Lattice Models and Schur Function Identities

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Famous Identities

Identity

Littlewood-Richardson Rule (Wheeler, Zinn-Justin 2015)

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

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Identity

Dual Cauchy Identity(Bump, McNamara and Nakasuji 2011)

$$\sum_\lambda s_\lambda(x) s_{\lambda'}(y) = \prod_{i,j} (1 + x_i y_j)$$

Famous Identities

Identity

Cauchy Identity: UMN combinatorics REU 2020

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

5-vertex Model

- Start with a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$

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$$\rho = (n-1, n-2, ..., 1, 0) \tag{1}$$

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- Each vertex also has a weight, uniquely determined by adjacent spins

a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2
$+$ / $+$ — z_i — $+$ / $+$	$-$ / $-$ — z_i — $-$ / $-$	$-$ / $+$ — z_i — $+$ / $-$	$+$ / $-$ — z_i — $-$ / $+$	$+$ / $-$ — z_i — $+$ / $-$	$-$ / $+$ — z_i — $-$ / $+$	$-$ / $-$ — z_i — $+$ / $+$	$+$ / $+$ — z_i — $-$ / $-$
1	z_i	0	z_i	0	0	z_i	1

Figure: weights for a vertex v in the i-th row

Partition Function

- The set of all states is denoted $\mathfrak{S}_\lambda$

Definition

The weight of a state $\mathfrak{s} \in \mathfrak{S}_\lambda$ is defined as

$$\mathrm{wt}(\mathfrak{s}) := \prod_{v \in \mathfrak{s}} \mathrm{wt}(v)$$

Definition

The partition function of a model, $\mathfrak{S}_\lambda$, is defined as

$$\mathcal{Z}(\mathfrak{S}_\lambda) := \sum_{\mathfrak{s} \in \mathfrak{S}_\lambda} \mathrm{wt}(\mathfrak{s})$$

Example

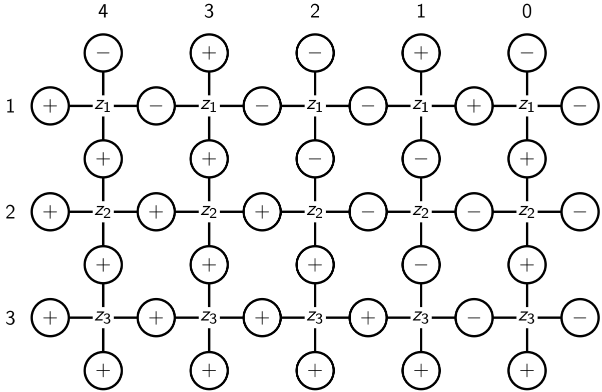
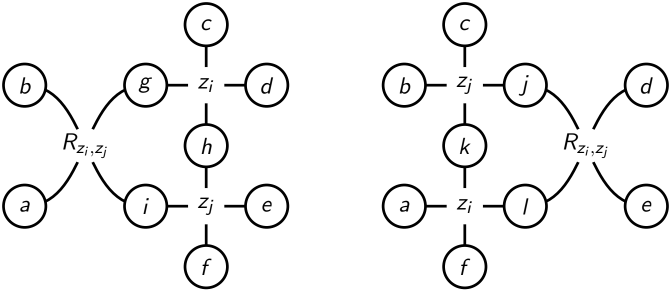


Figure: A state with non-zero weight for a five-vertex model system with $\lambda = (2, 1, 0)$, $n = 3$ $\rho = (2, 1, 0)$

- Goal to define rotated vertices R_{z_i,z_j} so that we have a local symmetry, i.e. partition functions of both sides are equal



A Solution

- If we define the weights of these rotated vertices as follows:

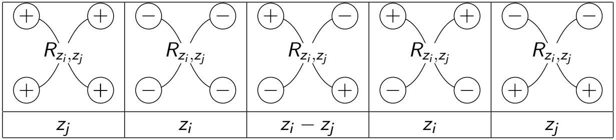


Figure: Boltzmann weights for the rotated vertices which satisfy the YBE

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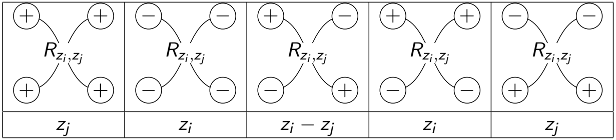
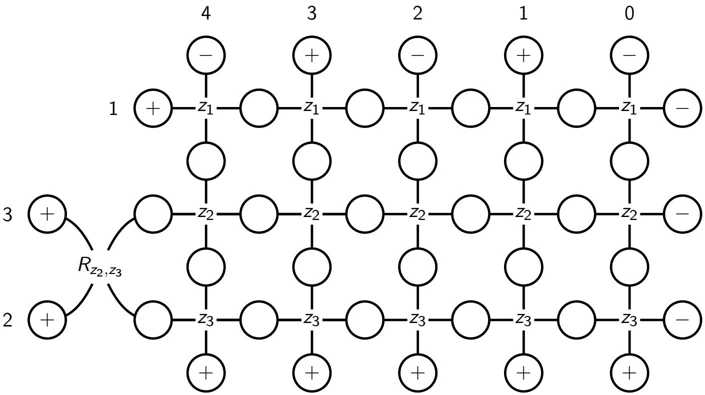


Figure: Boltzmann weights for the rotated vertices which satisfy the YBE

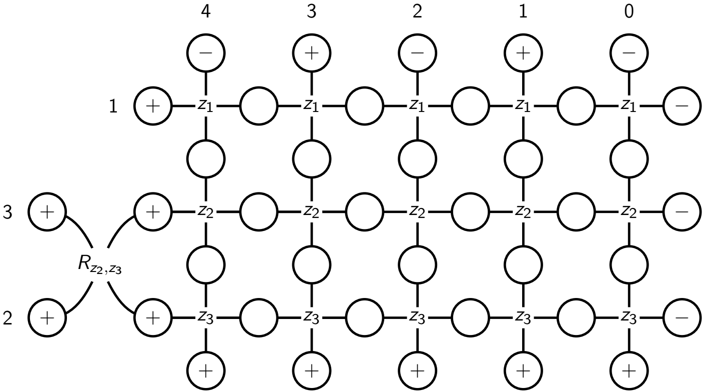
- We utilize the YBE in a method called the train argument

- Attach a rotated vertex to the models for our running example $\lambda = (2, 1, 0)$ to get a new boundary value problem

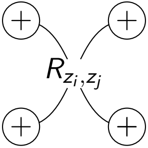


Train Argument

- This model is not so different than the $\mathfrak{S}_{(2,1,0)}$ model



The weight of the rotated vertex

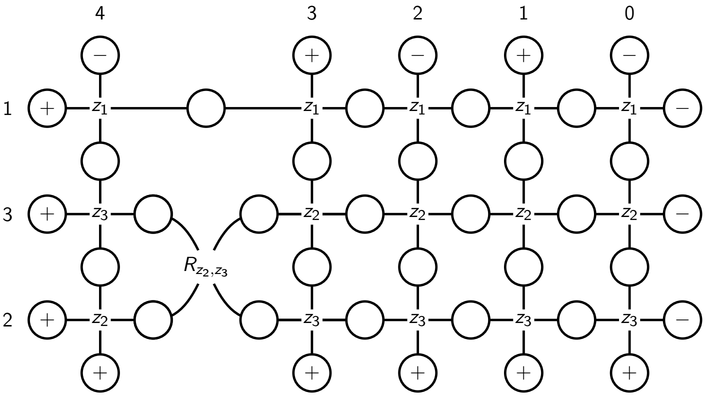


is given by z_j. So we have the partition function of our new models is given by

$$z_3 \mathcal{Z}(\mathfrak{S}_{(2,1,0)})$$

Train Argument

- Apply the YBE to obtain a new model with equal partition function



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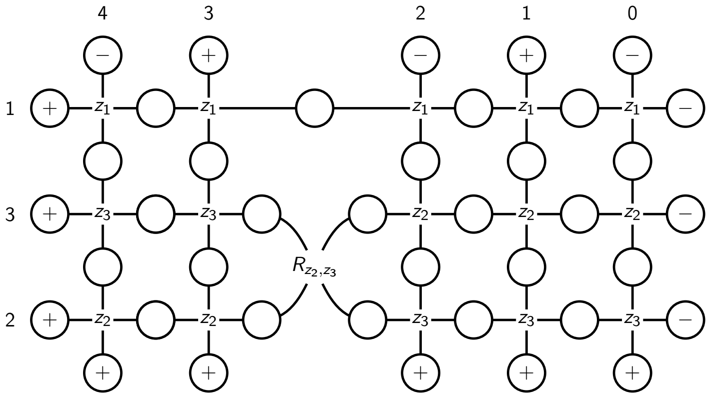

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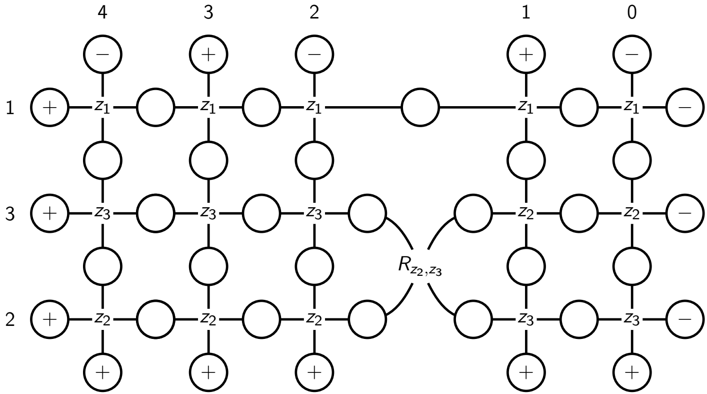

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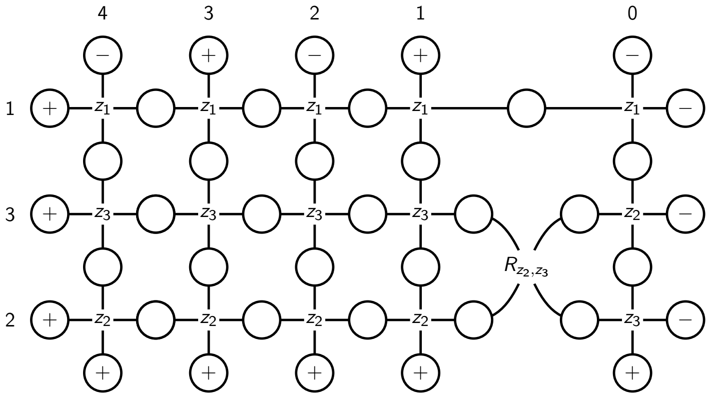
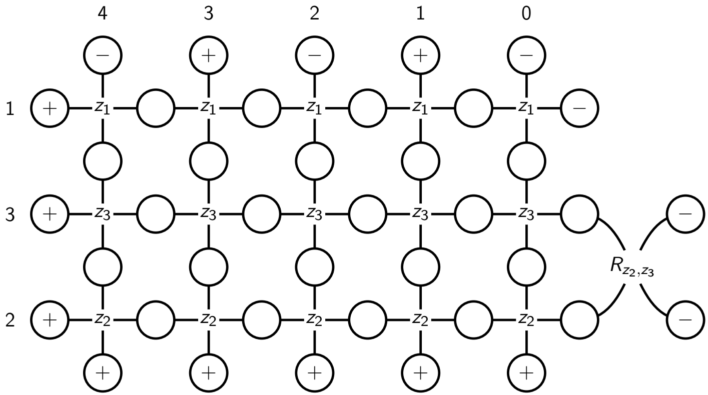
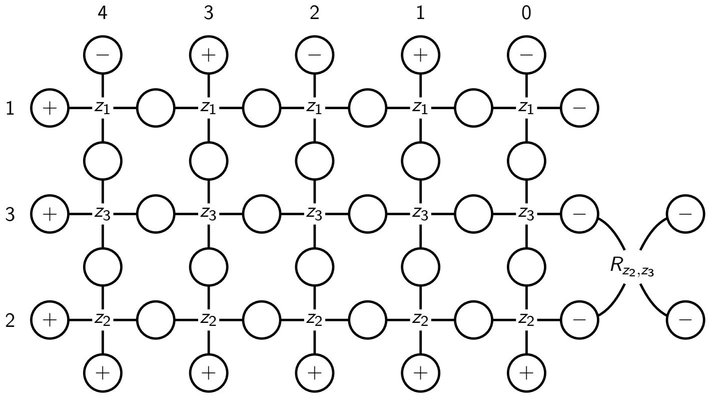


- Finally we obtain equality with following model's partition function



- This model is again very similar to the $\mathfrak{S}_{(2,1,0)}$ model



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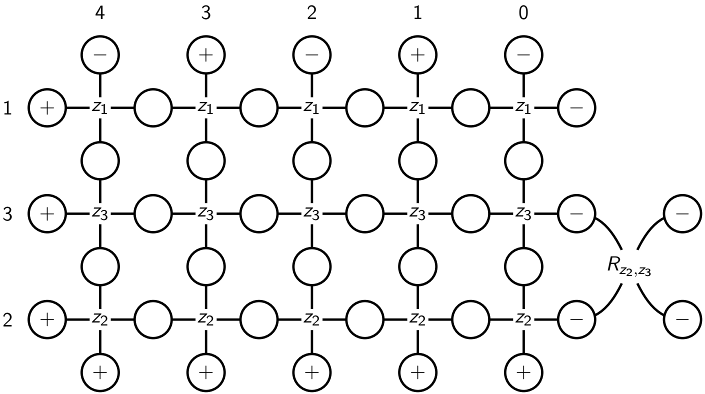
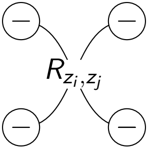


The weight of the rotated vertex



is given by z_i. So in our example with $\lambda = (2,1,0)$ the partition function of the final model is given by

$$z_2 \mathcal{Z}(\widehat{\mathfrak{S}}_{(2,1,0)})$$

Where the $\widehat{\mathfrak{S}}_{(2,1,0)}$ indicates that we have swapped the spectral indices $2 \leftrightarrow 3$.

Upshot

Since the partition functions of the initial and final model are equal we obtain

$$z_3\mathcal{Z}(\mathfrak{S}_{(2,1,0)}) = z_2\mathcal{Z}(\widehat{\mathfrak{S}}_{(2,1,0)})$$

For a arbitrary λ the same argument shows that for $1 \leq k \leq n-1$

$$z_{k+1}\mathcal{Z}(\mathfrak{S}_\lambda)$$

is invariant under $k+1 \leftrightarrow k$. This is very strong!

Relation with Schur Function

Proposition

$$\mathcal{Z}(\mathfrak{S}_\lambda) = z^\rho s_\lambda(z)$$

with $\rho = (n-1, n-2, \ldots, 0)$

Cauchy ID

Identity

Cauchy Identity

$$\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \prod_{i,j}(1 - x_i y_j)^{-1}$$

New Model

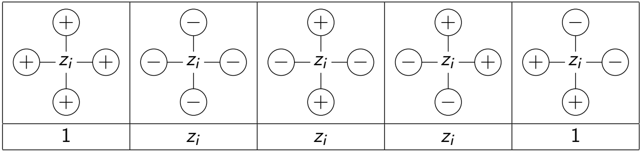


Figure: The Γ-weights for a vertex v in the i-th row

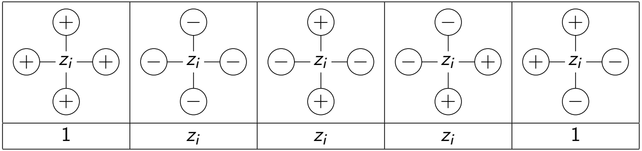


Figure: The Ω-weights for a vertex v in the i-th row

New Model $m = n$

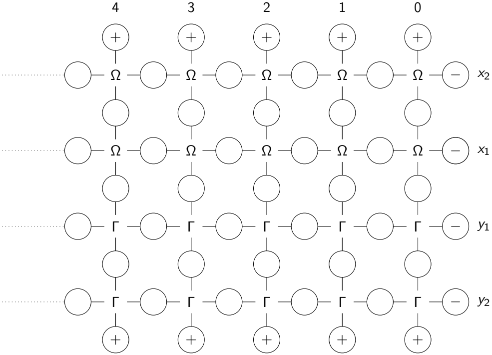


Figure: A diagram of $\mathfrak{S}_{\infty}^{\Omega\Gamma}$ with $n = m = 2$. The dotted lines indicate that the picture continues infinitely to the left.

New Model $m \neq n$

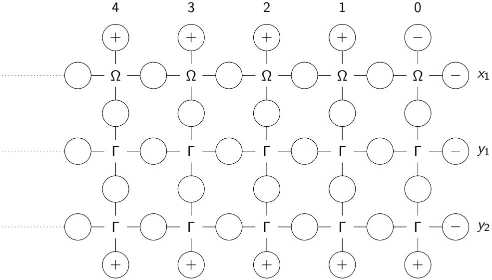


Figure: A diagram of $\mathfrak{S}_{\infty}^{\Omega\Gamma}$ with $n = 1, m = 2$. The dotted lines indicate that the picture continues infinitely to the left.

Partition Function (1)

Theorem

For two finite alphabets of variables $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_m)$ *with* $m \geq n$*, we have*

$$\mathcal{Z}(\mathfrak{S}_{\infty}^{\Omega\Gamma}) = x^{\rho+\kappa} y^{\rho} \sum_{\lambda} s_\lambda(x) s_\lambda(y) \tag{2}$$

where $\kappa = (\underbrace{k, k, \ldots, k}_{n})$ *with* $k = m - n$*. The sum is over all partitions* λ *with at most* $\min(n, m)$ *parts.*

Partition Function(2)

Proposition

We have

$$\mathcal{Z}(\mathfrak{S}_{\infty}^{\Omega\Gamma}) = \sum_{r \geq m} \mathcal{Z}(\mathfrak{S}_{r}^{\Omega\Gamma}) - \mathcal{Z}(\mathfrak{S}_{r-1}^{\Omega\Gamma})$$

Note that $\mathcal{Z}(\mathfrak{S}_{m-1}^{\Omega\Gamma}) = 0$.

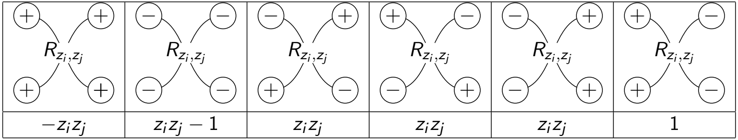


Figure: The Boltzmann weights for $\Gamma\Omega$.

Special Case m=n=1

Proposition

If $n = m = 1$ *then the partition function of the two row half-infinite lattice model* $\mathfrak{S}_{\infty}^{\Omega\Gamma}$*, with spectral parameters* x, y*, is given by*

$$Z(\mathfrak{S}_{\infty}^{\Omega\Gamma}) = \frac{1}{1 - xy}$$

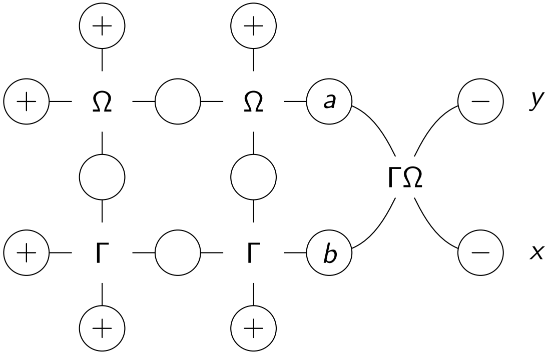
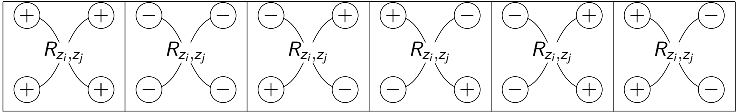


Figure: A diagram of our modified $\mathfrak{S}_r^{\Omega\Gamma}$ model when $n = m = 1$. For illustrative purposes we have taken $r = 2$.

YBE for Different Weight



$-z_i z_j$	$z_i z_j - 1$	$z_i z_j$	$z_i z_j$	$z_i z_j$	1

Figure: The Boltzmann weights for $\Gamma\Omega$.

Special Case m=n=1

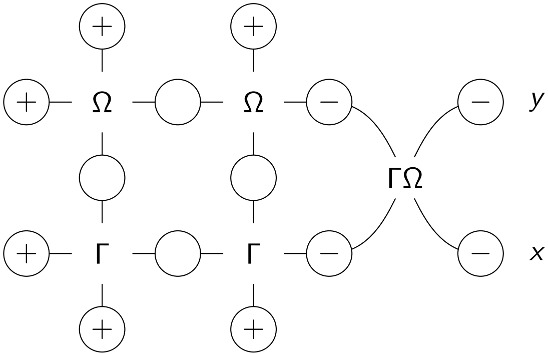


Figure: Weight $= (xy - 1)\mathfrak{S}_r^{\Omega\Gamma}$

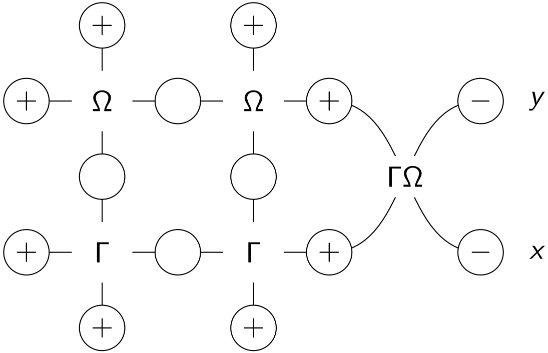


Figure: Weight = 1

Special Case m=n=1

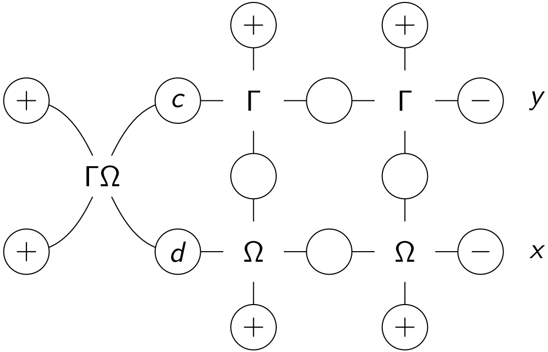
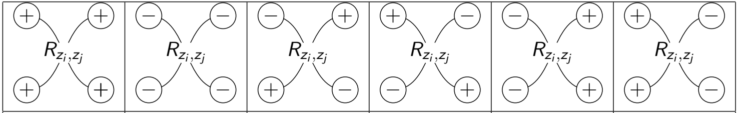


Figure: A diagram of our modified $\mathfrak{S}_r^{\Omega\Gamma}$ model when $n = m = 1$ after applying thm:Gamma Omega ybe r times. For illustrative purposes we have taken $r = 2$.

Special Case m=n=1



$-z_iz_j$	$z_iz_j - 1$	z_iz_j	z_iz_j	z_iz_j	1

Figure: The Boltzmann weights for $\Gamma\Omega$

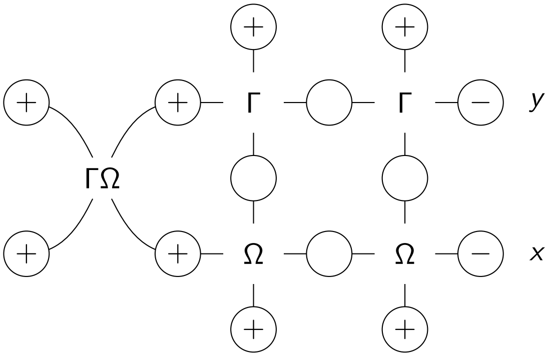


Figure: Weight is 0 because of no admissible state

Special Case m=n=1

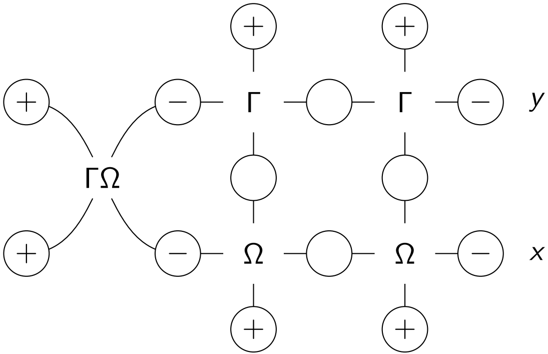


Figure: Weight= $(xy)^r$.

Special Case m=n=1

Proof:

$$1 + (xy - 1)Z(\mathfrak{S}_r^{\Omega\Gamma}) = (xy)^r$$

$$Z(\mathfrak{S}_r^{\Omega\Gamma}) = \frac{(xy)^r - 1}{xy - 1} = \sum_{j=0}^{r-1}(xy)^j$$

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = 1 + \sum_{r>1}\left(\sum_{j=0}^{r-1}(xy)^j - \sum_{j=0}^{r-2}(xy)^j\right)$$

$$= \sum_{r\geq 0}(xy)^r$$

$$= \frac{1}{1 - xy}$$

Braid R Vertex

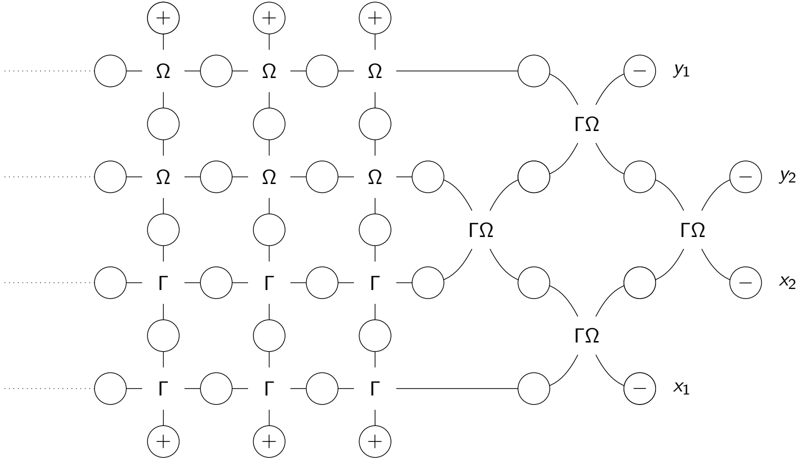


Figure: A diagram of the proposed augmented model in the rank 2 case, i.e. $n = m = 2$.

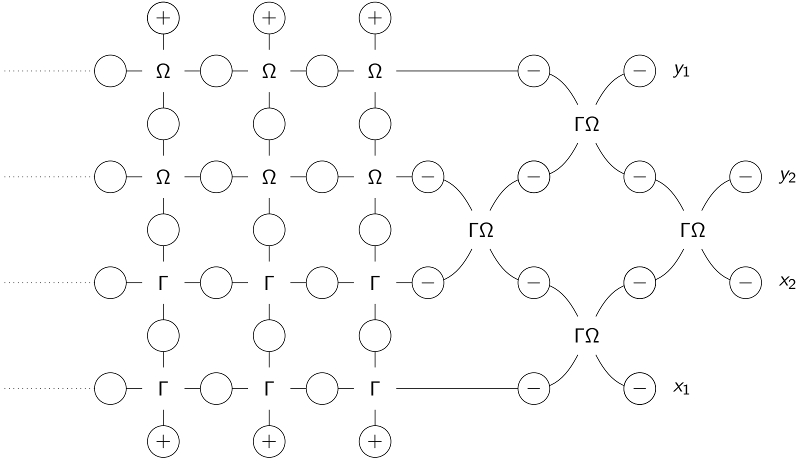


Figure: A diagram of the proposed augmented model in the rank 2 case, i.e. $n = m = 2$.

All Minus Spins

- If every spin on the braid is minus sign, then the weight for the braid is $\prod_{i,j}(1 - x_i y_j)^{-1}$
- If every spin on the braid is minus sign, then the weight of lattice attached to the braid is what we want.

Further Possible Steps

- Try to pair up the braids to find nice cancellation.
- Push the braid to the left to permute the parameter.
- Expand the result from $m = n$ to $m \neq n$.

Reference

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- D. Bump, P.J. McNamara, and M.Nakasuji: Factorial Schur functions and the YangBaxter equation. (2011), arXiv: 1108.3087
- M. Wheeler, P. Zinn-Justin: *Refined Cauchy/Littlewood identities and six-vertex model partition functions: III. Deformed bosons* (2015),arXiv:1508.02236

Thank you!

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