Lattice Models, Differential Forms, and the Yang-Baxter Equation

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University of Minnesota REU 2020

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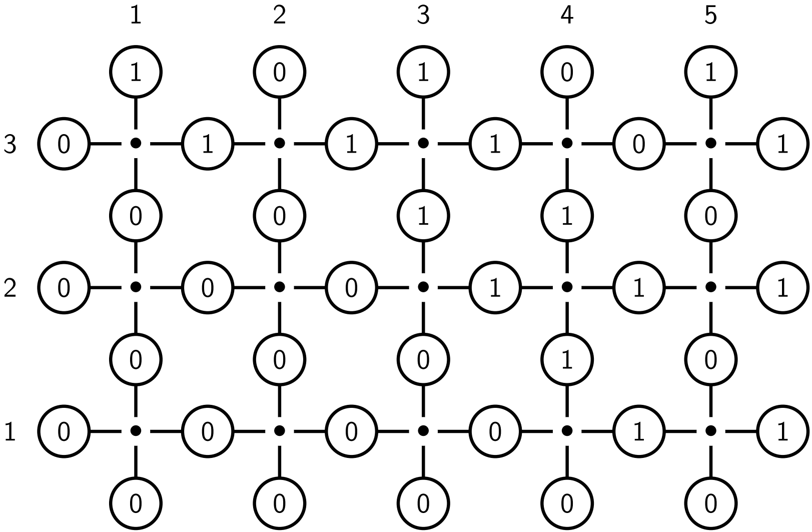
What is a Lattice Model?



What is a Lattice Model?

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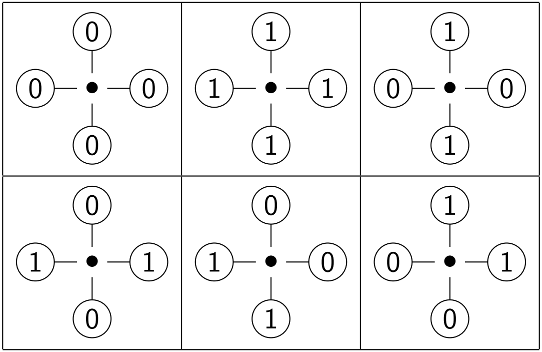
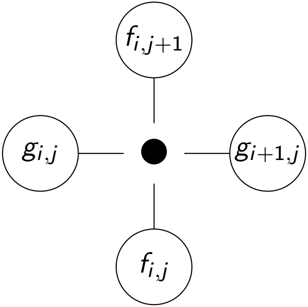
- Grid with labeled edges.
- Labelings around a vertex locally satisfy some property.

Six-Vertex Model



Six-Vertex Model

- Observation: A state



is admissible iff

$$f_{i,j+1} - f_{i,j} \equiv g_{i+1,j} - g_{i,j} \pmod 3.$$

Differential Forms

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\begin{aligned}
& f_{i,j+1} - f_{i,j} \equiv g_{i+1,j} - g_{i,j} \pmod 3 \\
& \Leftrightarrow D_y f = D_x g \\
& \Leftrightarrow f dx + g dy \text{ is closed.}
\end{aligned}
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- Admissible 1-form $fdx + gdy$: f and g only equal 0 and 1.
- So admissible states $\leftrightarrow$ closed admissible 1-forms.

Differential Forms

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Lemma

Every closed 1-form on $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ is exact.

3-Colorings

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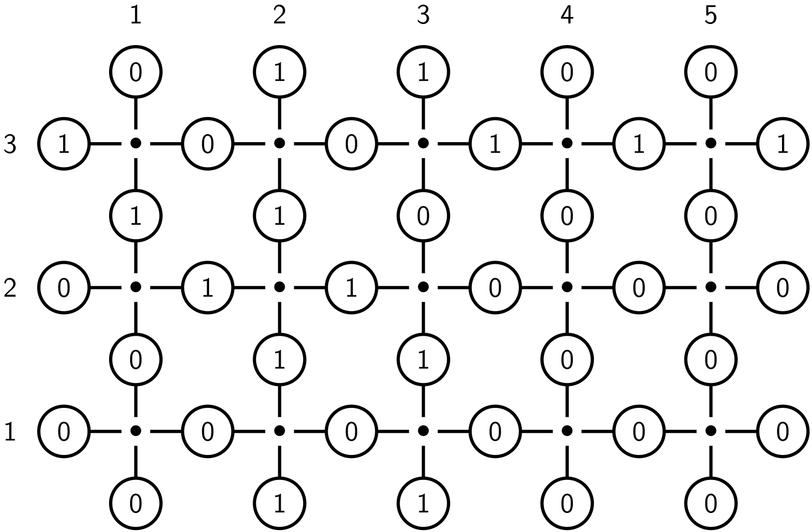
- Using this correspondence, we can prove

Theorem

We have a one-to-one correspondence

$$\{\textit{Admissible states}\} \leftrightarrow \{\textit{3-colorings of a rectangular grid}\} \times \mathbb{F}_3.$$

Toroidal Boundary Conditions



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Lemma

Every closed 1-form on the discrete torus can be written uniquely in the form

$$rdx + sdy + \omega,$$

where $r, s \in \mathbb{F}_3$ and ω is exact.

Toroidal Boundary Conditions

- 3-colorings of a rectangular grid $\leftrightarrow$ functions h such that $D_x h, D_y h \neq 0$, and $h_{1,1} = 0$.

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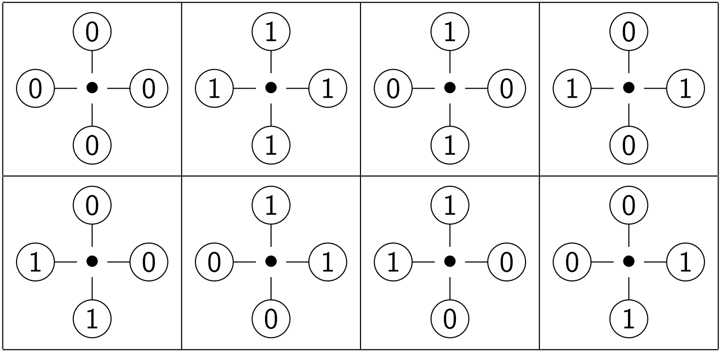
Toroidal Boundary Conditions

- 3-colorings of a rectangular grid $\leftrightarrow$ functions h such that $D_x h, D_y h \neq 0$, and $h_{1,1} = 0$.
- Call h *sparse* if neither $D_x h$ nor $D_y h$ are surjective, and $h_{1,1} = 0$.
- No nice correspondence with 3-colorings in toroidal case, but we have

Theorem

There is a one-to-one correspondence between sparse functions and admissible states of the six-vertex model with toroidal boundary conditions.

Eight-Vertex Model



Eight-Vertex Model

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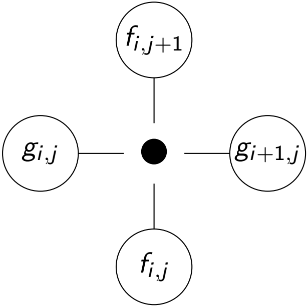

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- Set of admissible states is a vector space over $\mathbb{F}_2$.
- Everything is a linear condition.
- Easy to count the number of admissible states.

Theorem

The number of admissible states of the eight-vertex model is 2^{m+n+mn}.

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- By linear algebra, this essentially does not depend on what the boundary conditions are.
- Admissible states of "homogeneous lattice" $\leftrightarrow$ Admissible states of lattice with given boundary conditions.

$$L_0 \mapsto L_B + L_0$$

Eight-Vertex Boundary Conditions

- New question: when does a set of boundary conditions have an admissible state?

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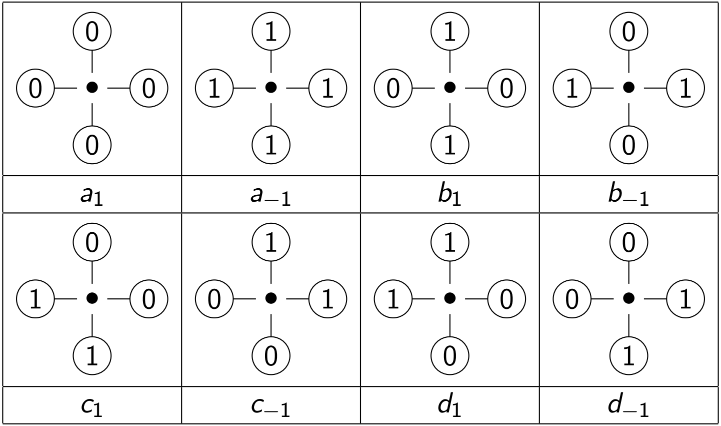
- New question: when does a set of boundary conditions have an admissible state?
- Answer: when the boundary values sum to 0.

Theorem

Let B be a set of boundary values that sum to 0. Then the number of admissible states with boundary conditions B is $2^{(m-1)(n-1)}$.

Adding Weights



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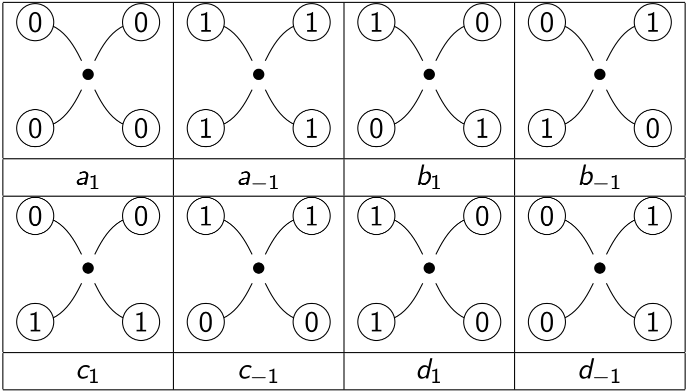
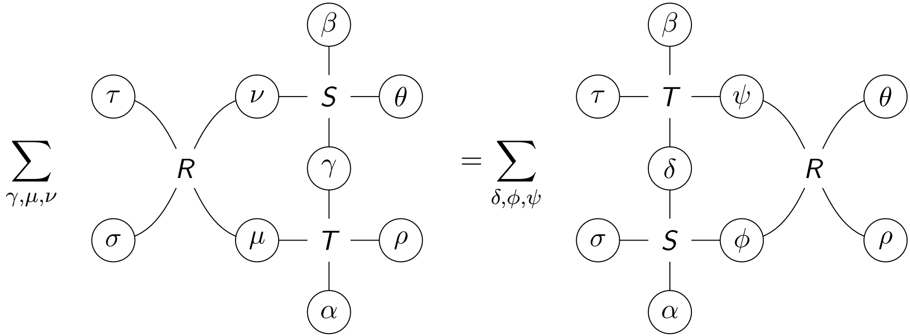


Yang-Baxter Equation



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Yang-Baxter Equation

- Question: Given S and T, when does there exist (nontrivial) R such that YBE holds?
- Galleas and Martins [2] answered this question in the case $c_1 = c_{-1}$ and $d_1 = d_{-1}$.
- YBE can be expressed as a matrix equation

$$R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12} = 0.$$

Explicit Computations

$$
\begin{aligned}
a_j(T)a_j(S)d_i(R) + d_i(T)c_i(S)a_{-j}(R) &= c_i(T)d_i(S)a_j(R) + b_{-j}(T)b_{-j}(S)d_i(R)\\
d_i(T)b_j(S)c_i(R) + a_j(T)d_i(S)b_{-j}(R) &= b_j(T)d_i(S)a_j(R) + c_{-i}(T)b_{-j}(S)d_i(R)\\
d_i(T)b_j(S)b_j(R) + a_j(T)d_i(S)c_{-i}(R) &= d_i(T)a_j(S)a_j(R) + a_{-j}(T)c_{-i}(S)d_i(R)\\
c_i(T)a_j(S)c_i(R) + b_j(T)c_i(S)b_{-j}(R) &= a_j(T)c_i(S)a_j(R) + d_{-i}(T)a_{-j}(S)d_i(R)\\
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b_{-j}(T)a_j(S)c_i(R) + c_{-i}(T)c_i(S)b_{-j}(R) &= d_{-i}(T)d_i(S)b_j(R) + a_j(T)b_{-j}(S)c_i(R)\\
c_1(T)c_{-1}(S)c_1(R) &= c_{-1}(T)c_1(S)c_{-1}(R)\\
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d_1(T)d_{-1}(S)c_1(R) &= d_{-1}(T)d_1(S)c_{-1}(R)
\end{aligned}
$$

Necessary Conditions

Theorem

Necessary conditions for a solution with $c_{-1}(R), c_1(R), d_{-1}(R), d_1(R)$ nonzero include

$$
\begin{aligned}
a_1(T)b_1(T)F(S) &= a_{-1}(T)b_{-1}(T)F(S) \\
a_1(S)b_1(S)F(T) &= a_{-1}(S)b_{-1}(S)F(T) \\
\frac{c_i(T)d_{-i}(T)}{c_{-i}(T)d_i(T)} G_i(S,T)^2 &= [a_1(T)b_1(T)F(S) - a_1(S)b_1(S)F(T)]^2 \\
\frac{c_1(T)c_{-1}(S)}{c_{-1}(T)c_1(S)} &= \frac{d_1(T)d_{-1}(S)}{d_{-1}(T)d_1(S)}.
\end{aligned}
$$

Acknowledgements

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