal{B}_\Gamma)$ are flag

(c) Prove conj: $\widetilde{N}(\overset{\circ}{1} - \overset{\circ}{2} \cdots - \overset{\circ}{n})$ and $Assoc_{n+1} = N(\overset{\circ}{1} - \overset{\circ}{2} \cdots \overset{\circ}{n} - \overset{\circ}{n+1})$
have the same f-vector $\forall n$.
Are they isomorphic?

3. f, h, γ vectors

f-vectors for $(d-1)$-spheres are too big and redundant!
e.g. they satisfy Euler's relation
$$-f_{-1} + f_0 - f_1 + f_2 - \cdots (-1)^{d-1} f_{d-1} = \tilde{\chi}(S^{d-1}) = (-1)^{d-1}$$
e.g. $Assoc_4$ $f = (1, 9, 21, 14) \rightsquigarrow -1 + 9 - 21 + 14 = (-1)^2 = 1$ ✓.

Thm (Dehn 1905 – Sommerville 1927) All linear relations among f-vectors of S^{d-1}'s are $h_i = h_{d-i}$ for $1 \le i \le \lfloor \frac{d}{2} \rfloor$
where $h = (h_0, \ldots, h_d)$ (h-vector) is defined by $\sum_{i=0}^{d} h_i (t+1)^{d-i} = \sum_{i=-1} f_i t^{d-1-i}$

⇕

h-polynomial: $\sum_{i=0}^{d} h_i t^i = \sum_{i=-1} f_i (t-1)^{d-1-i}$

e.g. $f = (1, 9, 21, 14) \rightsquigarrow (t-1)^3 + 9(t-1)^2 + 21(t-1)^1 + 14(t-1)^0$

$Assoc_4$ Kirkman-Cayley #s. $= t^3 + t^2(\underbrace{-3+9}_{6}) + t^1(\underbrace{3-18+21}_{6}) + t^0(\underbrace{-1+9-21+14}_{1})$

$\Rightarrow h = (1, 6, 6, 1)$

Narayana #s: $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ for $h(Assoc_n)$.

Rmk/ h-vectors are smaller and nonnegative:

Thm (Stanley 1970, [illegible] 1980) If $\Delta \cong S^{d-1}$ and more generally for Cohen-Macaulay Δ's := MFR for $\mathbb{K}[\Delta]$ is as short as possible i.e. # of vertices $- d$.
⇕ topological definition(s)

$\Rightarrow h = (h_0, \ldots, h_d)$ are nonnegative

REU Exercise 17 Show that the Hilbert series for $\mathbb{K}[\Delta]$, i.e. $\mathrm{Hilb}(\mathbb{K}[\Delta];t) = \sum_{i=0}^{\infty} \dim_{\mathbb{K}} (\mathbb{K}[\Delta])_i t^i$ (ring theoretic ith graded piece, $\deg(x_i) = 1\ \forall i$)

has two expressions:

$$\mathrm{Hilb}(\mathbb{K}[\Delta], t) \overset{(a)}{=} \sum_{i \geq -1} f_i \frac{t^{i+1}}{(1-t)^{i+1}}$$

$$\overset{(b)}{=} \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d}$$

For flag simplicial spheres $\Delta \cong \mathbb{S}^{d-1}$, even the h-vector is too big (conjecturally)

write the h-polynomial $h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d$ (with $h_0 = 1$, $h_d = 1$)

$$= (1+t)^d + \gamma_1 t (1+t)^{d-2} + \gamma_2 t^2 (1+t)^{d-4} + \cdots$$

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{d-1}{2} \rfloor})$$

e.g. $h(\mathrm{Assoc}_5) = (1, 10, 20, 10, 1)$

$$1 + 10t + 20t^2 + 10t^3 + t^4 = 1(1+t)^4 + 6t(1+t)^2 + 2t^2(1+t)^0$$

(with $\gamma_0 = 1$, $\gamma_1 = 6$, $\gamma_2 = 2$) so $\gamma = (1, 6, 2)$.

1	10	20	10	1
1	4	6	4	1
	6	12	6	
		2.		

CONJ (S. Gal, 2005). Flag $\Delta \cong \mathbb{S}^{d-1}$ have γ nonnegative

(proven for flag $N(\mathcal{B})$ by V Volodin, 2009
N. Aisbett 2012)

REU Problem 7(d)

- Study h-vectors, γ-vectors (when they're flag) of $\tilde{N}(\mathcal{B})$
- How do they relate to those of $N(\mathcal{B})$?
- Do they have combinatorial 'interpretations' similar to those for $N(\mathcal{B})$ and $N(\mathcal{B}_{\text{vertices}})$ as in Postnikov-Williams-R