# ASM Polymath REU

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## 1 Introduction

## 2 Historical Development

### 2.1 Alternating Sign Matrices

An **alternating sign matrix** (ASM) is a square matrix consisting of 0's, 1's and $-1$'s such that the entries in each row and each column sum to 1 and the nonzero entries in each row and each column alternate in sign. An example is shown:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

As a consequence of its definition, the first row of an alternating sign matrix contains exactly one 1 and no $-1$'s. Therefore, the set of $n \times n$ alternating sign matrices $A_n$ can be partitioned into $n$ sets

$$A_{n,1}, A_{n,2}, \ldots, A_{n,n}$$

where $A_{n,r}$ is the set of $n \times n$ alternating sign matrices such that the position of the lone 1 in the first row is the $r$th entry.

### 2.2 Lattice models for ASMs

Shortly after Zielberger presented the first proof of the ASM conjecture in early 1995, Kuperberg presented a much shorter proof in December of the same year that drew inspiration from physics and statistical mechanics. For years, physicists have studied lattice models because of their use in describing physical systems. A lattice model called the square ice model with domain wall boundary conditions was noted in 1992 [citation here] to be in bijection with alternating sign matrices.

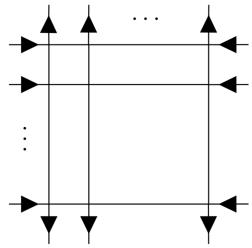

Figure 1: Square Ice

A state of this lattice model corresponds to a filling of the lattice with arrows, where each vertex is required to have exactly 2 inward facing arrows. The boundary arrows are fixed and refer to the 'domain wall boundary conditions.' There are 6 types of vertices in the square ice model:

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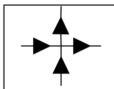
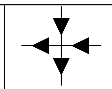
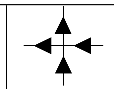
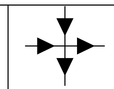
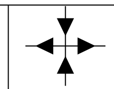
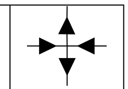
|---|---|---|---|---|---|
|  |  |  |  |  |  |
| $a_1$ | $a_2$ | $b_1$ | $b_2$ | $c_1$ | $c_2$ |

Figure 2: Square Ice Vertices

The following correspondence gives a bijection between square ice with domain wall boundary conditions and alternating sign matrices.

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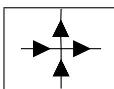
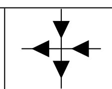
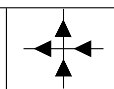
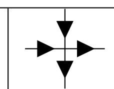
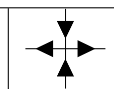
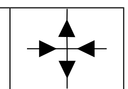
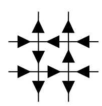
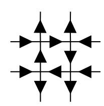
|---|---|---|---|---|---|
|  |  |  |  |  |  |
| 0 | 0 | 0 | 0 | -1 | 1 |

Figure 3: ASM and Square Ice bijection

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow$$  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow$$ 

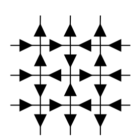
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$ 

Figure 4: ASM and Square Ice examples

Kuperberg considered something called the *partition function* on the square ice model. First, parameterize square ice by $\vec{x}, \vec{y} \in \mathbb{R}^n$.

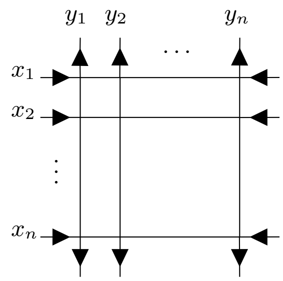



Figure 5: Parameterized Square Ice

Next, assign a weight to each vertex based on its type and position. For example, the weight of a vertex $v$ of type $a_1$ in row $i$ and column $j$ is given by $w(v) = a_1(x_i, y_j)$. Define the weight of a filling to be the product of the weights of the vertices it contains. A weighting system with this property is called Boltzmann.

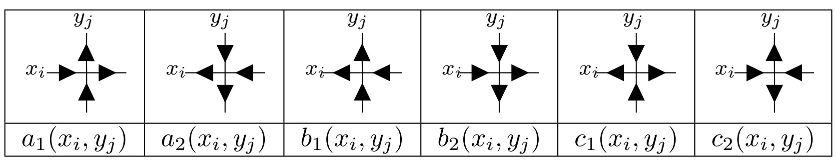



Figure 6: Parameterized Boltzmann Weights

Define the partition function to be

$$Z(x_1 \ldots, x_n; y_1, \ldots, y_n) := \sum_{\text{fillings } F} w(F) = \sum_{\text{fillings } F} \prod_{\text{vertices } v \in F} w(v).$$

Physicists considered the following Boltzmann weights, where $Z$ is further parameterized by $q$.

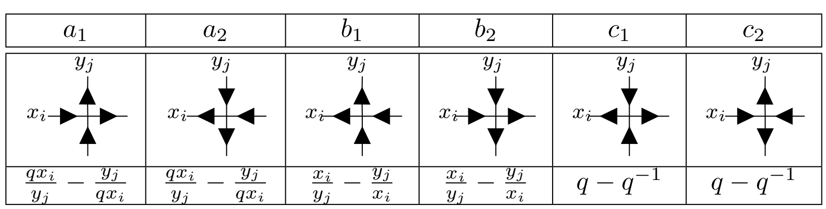



Figure 7: Some Boltzmann Weights

With these Boltzmann weights, $Z$ has two remarkable properties.

1. $Z$ is symmetric in $\vec{x}$ and $\vec{y}$. That is,

   $$Z(x_1, \ldots, x_i, \ldots, x_j, \ldots x_n; y_1, \ldots, y_n) = Z(x_1, \ldots, x_j, \ldots, x_i, \ldots x_n; y_1, \cdots y_n)$$

   and

   $$Z(x_1, \ldots x_n; y_1, \ldots, y_k, \ldots, y_\ell, \ldots y_n) = Z(x_1, \ldots x_n; y_1, \ldots, y_\ell, \ldots, y_k, \ldots y_n).$$

2. $Z$ is completely determined recursively.

Property 1 is another way of saying that the weights satisfy the Yang-Baxter equation.

#### 2.2.1 The Yang-Baxter Equation

Define a new "rotated" vertex $R$ and assign Boltzmann weights $a_1(R), \ldots, c_2(R)$ to each configuration.

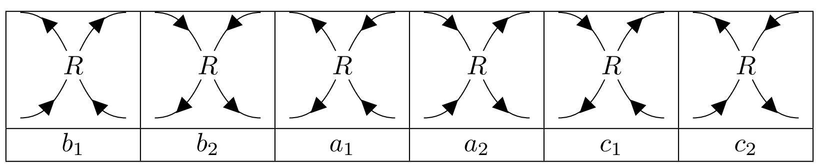



Figure 8: $R$ weights

Let $a_1(S), \ldots, c_2(S)$ and $a_1(T), \ldots, c_2(T)$ denote the (no longer parameterized) Boltzmann weights of vertex $S$ and $T$ configurations, as in Figure 7. We say that vertices $R$, $S$, and $T$ satisfy the YBE if, for all fixed combinations of in/out arrows $a, \ldots, f$, we have

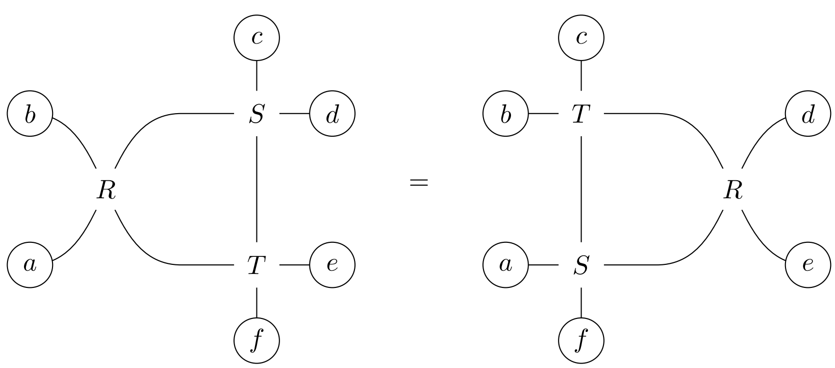



Figure 9: Yang-Baxter Equation

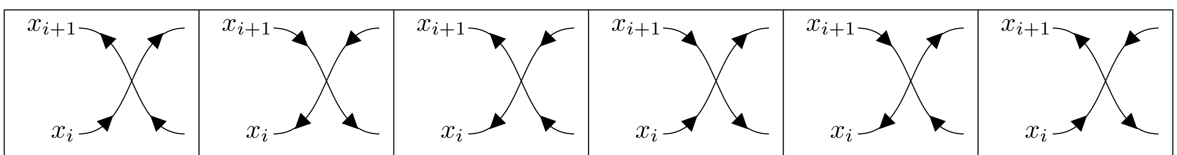
Equality here is in the sense that the partition function of the left hand side is equal to the partition function of the RHS:

$$\sum_{\text{Fillings of LHS}} (\text{filling weight}) = \sum_{\text{Fillings of RHS}} (\text{filling weight}).$$

Since the weights are Boltzmann, the the weight of a filling is simply the product of the weights of the vertices it contains. Exactly 3 of $a, \ldots, f$ must be pointing inward, and therefore there are $\binom{6}{3} = 20$ equations that must be satisfied. In the field-free case, i.e. $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$, 10 equations are eliminated since weights are unaffected by rotation by 180°.

### 2.2.2 Symmetry of $Z$

We will use the YBE to show that, for the weights in Figure 7, $Z$ is symmetric in $\vec{x}$ and $\vec{y}$ separately. To show that $Z$ is symmetric in $\vec{x}$, we examine the horizontal YBE, which was the formulation of YBE in the previous section. Symmetry in $\vec{y}$ is similar, except we examine the vertical YBE, which is the same as the horizontal YBE but rotated by 90°. One can check that if we let the weight of a rotated vertex be the same as the weight of the regular vertex obtained by rotating 90° clockwise, then the weights satisfy the horizontal YBE. Explicitly, we let




| | | | | | |
|---|---|---|---|---|---|
| $\frac{x_{i+1}}{x_i} - \frac{x_i}{x_{i+1}}$ | $\frac{x_{i+1}}{x_i} - \frac{x_i}{x_{i+1}}$ | $\frac{qx_{i+1}}{x_i} - \frac{x_i}{qx_{i+1}}$ | $\frac{qx_{i+1}}{x_i} - \frac{x_i}{qx_{i+1}}$ | $q - q^{-1}$ | $q - q^{-1}$ |

Figure 10: $R$ weights satisfying YBE

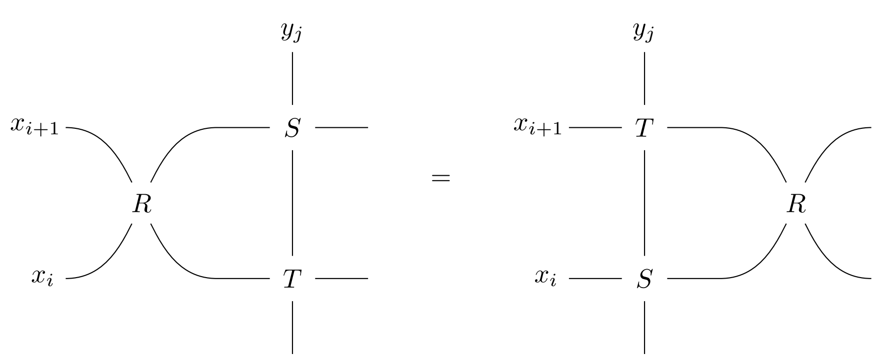



Figure 11: Parameterized horizontal YBE

### 2.2.3 Evaluation

(discussion of recursion)
Korepin and Izergin [cite] explicitly formulated the partition function for the square ice model with these weights.

**Theorem 2.1** (Izergin, 1987 [Ize87])**.**

$$Z = \frac{\prod_{i,j=1}^{n}(x_j/y_i - y_i/x_j)(qx_j/y_i - q^{-1}y_i/y_j)}{\prod_{1\le i<j\le n}(x_i/x_j - x_j/x_i)(y_i/y_j - y_j/y_i)} \det_{i,j=1\ldots n}\left(\frac{q-q^{-1}}{(x_j/y_i - y_i/x_j)(qx_j/y_i - q^{-1}y_i/y_j)}\right)$$

Kuperberg pointed out that given particular parameters the partition function enumerates states of the ice model. Evaluating Korepin and Izergin's formula with Kuperberg's parameters was tricky, but Okada and Stroganoff developed a simpler method that involved identifying the partition function with a Schur polynomial. Remarkably, the number of nxn alternating sign matrices is equal to a certain Schur polynomial evaluated at $(1, \ldots, 1)$ multiplied by a simple factor.

## 2.3 Bijections with ASMs

[You can probably largely look at the Propp paper for this, although you should mention if your exposition is going to closely follow his exposition and should reference the original papers for each bijection]

## 2.4 The ASM polytope

The $n$th Birkhoff Polytope, $B_n$, is defined as the convex hull of all $n \times n$ permutation matrices in $\mathbb{R}^{n^2}$. This polytope is has been very well studied. A major result about the Birkhoff Polytope is the Birkhoff-Von Neumann Theorem, which states that $B_n$ is the set of all $n \times n$ *doubly stochastic matrices*, matrices with entries in $[0, 1]$ and all of whose rows and columns sum to 1.

Another result is that for any $n$-vector $v$, $B_n \cdot v = P_v$, where $P_v$ is the *permutohedron* of $v$: the convex hull of all permutations of $v$.

Striker [insert citation here] defined the $n$th ASM polytope, denoted $ASM_n$, as the convex hull of all $n \times n$ alternating sign matrices in $\mathbb{R}^{n^2}$. Since permutation matrices are special cases of ASMs, $B_n \subseteq ASM_n$. Striker was able to extend many of the results about the Birkhoff polytope to the ASM polytope.

The inequality representation of the ASM polytope is

$$\begin{aligned}
0 \leq \sum_{i=1}^{i'} x_{ij} \leq 1 \qquad & \forall\ 1 \leq i' \leq n, 1 \leq j \leq n. \\
0 \leq \sum_{j=1}^{j'} x_{ij} \leq 1 \qquad & \forall\ 1 \leq j' \leq n, 1 \leq i \leq n. \\
\sum_{i=1}^{n} x_{ij} = 1 \qquad & \forall\ 1 \leq j \leq n. \\
\sum_{j=1}^{n} x_{ij} = 1 \qquad & \forall\ 1 \leq i \leq n.
\end{aligned}$$

That is to say, $ASM_n$ contains the matrix $(x_{ij})$ if it satisfies the above conditions. This is a loosening of the conditions for doubly stochastic matrices; negative entries are allowed, but partial row and column sums must be nonnegative.
Striker also proved that if $v$ is a strictly decreasing $n$ vector, $ASM_n \cdot v = P_v$. We extend these results to $\lambda$-ASMs in section 6 of the paper.

# 3 Generalized alternating sign matrices

[This section should contain basic definitions, etc, to set up the following sections.]

# 4 Lattice model for $\lambda$-ASMs

A $\lambda$-ASM is an row $\times$ column matrix such that:

1. row $\leq$ column
2. The sum of entries in any row equals 1 and the sum of entries in any column equals 0 or 1.
3. The non zero entries alternate signs.

Similarly, $\lambda$-lattice model is a six-vertex state model such that

1. The leftmost edge points ¿ and the rightmost edge points ¡.
2. The top edge of every column points either $\wedge$ or $\vee$.
3. The bottom edge of every column points $\vee$.

To form a lattice model with r rows, we first define $\lambda$=($\lambda$1,...,$\lambda$r)such that it is a set of weakly decreasing integers. Now, if we let, $\rho$=(r-1,r-2,...,2,1,0) such that

we compute $\lambda+\rho$. Now, a lattice model can be made with r rows and at least $\lambda 1$+r columns. The convention is that if the column's index is the same as one of the values in the set defined by $\lambda$ then it is going to point $\wedge$, else it is going to point $\vee$. After this convention is set up, the remaining arrows are chosen using the general rule: at each vertex, two arrows point towards the vertex and two arrows point away from the vertex. The idea is that even after using the convention that the column edges corresponding to our $\lambda$ set, the general rule of two sets of inward/outward arrows still holds.
H —— O —— H

### 4.1 Evaluating partition functions

## 5 Other Bijections with $\lambda$-ASMs

### 5.1 Monotone Triangles

When we consider the ice model lattice from a $\lambda_A SM$, we see that the latter row must have exactly one fewer up-arrow than the previous one does, and there must be exactly one up-arrow in the latter row between the columns that have two consecutive up-arrows in the previous row. Therefore we would have these number, where $a_{i,j}$'s are the position to put the up-arrows in row *ith*:

$$\begin{array}{ccccccccccc}
\lambda_n & & \lambda_{n-1} & & & & \dots & & \lambda_1 & & 0 \\
& a_{1,n} & & a_{1,n-1} & & & & \dots & & a_{1,0} & \\
& & a_{2,n-1} & & & a_{2,n-2} & & & \dots & & \\
& & & \dots & & & & \dots & & & \\
& & & & & a_{n,0} & & & & &
\end{array}$$

This exactly satisfies the conditions for an upside-down monotone triangle with prescribed bottom row $\lambda = (\lambda_n, \lambda_{n-1}, \dots, \lambda_0)$. The number of those monotone triangle was presented by a recursion in Fischer 2005. We have attempted to count those number of monotone triangles or Gelfand-Tsetlin patterns by some techniques in the hope of finding such beautiful formula as the original ASMs problem, but it has not been done yet. It seems like it is too complicated to conjecture such formula.

### 5.2 Monotone Trapezoids

Let $\lambda = (\lambda_1, ..., \lambda_{n-1}, 0)$, $\bar{\lambda}$ is the strictly decreasing tuple consisting of all positive integers from 0 to $\lambda_1$ that does not appear in $\lambda$ and $\bar{\lambda}'$ is the vector reversing the order of elements and then plus 1 for each entry in $\bar{\lambda}$. For example, if $\lambda = (4, 2, 0)$, then $\bar{\lambda} = (3, 1)$ $\bar{\lambda}' = (2, 4)$.

**Theorem** (5.2)**.** *$\lambda$-Alternating Sign Matrix is in bijection to Monotone Trapezoid with prescribed top row $\bar{\lambda}'$.*

*Proof.* We divide our proofs into several steps. If $r = \lambda_r$, then the claim reduces to the famous bijection between ASM and Monotone Triangle. We now assume $r < \lambda_r$

- Adding an auxiliary row $R_0$ on the top of $\lambda$-ASM to get a new $(r+1)$-by-$\lambda_r$ matrix $A$.

  We fix $R_0 = 1_{\bar{\lambda}}$ such that $R_0$ is a row vector of length $\lambda_r$ where entries equal to 1 if the column index is the same as the entries of $\bar{\lambda}$, otherwise 0. In $\lambda$-ASM we have row sums to 1 and column sums to the indicator function of $\lambda$. After adding the auxiliary row on the top, matrix $A$ row sums to 1 and column sums to 1.

- Transforming $A$ to its corresponding HFM $A'$.

  Denote the $(r+1)$ rows and $\lambda_r$ columns of $A'$ as $(R_0, R_1, ..., R_r)^T$ or $(C_1, C_2, ..., C_{\lambda_r})$ respectively. That is, $R_i$ refers to the $i$-th row and $C_j$ refers to the $j$-th column. Then

  – For $1 \leq i \leq r+1$, (#1's in $R_i$) $-$ $(\#1's$ in $R_{i-1}) = 1$.

    This is because every row in matrix $A$ sums to 1 and the alternating nature of $A$.

  – The entries of last row $R_r$ are all 1's.

    Let $|\bar{\lambda}|$ denote the number of parts of $\bar{\lambda}$. Then the number of 1's in $R_0$ equals to $\lambda_r - r$. From the relation of neighboring rows, we have the (#1's in $R_r$) $= \lambda_r$ which is the same as the number of columns.

- Transforming $A'$ to $\lambda$-Monotone Trapezoid.

  We transform $A'$ to $\lambda$-Monotone Trapezoid by recording the position of 1's. $a_{ij}, 1 \leq j \leq i$, denotes the $j$th entry in row $i$, counted from the top. We have:

  – $a_{ij} < a_{i,j+1}, 1 \leq j < i$. This is clear.

  – $a_{ij} \leq a_{i-1,j} \leq a_{i,j+1}, 1 \leq j < i$. This is essentially the same argument of why ASM is in bijection to Monotone Triangle.

The converse transformation from $\lambda$-Monotone Trapezoid to $\lambda$-Alternating Sign Matrix is then easy to see. The proof is now complete. For a detailed example, please check the file below called "Complete examples of $\lambda$-ASMs". □

# 6 Results on $\lambda$-ASM polytopes

For a given $\lambda$, we define the $\lambda$-ASM polytope as the convex hull of all $\lambda$-ASMs.

## 6.1 H-representation

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 0)$ be a strict partition, and let $m = \lambda_1 + 1$. Recall that a $\lambda$-ASM is an $n$ by $m$ matrix. The following theorem gives the H-representation of the $\lambda$ polytope.

**Theorem** (6.1)**.** *For a given $\lambda$, the $n$ by $m$ matrix $X = \{x_{ij}\}$ is in the $\lambda$-ASM polytope if and only if*

$$0 \leq \sum_{i=1}^{i'} x_{ij} \leq 1 \qquad \forall\ j \in \lambda, 1 \leq i' \leq n. \tag{1}$$

$$-1 \leq \sum_{i=1}^{i'} x_{ij} \leq 0 \qquad \forall\ j \notin \lambda, 1 \leq i' \leq n. \tag{2}$$

$$0 \leq \sum_{j=1}^{j'} x_{ij} \leq 1 \qquad \forall\ 1 \leq j' \leq m, 1 \leq i \leq n \tag{3}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall\ j \in \lambda. \tag{4}$$

$$\sum_{i=1}^{n} x_{ij} = 0 \qquad \forall\ j \notin \lambda. \tag{5}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall\ 1 \leq i \leq n. \tag{6}$$

It is relatively quick to see that every matrix in the $\lambda$-ASM polytope satisfies these inequalities.

Let $P(\lambda)$ denote the polytope defined by (1)-(6). $P(\lambda)$ is convex since it is an intersection of half-planes. Further, note that any $\lambda$-ASM satisfies these inequalities[1].

By definition, the $\lambda$-ASM polytope is the smallest convex polytope containing each $\lambda$-ASM. Therefore, the $\lambda$-ASM polytope is contained within $P(\lambda)$. The following proof shows the other direction of containment.

**Proof of Theorem 6.1**. This follows the proof by Striker with small modifications. Let $\lambda$ be any partition and let $X$ be any matrix in $P(\lambda)$. Define the

[1] This comes from the fact that, since nonzero terms in a row alternate between 1 and $-1$, partial row sums bounce between 0 and 1. The same holds for columns that are in $\lambda$. For columns not in $\lambda$, the first nonzero element (if there is one) will be a $-1$, meaning partial sums bounce between $-1$ and 0.

*modified partial sums* of $X$ (which we will refer to simply as partial sums) as follows

$$r_{ij} = \sum_{j'=1}^{j} x_{ij'} \qquad \forall\ 1 \le i \le n, 1 \le j \le m$$

$$c_{ij} = \sum_{i'=1}^{i} x_{i'j} \qquad \forall\ j \in \lambda, 1 \le i \le n$$

$$c_{ij} = 1 + \sum_{i'=1}^{i} x_{i'j} \qquad \forall\ j \notin \lambda, 1 \le i \le n$$

With this definition, the conditions that define $P(\lambda)$ are equivalent to

$$0 \le c_{ij} \le 1, \quad 0 \le r_{ij} \le 1, \quad r_{im} = c_{nj} = 1 \qquad \forall 1 \le i \le n, 1 \le j \le m$$

For consistency, we will define $r_{i0} = 0$, $c_{0j} = 0$ if $j \in \lambda$ and $c_{0j} = 1$ if $j \notin \lambda$. As Striker does, we will construct a matrix by interleaving these partial sums between the elements of $x$ as shown.

$$\begin{pmatrix}
 & c_{01} & & c_{02} & & & c_{0m} & \\
r_{10} & x_{11} & r_{11} & x_{12} & r_{13} & \cdots & x_{1m} & r_{1m} \\
 & c_{11} & & c_{12} & & & c_{1m} & \\
r_{20} & x_{21} & r_{21} & x_{22} & r_{23} & \cdots & x_{2m} & r_{2m} \\
 & \vdots & & \vdots & & \ddots & \vdots & \\
 & c_{n-1,1} & & c_{n-1,2} & & & c_{n-1,m} & \\
r_{n0} & x_{n1} & r_{n1} & x_{n2} & r_{n3} & \cdots & x_{nm} & r_{nm} \\
 & c_{n1} & & c_{n2} & & & c_{nm} &
\end{pmatrix}$$

For example if $\lambda = (1, 1, 0, 0)$, this process could look like

$$\begin{pmatrix}
.5 & .2 & -.5 & .5 & .3 \\
0 & .8 & -.3 & 0 & .5 \\
.3 & 0 & .1 & .4 & .2 \\
.2 & 0 & .7 & .1 & 0
\end{pmatrix}
\Longrightarrow
\begin{pmatrix}
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & .5 & .5 & .2 & .7 & -.5 & .2 & .5 & .7 & .3 & 1 \\
 & .5 & & .2 & & .5 & & .5 & & .3 & \\
0 & 0 & 0 & .8 & .8 & -.3 & .5 & 0 & .5 & .5 & 1 \\
 & .5 & & 1 & & .2 & & .5 & & .8 & \\
0 & .3 & .3 & 0 & .3 & .1 & .4 & .4 & .8 & .2 & 1 \\
 & .8 & & 1 & & .3 & & .9 & & 1 & \\
0 & .2 & .2 & 0 & .2 & .7 & .9 & .1 & 1 & 0 & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 &
\end{pmatrix}$$

Call a partial sum $\alpha$ *inner* if $0 < \alpha < 1$. If a matrix $X$ has no inner partial sums, then it is a $\lambda$-ASM. Otherwise, notice that $x_{ij} = r_{ij} - r_{i,j-1} = c_{ij} - c_{i-1,j}$, so

$$r_{ij} + c_{i-1,j} = c_{ij} + r_{i,j-1}$$

This equation guarantees that if an entry of $X$ has an adjacent inner partial sum, it has at least one other adjacent partial sum that is also inner. This means we can construct a path in $X$ by moving between entries with an inner partial sum between them. Since none of the partial sums along the edge of the matrix are inner, this path will stay inside the matrix. Further, because the matrix has a finite number of entries, this path must eventually reach an entry it has already passed, creating a circuit in $X$ of inner partial sums. An example is shown in Figure 12.

$$
\left(\begin{array}{ccccccccccc}
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & .5 & .5 & \mathbf{.2} & \mathbf{.7} & \mathbf{-.5} & \mathbf{.2} & \mathbf{.5} & .7 & .3 & 1 \\
 & .5 & & \mathbf{.2} & & .5 & & \mathbf{.5} & & .3 & \\
0 & 0 & 0 & \mathbf{.8} & \mathbf{.8} & \mathbf{-.3} & .5 & \mathbf{0} & .5 & .5 & 1 \\
 & .5 & & 1 & & \mathbf{.2} & & \mathbf{.5} & & .8 & \\
0 & .3 & .3 & 0 & .3 & \mathbf{.1} & \mathbf{.4} & \mathbf{.4} & .8 & .2 & 1 \\
 & .8 & & 1 & & .3 & & .9 & & 1 & \\
0 & .2 & .2 & 0 & .2 & .7 & .9 & .1 & 1 & 0 & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}\right)
$$

Figure 12: Each of the partial sums (shown in blue) in the circuit are inner.

Next, we will label the corners of the circuit alternately with $+$ and $-$. Let $R_+$ denote the set of all row partial sums in the circuit to the right of a corner labeled $+$, $R_-$ denote the set of all row partial in the circuit sums to the right of a $-$. Similarly, define $C_+$ and $C_-$ be the sets of all column sums in the circuit below a corner labeled $+$ and $-$, respectively.

$$
\left(\begin{array}{ccccccccccc}
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & .5 & .5 & + & \mathbf{.7} & \mathbf{-.5} & \mathbf{.2} & - & .7 & .3 & 1 \\
 & .5 & & \mathbf{.2} & & .5 & & \mathbf{.5} & & .3 & \\
0 & 0 & 0 & - & \mathbf{.8} & + & .5 & \mathbf{0} & .5 & .5 & 1 \\
 & .5 & & 1 & & \mathbf{.2} & & \mathbf{.5} & & .8 & \\
0 & .3 & .3 & 0 & .3 & - & \mathbf{.4} & + & .8 & .2 & 1 \\
 & .8 & & 1 & & .3 & & .9 & & 1 & \\
0 & .2 & .2 & 0 & .2 & .7 & .9 & .1 & 1 & 0 & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}\right)
$$

Figure 13: In this example, $R_+ = \{.7, .2\}, R_- = \{.8, .4\}, C_+ = \{.5\}$ and $C_+ = \{.2\}$

Define

$$k' = \min\{R_-, 1 - R_+, C_-, 1 - C_+\}$$

Notice that $k' \geq 0$. We will construct a new matrix $X'$ by adding $k'$ to all entries labeled $+$ and subtracting $k'$ from all entries labeled $-$. In our example, $k' = .3$, so

$$X' = \begin{pmatrix} .5 & .5 & -.5 & .2 & .3 \\ 0 & .5 & 0 & 0 & .5 \\ .3 & 0 & -.2 & .7 & .2 \\ .2 & 0 & .7 & .1 & 0 \end{pmatrix}$$

We claim that $X'$ is an element of $P(\lambda)$ with at least one fewer inner partial sum.

Let $r'_{ij}$ and $c'_{ij}$ denote the partial sums of $X'$. First, notice that each row and column of $X$ has the same number of entries labeled $+$ and $-$. This implies that the row and column sums of $X'$ are the same as the row and column sums of $X$, meaning $r'_{im} = c'_{nj} = 1$ as desired. Also, $k' \leq \min\{R_-\}$ implies $r_{ij} \geq 0$ and $k' \leq \min\{1 - R_+\}$ implies $r_{ij} \leq 1$ for all $i$ and $j$. Similarly, $k' \leq \min\{C_-\}$ implies $c_{ij} \geq 0$ and $k' \leq \min\{1 - C_+\}$ implies $c_{ij} \leq 1$. Therefore, $X' \in P(\lambda)$. Finally, notice that if $k' = r_{ij} \in R_-$, then $r'_{ij} = 0$, so $r'_{ij}$ is not inner. Similar results hold if $k'$ takes its value from $1 - R_+$, $C_-$, or $1 - C_+$.

Hence, $X'$ is an element of $P(\lambda)$ with at least one fewer inner partial sum than $X$. We could redo the exact same process reversing the $+$ and $-$ labels to obtain a new positive constant $k''$ and matrix $X''$ in $P(\lambda)$ that also has at least one fewer inner partial sum than $X$. Furthermore, our construction guarantees

$$X = \frac{k''}{k' + k''} X' + \frac{k'}{k' + k''} X''$$

That is to say, $X$ is a convex combination of $X'$ and $X''$.

In our example, we get $k'' = .2$, and this decomposition is

$$X = \frac{2}{5} \begin{pmatrix} .5 & .5 & -.5 & .2 & .3 \\ 0 & .5 & 0 & 0 & .5 \\ .3 & 0 & -.2 & .7 & .2 \\ .2 & 0 & .7 & .1 & 0 \end{pmatrix} + \frac{3}{5} \begin{pmatrix} .5 & 0 & -.5 & .7 & .3 \\ 0 & 1 & -.5 & 0 & .5 \\ .3 & 0 & .3 & .2 & .2 \\ .2 & 0 & .7 & .1 & 0 \end{pmatrix}$$

Notice that $X'$ and $X''$ each have fewer inner partial sums than $X$. By iterating this process, we could write $X$ as a convex combination of matrices with no inner partial sums, which are precisely the $\lambda$-ASMs. Therefore, $X$ is in the $\lambda$-ASM polytope. $\square$

## 6.2 Projections

For a vector $v \in \mathbb{R}^n$, the *permutohedron*, $P_v$, is defined as the convex hull of all permutations of the components of $v$ in $\mathbb{R}^n$. Striker proved that if $v$ is an $n$-long decreasing vector with distinct components, then

$$ASM_n \cdot v = P_v$$

We extend this result to $\lambda$-ASMs.

**Definition.** For a vector $z = (z_1, z_2, \ldots, z_n)$, let $(z_{[1]}, z_{[2]}, \ldots z_{[n]})$ be the vector composed of the elements of $z$ in decreasing order. For two $n$-long vectors $u$ and $v$, we say $u \preceq v$ if

$$\sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]} \qquad \text{for all } 1 \leq k \leq n, \text{ and} \tag{7}$$

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i \tag{8}$$

An important result proved by Rado [citation needed] is that $u \in P_v$ if and only if $u \preceq v$.

**Theorem** (6.2)**.** *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, 0)$ be an $n$-long partition, and let $m = n + 1$. If $\mathfrak{A}$ is the $\lambda$-ASM polytope, and $v$ is a decreasing $m$-long vector with distinct parts, then*

$$\mathfrak{A} \cdot v = P_{v'}$$

*where $v'$ is the $n$-long sub-vector of $v$ indexed by $\lambda$ in reverse order. That is,*

$$v' = (v_{\lambda_n}, v_{\lambda_{n-1}} \ldots, v_{\lambda_2}, v_{\lambda_1})$$

Note that reversing the order of $\mu$ is not strictly necessary; it is done in order to make $v'$ a decreasing vector.

**Proof of Theorem 6.2.** Fix a decreasing $n$-vector with distinct parts $v$. We will first show that $P_{v'} \subseteq \phi_v(\mathfrak{A})$.

Let $w$ be any vertex of $P_{v'}$. There exists a permutation $\sigma$ such that $w = (v'_{\sigma(1)}, v'_{\sigma(2)}, \ldots, v'_{\sigma(n)})$ be any vertex of $P_{v'}$, where $\sigma$ is a permutation. This vector may be rewritten as $w = (v_{\mu_{\sigma(1)}}, v_{\mu_{\sigma(2)}}, \ldots, v_{\mu_{\sigma(n)}})$. Notice that the matrix $A = (a_{ij})$ given by

$$a_{ij} = \begin{cases} 1 & j = \mu_{\sigma(i)} \\ 0 & \text{otherwise} \end{cases}$$

is a $\lambda$-ASM satisfying $Av = w$. Therefore, $w \in \mathfrak{A} \cdot v$. Since every vertex of $P_{v'}$ is in $\mathfrak{A} \cdot v$, convexity implies $P_{v'} \subseteq \mathfrak{A} \cdot v$.

To prove $\mathfrak{A} \cdot v \subseteq P_{v'}$, we will appeal to the result by Rado. We must show that for any $\lambda$-ASM $X$, $Xv \preceq v'$.

We can verify (8) by noting that

$$\sum_{i=1}^{n}(Xv)_i = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}v_j = \sum_{j=1}^{m}\left(v_j \sum_{i=1}^{n} x_{ij}\right)$$

Recall that $\sum_{i=1}^{n} x_{ij} = 1$ if $j \in \lambda$ and 0 if $j \notin \lambda$. Hence, the expression above becomes

$$\sum_{i=1}^{n}(Xv)_i = \sum_{j\in\mu} v_j = \sum_{i=1}^{n} v'_i$$

as desired.

We must now verify (7), which is

$$\sum_{i=1}^{k}(Xv)_{[i]} \leq \sum_{i=1}^{k} v'_{[i]} \qquad \text{for all } 1 \leq k \leq n$$

Note that since $v'$ is a decreasing vector, $v'_{[i]}$ is simply $v'_i$. We will show that for any subset $I \subseteq \{1, 2, \ldots, n\}$,

$$\sum_{i\in I} (Xv)_i \leq \sum_{i=1}^{|I|} v_i$$

Inequality (7) follows from this. Notice that

$$\sum_{i\in I}\sum_{j=1}^{m} x_{ij} = \sum_{i\in I} 1 = |I|. \tag{9}$$

Additionally,

$$\sum_{i\in I}\sum_{j=1}^{k} x_{ij} \leq \min\{|I|, k\} \qquad \text{for all } 1 \leq k \leq m \tag{10}$$

since

$$\sum_{i\in I}\sum_{j=1}^{k} x_{ij} \leq \sum_{i\in I}\sum_{j=1}^{m} x_{ij} = |I|$$

and

$$\sum_{i\in I}\sum_{j=1}^{k} x_{ij} = \sum_{j=1}^{k}\sum_{i\in I} x_{ij} \leq \sum_{j=1}^{k} 1 = k.$$

Using these two facts, we see that

$$
\begin{aligned}
\sum_{i\in I}(Xv)_i &= \sum_{i\in I}\sum_{j=1}^{m} x_{ij}v_j \\
&= \sum_{j=1}^{m}\left(v_j\sum_{i\in I}x_{ij}\right) \\
&= \sum_{k=1}^{m-1}\left((v_j - v_{j+1})\sum_{j=1}^{k}\sum_{i\in I}x_{ij}\right) + v_m\sum_{j=1}^{m}\sum_{i\in I}x_{ij} \\
&= \sum_{k=1}^{m-1}\left((v_k - v_{k+1})\sum_{j=1}^{k}\sum_{i\in I}x_{ij}\right) + v_m|I| && \text{by (9)} \\
&= \sum_{k=1}^{|I|-1}\left((v_k - v_{k+1})\sum_{j=1}^{k}\sum_{i\in I}x_{ij}\right) + \sum_{k=|I|}^{m-1}\left((v_k - v_{k+1})\sum_{j=1}^{k}\sum_{i\in I}x_{ij}\right) + v_m|I| \\
&\leq \sum_{k=1}^{|I|-1}(v_k - v_{k+1})k + \sum_{k=|I|}^{m-1}(v_k - v_{k+1})|I| + v_m|I| && \text{by (10)} \\
&= \left(\sum_{k=1}^{|I|-1}v_k\right) - (|I|-1)v_{|I|} + |I|(v_{|I|} - v_m) + v_m|I| \\
&= \left(\sum_{k=1}^{|I|-1}v_k\right) + v_{|I|} \\
&= \sum_{k=1}^{|I|}v_k
\end{aligned}
$$

Hence, (7) is verified. We have shown that for each $\lambda$-ASM X, $Xv \in P_{v'}$. By convexity, $\mathfrak{A}\cdot v \subseteq P_{v'}$.

We have shown containment both ways, so $\mathfrak{A}\cdot v = P_{v'}$ □

# 7 Enumerating $\lambda$-ASMs

## 7.1 Counting by cases

### 7.1.1 Partitions with two parts

### 7.1.2 Partitions with three parts

Let $\mu$ be any parition of the form $(A, B, 0)$, where $A > B \geq 1$. Partitions with three parts will always correspond to an ASM with three rows. Since the first column will have exactly one $+1$ and two 0s, we can use this to split the $\lambda$-ASMs into three main categories, each with their own subcategories, as shown below.

1. There is a $+1$ in the upper left corner of the $\lambda$-ASM
   a. There are no other $+1$s in the top row
   b. There is an additional $+1$ at the top of column 0
   c. There is an additional $+1$ at the top of column $B$
   d. There are $+1$s at the top of columns $B$ and 0
2. There is a $+1$ in the in the left-most spot of the middle row
   a. There are no $+1$s in the middle of column $B$ or 0 (but maybe in the middle row of other columns).
   b. There is an additional in the middle of column 0
   c. There is an additional in the middle of column $B$
   d. There are $+1$s in the middle of columns $B$ and 0
3. There is a $+1$ in the bottom left corner

Case 1.a With the parameters mentioned above, the top row of a lattice model becomes fixed and the second and third rows of the lattice model become fixed until column $B$. The boundary edges surrounding the incomplete portion of the lattice model mirror those of a 2-part partition, allowing us to conclude that the number of $\lambda$-ASMs of this form is $B+1$.

Case 1.b The top row of an ice model will be fixed for each possible location of the $-1$. Let $C$ represent the column containing the $-1$. The second and third rows will be fixed on both the left and right until column $B$ or column $C$, depending on which column comes first. This leaves $A-B-1$ possibilities for the $-1$ to the left of $B$ and $B-1$ possibilities for the $-1$ to the right of $B$. We sum all of the number of possible $\lambda$-ASMs for each possible $C$ on either side of $B$, which yields the following formula for the total number of $\lambda$-ASMs of this form:

$$\sum_{D=1}^{A-B-1}(D+1)+\sum_{D=2}^{B}(D)=\frac{1}{2}(A-B-1)(A-B+2)+\frac{1}{2}(B+2)(B-1)=\frac{1}{2}(A^2+A)+B^2-AB-2.$$

Case 1.c There are $A-B-1$ possibilities for the location of the $-1$. Let the column containing the $-1$ be represented by $C$, where $A>C>B$. For each possible ice model with these restrictions, the top row becomes fixed and the second and third rows become fixed from the left up to $C$. The boundary edges around the unfinished portion of the lattice model will once again mirror those of a 2-part partition, so there are $C+1$ $\lambda$-ASMs. We sum this for all possible $C$s, and get

$$\sum_{C=B+1}^{A-1}(C+1)=\frac{1}{2}(A^2-B^2+A-3B-2)$$

## 7.2 Counting $\lambda$-ASMs with two and three parts by monotone triangles or Gelfand-Tsetlin patterns

Let $\lambda = (\lambda_1, \lambda_2, \ldots, 0)$ be any 'distinct' partition: $\lambda_1 > \lambda_2 > \ldots \geq 1$. As usual, we consider only those $\lambda$-ASMs and skip the formalities of sliding our partition over.

As presented in the bijection between $\lambda$-ASMs and monotone triangles or strict Gelfand-Tsetlin patterns, we find how many ways we can choose "between numbers" between the $\lambda_i's$ as positions in the below rows to put up-arrows in.

Belows are some notations and remarks about $q - binomials$ and integer partitions in form of Young tableaus or Ferrers' diagrams:

a. $[n]!_q = (1-q)\ldots(1-q^n)$

b. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!_q}{[k]!_q[n-k]!_q}$

c. $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$

d. The generating function for Young tableaus or Ferrers' diagrams that fit in a rectangle $n \times k$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

### 7.2.1 Two-part partition

Let $\lambda = (\lambda_1, 0)$, then we need to find $\alpha_1$ such that $\lambda_1 \geq \alpha_1 \geq 0$, which is the position to put an up-arrow on the second row. Here we count the number of partitions with one part, size between 0 and $\lambda_1$. So its Ferrers' diagram fit in a $\lambda_1 \times 1$ rectangle. Generating function:

$$S_2(q) = \begin{bmatrix} \lambda_1 + 1 \\ 1 \end{bmatrix}_q$$

To have the total number of two part $\lambda - ASMs$, set $q = 1$. We have $\lambda_1 + 1$.

### 7.2.2 Three-part partition

Let $\lambda = (\lambda_1, \lambda_2, 0)$, then we need to find $\alpha_1 > \alpha_2$ and $\beta_1$ such that $\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq 0$ and $\alpha_1 \geq \beta_1 \geq \alpha_2$. $\alpha_i's$ are the positions to put up-arrows on the second row. $\beta_1$ is the position to put an up-arrow on the third row. Here we count the number of partitions with three parts, $(\alpha_1, \beta_1, \alpha_2)$. We have two cases:

Case 1: $\beta_1 > \lambda_2$. If we take out $\lambda_2 + 1$ from each $\alpha_1$ and $\beta_1$ then the Ferrers' diagram of $(\alpha_1 - \lambda_1 - 1, \beta_1 - \lambda_1 - 1)$ fits in $(\lambda_1 - \lambda_2 - 1) \times 2$ rectangle. $\alpha_2$ as in the two-part partition section. Generating function for this case is

$$q^{2\lambda_2} \cdot q^2 \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 2 \end{bmatrix}_q \begin{bmatrix} \lambda_2 + 1 \\ 1 \end{bmatrix}_q$$

Case 2: $\beta_1 < \lambda_2$. If we take out $\lambda_1$ from $\alpha_1$ then Ferrers' diagram of $(\alpha_1 - \lambda_1)$ fits in $(\lambda_1 - \lambda_2) \times 1$ rectangle. Ferrers' diagram of $(\beta_1, \alpha_2)$ fits in $(\lambda_2 - 1) \times 2$ rectangle. Generating function for this case is

$$q^{\lambda_2} \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} \lambda_2 + 1 \\ 2 \end{bmatrix}_q$$

Case 3: $\beta_1 = \lambda_2$. Then the generating function for this one is:

$$q^{\lambda_2} \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 1 \end{bmatrix}_q \cdot q^{\lambda_2} \begin{bmatrix} \lambda_2 + 1 \\ 1 \end{bmatrix}$$

However, we cannot have the case when $\alpha_1 = \beta_1 = \alpha_2$, so we need to take out $q^{3\lambda_2}$.

The generating function finally is:

$$S_3(q) = q^{2\lambda_2+2} \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 2 \end{bmatrix}_q \begin{bmatrix} \lambda_2 + 1 \\ 1 \end{bmatrix}_q + q^{\lambda_2} \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} \lambda_2 + 1 \\ 2 \end{bmatrix}_q + q^{2\lambda_2} \begin{bmatrix} \lambda_1 - \lambda_2 + 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} \lambda_2 + 1 \\ 1 \end{bmatrix} - q^{3\lambda_2}$$

As $q = 1$, we have the number of three-part-$\lambda - ASMs$, which is

$$\binom{\lambda_1 - \lambda_2 + 1}{2} \binom{\lambda_2 + 1}{1} + \binom{\lambda_1 - \lambda_2 + 1}{1} \binom{\lambda_2 + 1}{2} + \binom{\lambda_1 - \lambda_2 + 1}{1} \binom{\lambda_2 + 1}{1} - 1$$

### 7.3 Equivalence with $k$-enumerations of ASMs

# 8 Future Directions

[Conjectures and suggestions for future investigation go here.]

# 9 Appendix A: Software Implementations

[Code can be included here. For now, just include it as-is, but if there is time, it would be good to be sure it is well-documented.]

# References

[Ize87] A. G. Izergin. Partition function of a six-vertex model in a finite volume. *Dokl. Akad. Nauk SSSR*, 297(2):331–333, 1987.