Arborescences of Derived Graphs

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Arborescence Definitions

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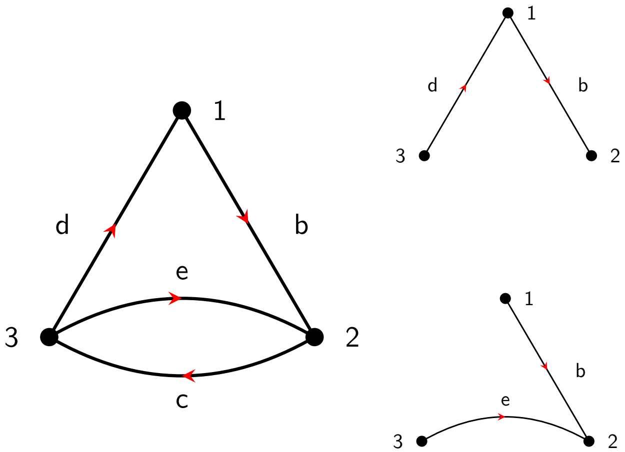
An *arborescence* T of Γ rooted at $v \in V$ is a spanning tree directed towards v. The *weight* of an arborescence wt(T) is the the product of the weights of its edges. We denote by $A_v(\Gamma)$ the sum of the weights of all arborescences of Γ rooted at v:

$$A_v(\Gamma) = \sum_{T \text{ an arborescence}} wt(T)$$

Arborescence Example



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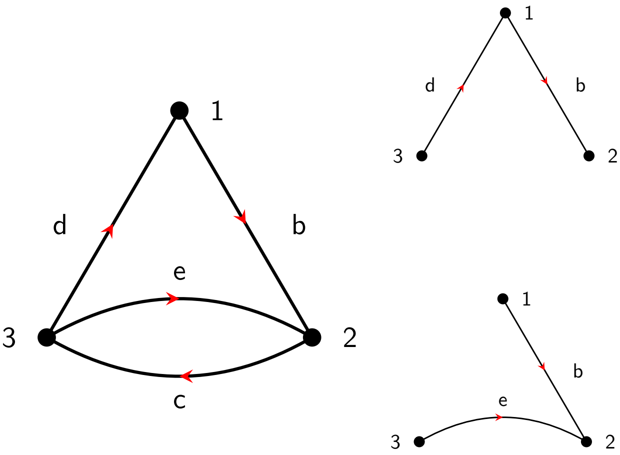


$$A_2(\Gamma) = bd + be$$

The Laplacian Matrix

Laplacian Matrix: $L(\Gamma) = D(\Gamma) - A(\Gamma)$

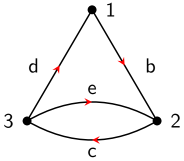
Weighted degree matrix:

$$d_{ii} = \sum_{e=(v_i,v_j)\in E} \mathrm{wt}(e).$$

Adjacency matrix:

$$a_{ij} = \sum_{e=(v_i,v_j)\in E} \mathrm{wt}(e),$$

Laplacian Example



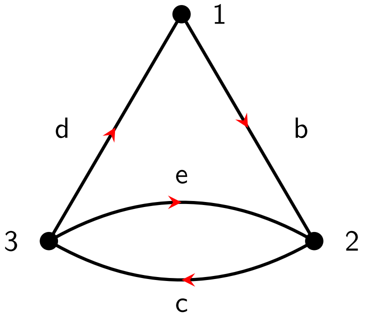
$$L(\Gamma) = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d+e \end{bmatrix} - \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ d & e & 0 \end{bmatrix}$$
$$= \begin{bmatrix} b & -b & 0 \\ 0 & c & -c \\ -d & -e & d+e \end{bmatrix}$$

Matrix Tree Theorem

Theorem (Kirchoff)

Given the Laplacian matrix of a graph Γ, $A_v(\Gamma)$ is the determinant of the matrix resulting from deleting its corresponding row and column of v.

Matrix Tree Theorem Example



$$L(\Gamma) = \begin{bmatrix} b & -b & 0 \\ 0 & c & c \\ -d & -e & d+e \end{bmatrix}$$

$$\begin{vmatrix} b & 0 \\ -d & d+e \end{vmatrix} = bd + be$$

Voltage Graphs and Derived Graphs

A *weighted G-voltage graph* $\Gamma = (V, E, \mathrm{wt}, \nu)$ is a directed, edge-weighted graph such that each edge e is also labeled by an element $\nu(e)$ of a finite group G. This labeling is called a *voltage* of Γ with respect to G.

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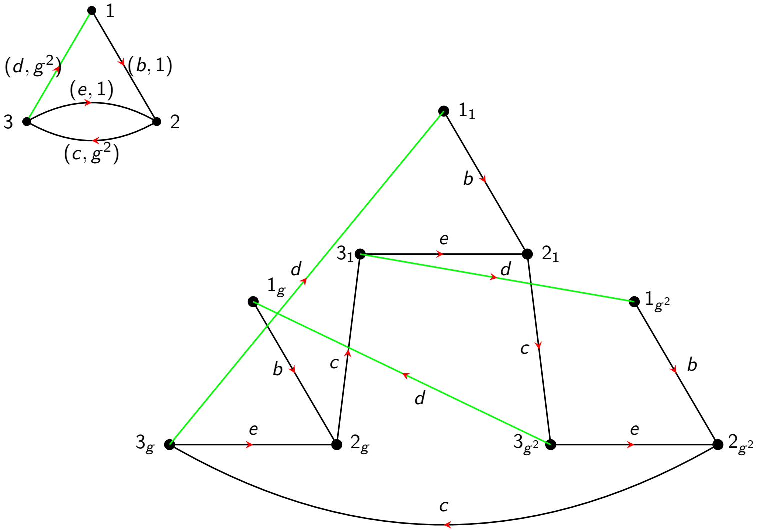
Given a G-voltage graph Γ, we can construct the *derived graph* $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ where

$$\tilde{V} := V \times G,$$

$$\tilde{E} := \{[v \times x, w \times (gx)] : x \in G, [v, w] \in E\}.$$

$\mathbb{Z}/3\mathbb{Z}$ Derived Graph Example



Invariance of Arborescence Ratio

Theorem (Galashin–Pylyavskyy, 2017)

If G is simple and strongly connected, then the ratio

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$$

is well-defined and independent of the choice of vertex v and its lift $\tilde{v}$.

The Voltage Laplacian

Voltage Laplacian: $\mathcal{L}(\Gamma) = D(\Gamma) - \mathcal{A}(\Gamma)$

Weighted degree matrix:

$$d_{ii} = \sum_{e=(v_i,v_j)\in E} \mathrm{wt}(e).$$

Voltage adjacency matrix:

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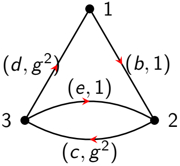
Voltage Laplacian Example



$$\mathcal{L}(\Gamma) = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d+e \end{bmatrix} - \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & \zeta_3^2 c \\ \zeta_3^2 d & e & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{bmatrix}$$

Our Conjecture

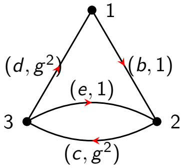
Conjecture (REU 2019)

Let G be a cyclic prime group of order p. Take any vertex v in Γ, and any lift of it in $\tilde{\Gamma}$, say $\tilde{v}$, then the following is true:

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{p} \prod_{i=1}^{p-1} \det \mathcal{L}(\Gamma, \zeta_p^i)$$

where $\mathcal{L}(\Gamma, \zeta_i)$ is the voltage Laplacian of Γ evaluated at certain powers of ζ_p.

Conjecture Example



$$\frac{1}{3}\prod_{i=1}^{3-1} \det \mathcal{L}(\Gamma, \zeta_3^i) = \frac{1}{3} * \begin{vmatrix} b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{vmatrix} * \begin{vmatrix} b & -b & 0 \\ 0 & c & -\zeta_3 c \\ -\zeta_3 d & -e & d+e \end{vmatrix}$$

$$= b^2c^2d^2 + b^2c^2e^2 + b^2c^2ef$$

$\mathbb{Z}/2\mathbb{Z}$ case

Theorem (REU 2019)

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{2} \det \mathcal{L}(\Gamma)$$

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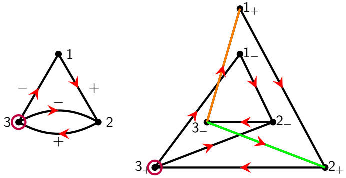
We've proven the special case of the general conjecture when $p = 2$:

$$\frac{\mathcal{A}_{\tilde{v}}(\tilde{\Gamma})}{\mathcal{A}_v(\Gamma)} = \frac{1}{p}\prod_{i=1}^{p-1}\det L(\Gamma, \zeta_p^i)$$

Easier to work with as a product identity:

$$2A_{\tilde{v}}(\tilde{\Gamma}) = A_v(\Gamma)\det \mathcal{L}(\Gamma)$$

$\mathbb{Z}/2\mathbb{Z}$ proof by induction sketch



Pick root to have ≥ 2 outgoing edges, then partition arborescences of cover into two classes (this step prevents generalization to $k > 2$, however)

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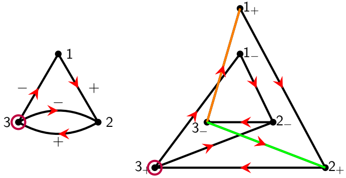

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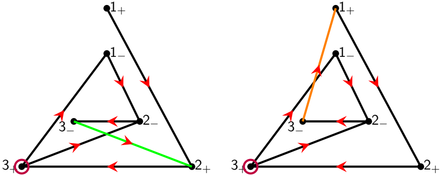
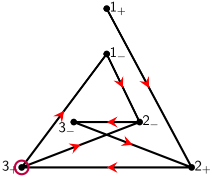


$\mathbb{Z}/2\mathbb{Z}$ proof by induction sketch

Remove other lift of edge as well, since it does not affect aborescences (its initial vertex is the root):



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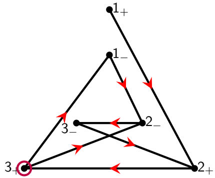
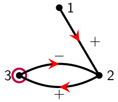


We end up with the derived graph of a signed graph with fewer edges:



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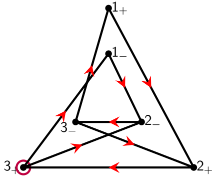

$$L(\tilde{\Gamma}) = \begin{bmatrix} a & -a & 0 & 0 & 0 & 0 \\ 0 & b & -b & 0 & 0 & 0 \\ 0 & 0 & c+d & -c & -d & 0 \\ 0 & 0 & 0 & a & -a & 0 \\ 0 & 0 & 0 & 0 & b & -b \\ -c & -d & 0 & 0 & 0 & c+d \end{bmatrix}$$

$$\det L_{3_+}^{3_+} = A_{3_+}(\tilde{\Gamma}) = a^2b^2c + a^2b^2d$$

Higher covers: progress towards $\mathbb{Z}/p\mathbb{Z}$

Lemma (REU 2019)

Under suitable change of basis, $L(\tilde{\Gamma})$ may be written in block matrix form

$$\begin{bmatrix} L(\Gamma) & * \\ 0 & [\mathcal{L}(\Gamma)]_{\mathbb{Q}} \end{bmatrix}$$

where $L(\Gamma)$ is the ordinary Laplacian matrix of Γ and $[\mathcal{L}(\Gamma)]_{\mathbb{Q}}$ is the voltage Laplacian of Γ written as a matrix with entries in $\mathbb{Q}$ (restriction of scalars).

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We know $\det[\mathcal{L}(\Gamma)]_{\mathbb{Q}}$ is equal to the norm of $\det \mathcal{L}(\Gamma)$, so this is very close to giving us the product formula we want:

$$A_v(\Gamma) N_{\mathbb{Q}(\zeta_p):\mathbb{Q}}(\mathcal{L}(\Gamma)) = p\mathcal{L}_{\tilde{v}}(\tilde{\Gamma})$$

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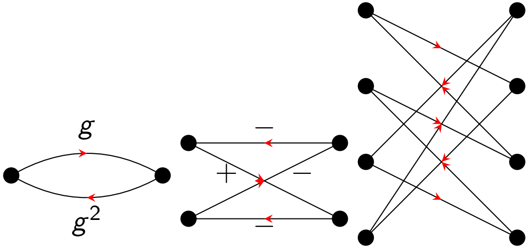
However, we don't quite know how to account for taking minors, since change of basis and taking minors do not commute (and we need a factor of p somewhere).

Higher covers: arbitrary abelian groups

Can always build derived graph by iteratively taking *p*-fold covers.

$G = \mathbb{Z}/4\mathbb{Z} = \{1, g, g^2, g^3\}$: take two 2-fold covers



Higher covers: arbitrary abelian groups

Resulting formula isn't particularly nice, but (conditioning on our the conjecture for prime cyclic G) we can say

Conjecture (REU 2019)

If Γ is G-volted with G abelian, then the ratio $\frac{A\tilde{v}(\tilde{\Gamma})}{A_v(\Gamma)}$ is a polynomial in the edge weights of Γ, and it has positive integer coefficients.

Special thanks

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References

Questions?

- S. Chaiken, *A combinatorial proof of the All-Minors Matrix Tree Theorem*, SIAM J. Alg Disc. Meth. 3(3), 319-329, 1982.
- P. Galashin and P. Pylyavskyy, *R-systems*, arXiv:1709.00578, 2017.
- V. Reiner, D. Tseng, *Critical groups of covering, voltage, and signed graphs*, Discrete Mathematics 318, 10-40, 2014.
- D. Stanton, D. White, *Constructive Combinatorics*, 1986.