F-POLYNOMIALS FOR THE R-KRONECKER QUIVER

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1. Introduction

Fomin and Zelevinsky introduced cluster algebras in 2001 [FZ01] and F-polynomials in 2006 [FZ06]. Cluster algebras are commutative algebras that can be defined by a quiver. We avoid going into the definition of cluster algebras, but briefly review how to get an F-polynomial from a quiver and a mutation sequence.

Definition 1.1 ((Framed) Quiver)**.** A *quiver* is a directed graph with no 2-cycles, where multiple edges are allowed. Given a quiver Q with n vertices labelled $\{1, 2, \ldots, n\}$, the corresponding *framed quiver* $\widetilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ is the original quiver Q with n additional vertices $\{1', 2', \ldots, n'\}$ and one directed edge from i' to i for each $1 \leq i \leq n$.

Example 1.2 (r-Kronecker Quiver)**.** For $r \geq 2$, *the r-Kronecker* quiver is the quiver with two vertices and r arrows pointing from 1 to 2. The r The following diagram shows the r-Kronecker quiver and the corresponding framed quiver when $r = 3$:

$$\begin{tikzcd} 1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 2 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 2 \\ 1' \arrow[u] & 2' \arrow[u] \end{tikzcd}$$

The 2-Kronecker quiver is also called *the Kronecker quiver*.

A *cluster seed* is an assignment of Laurent polynomials to the vertices of $\widetilde{Q}$. These Laurent polynomials are called *cluster variables*. Starting with $\widetilde{Q}$ and a cluster seed, we may mutate the quiver at a vertex.

Definition 1.3 (Mutation at vertex i)**.** Given a framed quiver $\widetilde{Q}$ with vertices $\{1, \ldots, n, 1', \ldots, n'\}$ and associated cluster variables $x_1, \ldots x_n, x_{1'}, \ldots, x_{n'}$, mutation at vertex i consists of the following steps:

(1) For every path $j \to i \to k$, draw an edge $j \to k$;
(2) Reverse the direction of all edges incident to i;
(3) Delete all 2-cycles;
(4) Update the cluster variable at vertex i to be
$$x_i' = \frac{\prod_{j \to i} x_j + \prod_{i \to k} x_k}{x_i}.$$

By the Laurent phenomenon, the cluster variables we obtain will still be Laurent polynomials. So after a mutation, we obtain a different quiver and cluster seed that we can perform another mutation on.

A mutation sequence is denoted $\mu_{i_1}\mu_{i_2}\mu_{i_3}\ldots$ where $i_1, i_2, i_3, \ldots$ is a (potentially infinite) sequence of vertices of a quiver Q. Note in particular that the primed vertices are not allowed in a mutation sequence. A mutation sequence $\mu_{i_1}\mu_{i_2}\mu_{i_3}\ldots$ is applied to an initial framed quiver and cluster seed by mutating at i_1, then $i_2, i_3, \ldots$.

Definition 1.4 (F-polynomial)**.** Given a mutation sequence $\mu_{i_1}\mu_{i_2}\mu_{i_3}\ldots$ and a framed quiver $\widetilde{Q}$, the ℓ-th *F-polynomial* is the cluster variable at the vertex i_ℓ after applying the mutations $\mu_{i_1}\ldots\mu_{i_\ell}$ to the framed quiver with initial cluster seed $x_i = 1$, $x_{i'} = y_i$.

Example 1.5. We show the first few steps of mutations of the 3-Kronecker quiver. Since the number of edges gets large, we collect edges of the same direction and label them with their multiplicity whenever the multiplicity is greater than one.

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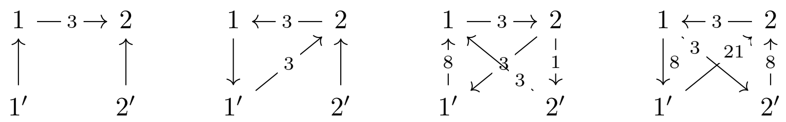

FIGURE 1. 3-Kronecker quiver under mutations $\mu_1\mu_2\mu_1$

If we set our initial cluster seed to be $x_1, x_2 = 1$ and $x_{1'} = y_1, x_{2'} = y_2$, then at the last step, we would obtain the third F-polynomial, which is the updated cluster variable at 1. By definition and inspecting the last quiver shown, we get

$$F_3 = \frac{F_2^3 + y_1^8 y_2^3}{F_1},$$

where F_2 is the cluster variable at 2, F_1 is the previous cluster variable at 1. This pattern holds in general, which implies the following recurrence relation for the F-polynomials of the r-Kronecker quiver:

$$F_{\ell+1}F_{\ell-1} = F_\ell^r + y_1^{a_{\ell+1}} y_2^{a_\ell}.$$

In this report, we investigate the F-polynomials of the r-Kronecker quiver associated to an alternating sequence of mutations, which to our knowledge have two closed-form formulas for general $r \geq 2$, one by [Gup18] and one by [Lee12]. In [Gup18], Gupta gives a formula for the F-polynomial of a generalized framed quiver and an arbitrary sequence of mutations, which we specialize to the r-Kronecker below. In [Lee12], Lee gives a formula for the Laurent expansion of cluster variables for rank-2 cluster algebras, which can be modified and specialized to get an expression for the F-polynomials of r-Kronecker quivers. The two formulas look strikingly similar. To state their formulas, we must first define some notation.

We use two kinds of binomial coefficients and use parentheses and brackets to distinguish them.

Definition 1.6. For $N, s \in \mathbb{Z}$,

$$\binom{N}{s} := \begin{cases} \prod_{i=1}^{s} \frac{N-i+1}{i}, & \text{if } s > 0, \\ 1, & \text{if } s = 0, \\ 0, & \text{if } s < 0. \end{cases}$$

$$\begin{bmatrix} N \\ s \end{bmatrix} := \begin{cases} \prod_{i=0}^{N-s-1} \frac{N-i}{N-s-i}, & \text{if } N > s, \\ 1, & \text{if } N = s, \\ 0, & \text{if } N < s. \end{cases}$$

The $\binom{N}{s}$ coefficients are the standard generalized binomial coefficients, whereas the $\begin{bmatrix} N \\ s \end{bmatrix}$ coefficients are not standard. In general,

$$\binom{N}{s} = \begin{bmatrix} N \\ N-s \end{bmatrix}.$$

In [*Lee*12], the formula for F-polynomials is stated in terms of the bracketed binomial coefficients to simplify notation. We opt to retain this simplication and keep two kinds of binomial coefficients in our report.

Definition 1.7. Given $r \geq 2$, let $a_{n,r}$ be the sequence defined by the following recurrence:

$$a_0 = 0, a_1 = 1, a_{n+2} = ra_{n+1} - a_n \text{ for all } n \in \mathbb{Z}.$$

Because the parameter r is generally clear from context, we typically omit it and simply write a_n.

Note that we allow negative n in the above definition, though negative indices appear only once in this report in the proof of a lemma in Section 2.

We are now ready to state Lee's and Gupta's formulas.

Theorem 1.8 (Theorem 3.1, [Gup18])**.** For the r-Kronecker quiver and the mutation sequence $\mu_1\mu_2\mu_1\mu_2\ldots$, the F-polynomial has the following formula:

$$(1) \qquad F_\ell(y_1, y_2) = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \prod_{i=1}^{\ell} \binom{a_{\ell-i+1} - r\sum_{j=i+1}^{\ell} a_{j-i}m_j}{m_i} y_1^M y_2^N.$$

where

$$M = a_1m_1 + a_2m_2 + \cdots + a_\ell m_\ell,$$
$$N = a_1m_2 + a_2m_3 + \cdots + a_{\ell-1}m_\ell.$$

Theorem 1.9 (Theorem 2.1, [Lee12])**.** For the r-Kronecker quiver and the mutation sequence $\mu_1\mu_2\mu_1\mu_2\ldots$, the F-polynomial has the following formula:

$$F_\ell(y_1, y_2) = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}^\ell} \prod_{i=1}^{\ell} \left[\begin{matrix} a_{\ell-i+1} - r\sum_{j=i+1}^{\ell} a_{j-i}m_j \\ m_i \end{matrix} \right] y_1^M y_2^N \tag{2}$$

where the summation is taken over tuples $(m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell$ such that

$$a_1m_1 + a_2m_2 + \cdots + a_\ell m_\ell = M$$
$$a_1m_2 + a_2m_3 + \cdots + a_{\ell-1}m_\ell = N$$
$$a_\ell N \le a_{\ell-1}M$$
$$\text{for } 2 \le i \le \ell,\ 0 \le m_i \le a_{\ell-i+1} - r\sum_{j=i+1}^{\ell} a_{j-i}m_j$$

Notably, while Gupta's formula requires that all $m_i \ge 0$, Lee's formula potentially allows $m_1 < 0$. Lee's formula sums over a finite number of tuples, implying in a straightforward manner that one recovers a polynomial from his formula. However, in the case of Gupta's formula, polynomiality is a consequence of nontrivial cancellations (discussed in more detail in Section 5). Both formulas leave the positivity of some of the coefficients as mysterious consequences.

Outline. We focus mainly on Gupta's formula because her expression for the F-polynomial coefficients satisfies a nice recurrence. We state a conjecture that makes clear exactly when the coefficients are positive, which we prove partially using this recurrence. We then focus on the $r = 2$ case. There is a well-known formula (Theorem 4.1 of [CZ06], Theorem 2.2 of [Zel06]) for the F-polynomials of the Kronecker quiver. We show that the coefficients given by the well-known formula satisfy the recurrence implied by Gupta's result, which then recovers this formula from Gupta's. We do so in two ways: an algebraic method that uses an identity of hypergeometric series, and a combinatorial method by interpreting the coefficients as counting certain subsets of integers (Theorem 2 and 3 of [MP06]).

2. r-Kronecker Quiver

Let

$$C_{M,N}^{(\ell,r)} := \sum_{\substack{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\ge 0}^\ell \\ a_1m_1+a_2m_2+\cdots+a_\ell m_\ell = M \\ a_1m_2+a_2m_3+\cdots+a_{\ell-1}m_\ell = N}} \prod_{i=1}^{\ell} \binom{a_{\ell-i+1} - r\sum_{j=i+1}^{\ell} a_{j-i}m_j}{m_i},$$

which is the coefficient of $y_1^M y_2^N$ in the F-polynomial. By pulling out the first binomial and reindexing the m_i's, we can show that these coefficients satisfy the following recurrence:

$$C_{M,N}^{(\ell,r)} = \sum_{k\ge 0} \binom{a_\ell - rN}{k} C_{N,rN-M+k}^{(\ell-1,r)}. \tag{3}$$

We have two conjectures regarding these coefficients.

Conjecture 1 (Polynomiality)**.** $C_{M,N}^{(\ell,r)} \ne 0$ if and only if $0 \le M \le a_\ell$, $0 \le N \le \frac{a_{\ell-1}}{a_\ell}M$.

Conjecture 2 (Positivity)**.** When $0 \le M \le a_\ell$, $0 \le N \le \frac{a_{\ell-1}}{a_\ell}M$, $C_{M,N}^{(\ell,r)} > 0$.

We know a lot about only-if direction of Conjecture 1 by appealing to properties of F-polynomials. Based on the F-polynomial recurrence for the r-Kronecker quiver discussed in Example 1.5, by induction we can show that the highest degree in y_1 is indeed a_ℓ. We also know that in general F-polynomials have positive coefficients, which would imply the nonnegativity of the $C_{M,N}^{(\ell,r)}$'s. However, we have not been able to completely recover these results from the formula for the coefficients $C_{M,N}^{(\ell,r)}$ given above, which would be an interesting result that leads to deeper understanding of these coefficients. In addition, we don't know much about the if direction of Conjecture 1.

We prove the only-if direction of Conjecture 1 partially.

Theorem 2.1. For all $M \geq 0$, $C^{(\ell,r)}_{M,N} \neq 0$ only if $0 \leq N \leq \frac{a_{\ell-1}}{a_\ell} M$.

Proof. We have $N \geq 0$ by definition. To show that $N \leq \frac{a_{\ell-1}}{a_\ell} M$ is needed to ensure $C^{(\ell,r)}_{M,N} \neq 0$, we proceed by induction on ℓ. Suppose that the claim is true for ℓ. We want to show that for any $M \geq 0$, if $N > \frac{a_\ell}{a_{\ell+1}} M$, then $C^{(\ell+1,r)}_{M,N} = 0$. Using the recurrence (3),

$$C^{(\ell+1,r)}_{M,N} = \sum_{k \geq 0} \binom{a_{\ell+1} - rN}{k} C^{(\ell,r)}_{N,rN-M+k}.$$

But

$$rN - M > (r - \frac{a_{\ell+1}}{a_\ell})N = \frac{a_{\ell-1}}{a_\ell} N.$$

By the inductive hypothesis, since $rN - M + k \geq rN - M > \frac{a_{\ell-1}}{a_\ell} N$, $C^{(\ell,r)}_{N,rN-M+k} = 0$ for all $k \geq 0$. So $C^{(\ell+1,r)}_{M,N} = 0$ as desired. $\square$

Given the bounds on N above, we may rewrite the recurrence (3) as the following factorization of generating functions.

Lemma 2.2. For $\ell \geq 2$,

$$\sum_{M=\lceil \frac{a_\ell}{a_{\ell-1}} N \rceil}^{\infty} C^{(\ell,r)}_{M,N} x^{M - \lceil \frac{a_\ell}{a_{\ell-1}} N \rceil} = (1+x)^{a_\ell - rN} \sum_{L=0}^{\lfloor \frac{a_{\ell-2}}{a_{\ell-1}} N \rfloor} C^{(\ell-1,r)}_{N,L} x^{\lfloor \frac{a_{\ell-2}}{a_{\ell-1}} N \rfloor - L}.$$

If we further assume Conjecture 1, then we could replace the upper bound on M in the summation on the left hand side with a_ℓ.

Proof. It suffices to show that the coefficient of $x^{M - \lceil \frac{a_\ell}{a_{\ell-1}} N \rceil}$ on both sides are the same. We may write the coefficient of $x^{M - \lceil \frac{a_\ell}{a_{\ell-1}} N \rceil}$ on the left hand side as

$$C^{(\ell,r)}_{M,N} = \sum_{k \geq 0} \binom{a_\ell - rN}{k} C^{(\ell-1,r)}_{N,rN-M+k},$$

which is the coefficient of x^A on the right hand side, where

$$A = k + \lfloor \frac{a_{\ell-2}}{a_{\ell-1}} N \rfloor - (rN - M + k) = M + \lfloor (\frac{a_{\ell-2}}{a_{\ell-1}} - r) N \rfloor = M - \lceil \frac{a_\ell}{a_{\ell-1}} N \rceil$$

as desired. $\square$

For example, when $r = 3$, $\ell = 5$ and $N = 20$,

$$\sum_{M=\lceil \frac{a_\ell}{a_{\ell-1}} N \rceil}^{a_\ell} C^{(\ell,r)}_{M,N} x^{M - \lceil \frac{a_\ell}{a_{\ell-1}} N \rceil} = \sum_{M=\lceil \frac{55}{21} 20 \rceil}^{55} C^{(5,3)}_{M,20} x^{M - \lceil \frac{55}{21} 20 \rceil} = 15 + 39x + 20x^2.$$

On the right hand side,

$$\begin{aligned} &(1+x)^{a_\ell - rN} \sum_{L=0}^{\lfloor \frac{a_{\ell-2}}{a_{\ell-1}} N \rfloor} C^{(\ell-1,r)}_{N,L} x^{\lfloor \frac{a_{\ell-2}}{a_{\ell-1}} N \rfloor - L} \\ &= (1+x)^{55 - 3 \cdot 20} \sum_{L=0}^{\lfloor \frac{8}{21} 20 \rfloor} C^{(4,3)}_{20,L} x^{\lfloor \frac{8}{21} 20 \rfloor - L} \\ &= (1+x)^{-5} (15 + 114x + 366x^2 + 645x^3 + 675x^4 + 420x^5 + 144x^6 + 21x^7). \end{aligned}$$

Using the definition, we can compute the coefficients explicitly for special values of M and N.

Lemma 2.3. For $0 \leq N < r$,

$$C^{(\ell,r)}_{M,N} = \binom{a_\ell - rN}{M - rN} \binom{a_{\ell-1}}{N}$$

Proof. If for some $(m_1, \dots, m_\ell) \in \mathbb{Z}^\ell_{\geq 0}$,

$$\sum_{i=2}^{\ell} a_{i-1} m_i = N < r = a_2,$$

we must have $m_2 = N$ and $m_i = 0$ for $i > 2$. And if furthermore,

$$\sum_{i=1}^{\ell} a_i m_i = M,$$

then $m_1 = M - rm_2 = M - rN$. Therefore, there is only one possible tuple that $C^{(\ell,r)}_{M,N}$ sums over and

$$C^{(\ell,r)}_{M,N} = \binom{a_\ell - rN}{M - rN}\binom{a_{\ell-1}}{N}.$$

□

Lemma 2.4. For $r \geq 2$, $\ell \geq 1$,

$$C^{(\ell,r)}_{a_\ell, a_{\ell-1}} = 1.$$

Proof. Since

$$\sum_{i=1}^{\ell} a_i m_i = a_\ell$$

and

$$\sum_{i=1}^{\ell} a_{i-1} m_i = a_{\ell-1},$$

for all $k \geq 0$, induction shows that

$$\sum_{i=1}^{\ell} a_{i-k} m_i = a_{\ell-k}.$$

In particular, if we let $k = \ell - 1$, then

$$m_\ell = 1 - \sum_{i=1}^{\ell-1} a_{i-\ell} m_i.$$

Since $a_i < 0$ when $i < 0$, the right hand side is positive. But if $m_\ell > 0$ and

$$\sum_{i=1}^{\ell} a_i m_i = a_\ell,$$

we must have $m_\ell = 1$ and $m_i = 0$ for all $1 \leq i < m_\ell$. □

Theorem 2.5. When $a_\ell - rN > 0$,

$$C^{\ell,r}_{a_\ell,N} = \binom{a_{\ell-1}}{N}.$$

Assuming Conjecture 1, this is true for $0 \leq N \leq a_{\ell-1}$.

Proof. When $a_\ell - rN > 0$, summands of

$$C^{(\ell,r)}_{a_\ell,N} = \sum_{k=0}^{\infty} \binom{a_\ell - rN}{k} C_{N, rN - a_\ell + k}$$

is nonzero only if

$$rN - a_\ell + k \geq 0$$

and

$$k \leq a_\ell - rN.$$

So we must have $k = a_\ell - rN$. Hence

$$C^{(\ell,r)}_{a_\ell,N} = \binom{a_\ell - rN}{a_\ell - rN} C_{N, rN - a_\ell + a_\ell - rN} = C^{(\ell-1,r)}_{N,0} = \binom{a_\ell}{N}$$

by Lemma 2.3. If Conjecture 1 holds, then

$$\sum_{M=\lceil \frac{a_{\ell+1}}{a_\ell} a_\ell \rceil}^{\infty} C^{(\ell+1,r)}_{M,a_\ell} x^{M - \lceil \frac{a_{\ell+1}}{a_\ell} a_\ell \rceil} = \sum_{M=a_{\ell+1}}^{a_{\ell+1}} C^{(\ell+1,r)}_{M,a_\ell} x^{M - a_{\ell+1}} = C^{(\ell+1,r)}_{a_{\ell+1},a_\ell} = 1.$$

But by Lemma 2.2,

$$\sum_{M=\lceil \frac{a_{\ell+1}}{a_\ell} a_\ell \rceil}^{\infty} C^{(\ell+1,r)}_{M,a_\ell} x^{M-\lceil \frac{a_{\ell+1}}{a_\ell} a_\ell \rceil} = (1+x)^{a_{\ell+1}-ra_\ell} \sum_{L=0}^{\lfloor \frac{a_{\ell-1}}{a_\ell} a_\ell \rfloor} C^{(\ell,r)}_{a_\ell,L} x^{\lfloor \frac{a_{\ell-1}}{a_\ell} a_\ell \rfloor - L}$$
$$= (1+x)^{-a_{\ell-1}} \sum_{L=0}^{a_{\ell-1}} C^{(\ell,r)}_{a_\ell,L} x^{a_{\ell-1}-L}.$$

So

$$\sum_{L=0}^{a_{\ell-1}} C^{(\ell,r)}_{a_\ell,L} x^{a_{\ell-1}-L} = (1+x)^{a_{\ell-1}}$$

as desired. □

For the remainder of this report, we focus on the $r = 2$ case.

3. Hypergeometric Series

To derive a known formula for the $r = 2$ Kronecker quiver, we rely primarily on Saalschütz's Theorem, which is an identity of hypergeometric series. In this section, we review some background on hypergeometric series.

Geometric series are of the form $\sum_{n=0}^{\infty} a_n x^n$, where $\frac{a_{n+1}}{a_n}$ is a fixed number. Analogously, hypergeometric series allow $\frac{a_{n+1}}{a_n}$ to be a fixed rational function in n.

Definition 3.1 (Hypergeometric Series)**.** Given integers $p \geq 1$, $q \geq 0$,

$${}_pF_q \left[\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q \end{matrix} \,\middle|\, x \right] = \sum_{n=0}^{\infty} a_n x^n$$

where the coefficients a_n are defined by $a_0 = 1$ and the recurrence

$$\frac{a_{n+1}}{a_n} = \frac{(n+\alpha_1)\cdots(n+\alpha_p)}{(n+1)(n+\beta_1)\cdots(n+\beta_q)}.$$

We often care about the value of

$${}_pF_q \left[\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q \end{matrix} \,\middle|\, 1 \right] = \sum_{n=0}^{\infty} a_n.$$

For special values of p, q and $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$, there are various identities that evaluate the hypergeometric series to a much simpler expression. We state two identities here, which will be used later. Recall that $(a)_n = \prod_{t=1}^{n} (a - 1 + t)$.

Theorem 3.2 (Vandermonde's Identity)**.** For $n \geq 0$,

$${}_2F_1 \left[\begin{matrix} -n, & a \\ & c \end{matrix} \,\middle|\, 1 \right] = \frac{(c-a)_n}{(c)_n}.$$

Theorem 3.3 (Saalschütz's Theorem)**.** For $n \geq 0$ and real numbers a, b, c,

$${}_3F_2 \left[\begin{matrix} -n, \ a, \ b \\ c, \ 1-n+a+b-c \end{matrix} \,\middle|\, 1 \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

4. Derivation for the Kronecker Quiver

Let $C^{\ell}_{M,N} = C^{(\ell,2)}_{M,N}$. Then by definition,

$$C^{\ell}_{M,N} = \sum_{\substack{(m_1,\ldots,m_\ell) \in \mathbb{Z}^{\ell}_{\geq 0} \\ m_1 + 2m_2 + \cdots + \ell m_\ell = M \\ m_2 + 2m_3 + \cdots + (\ell-1) m_\ell = N}} \prod_{i=1}^{l} \binom{l + 1 - i - 2\sum_{j=i+1}^{l} (j-i) m_j}{m_i}. \tag{4}$$

We deduce the following formula for $C^{\ell}_{M,N}$, which proves Conjecture 1 for $r = 2$.

Theorem 4.1. For $\ell \geq 1$ and $M \geq N \geq 0$,

$$C^{\ell}_{M,N} = \binom{\ell - N}{\ell - M} \binom{M-1}{N}.$$

Otherwise, $C^{\ell}_{M,N} = 0$.

Recall that the $C^{(\ell,r)}_{M,N}$ coefficients satisfy a recurrence, which in the case where $r=2$ specializes to

$$C^{\ell}_{M,N}=\sum_{k\geq 0}\binom{\ell-2N}{k}C^{\ell-1}_{N,2N-M+k}.$$

To prove Theorem 4.1, we show that the product of two binomials formula satisfies the same recurrence as $C^{\ell}_{M,N}$.

Proof. We proceed by induction on ℓ. When $\ell=1$, by definition,

$$C^{1}_{M,N}=\sum_{\substack{m_1\in\mathbb{Z}_{\geq 0}\\ m_1=M}}\binom{1}{m_1}$$

is only nonzero when $N=0$, $M=0$ or $N=0$, $M=1$. In both cases, $C^{1}_{M,N}$ is equal to 1. This agrees with the formula, which says that

$$C^{1}_{M,N}=\binom{1-N}{1-M}\binom{M-1}{N}.$$

To prove the inductive step, it suffices to show that

$$\binom{\ell-N}{\ell-M}\binom{M-1}{N}=\sum_{k\geq 0}\binom{\ell-2N}{k}\binom{\ell-1-2N+M-k}{\ell-1-N}\binom{N-1}{2N-M+k}.$$

This is Theorem 4.2. □

Theorem 4.2. For $M\geq N\geq 0$,

$$\binom{\ell-N}{\ell-M}\binom{M-1}{N}=\sum_{k\geq 0}\binom{\ell-2N}{k}\binom{\ell-1-2N+M-k}{\ell-1-N}\binom{N-1}{2N-M+k}$$

If $M>N$, we may write the right hand side as

$$\binom{\ell-N}{\ell-M}\binom{M-1}{N}=\sum_{k=0}^{M-N-1}\binom{\ell-2N}{k}\binom{\ell-1-2N+M-k}{\ell-1-N}\binom{N-1}{2N-M+k}.$$

Proof. When $M=N$, if $N>0$, then the right hand side is zero because the factor $\binom{N-1}{2N-M+k}=\binom{N-1}{N+k}$ is zero. This agrees with the left hand side. If $N=0$, then the second factor

$$\binom{\ell-1-2N+M-k}{\ell-1-N}=\binom{\ell-1-N-k}{\ell-1-N}$$

vanishes for $k>0$ and one can check that the identity also holds.

Now we discuss the case where $M>N\geq 0$. We may realize the right hand side as a hypergeometric series. Let

$$a_k=\binom{\ell-2N}{k}\binom{\ell-1-2N+M-k}{\ell-1-N}\binom{N-1}{2N-M+k}$$

so that the right hand side is equal to $\sum_{k=0}^{\infty}a_k$. Since

$$\frac{a_{k+1}}{a_k}=\frac{(k+2N-\ell)(k+N-M)(k-M+N+1)}{(k+1)(k+2N-M-\ell+1)(k+2N-M+1)}$$

vanishes when $k=M-N-1$, we can rewrite the series so that it sums up to $M-N-1$.

$$\begin{aligned}\sum_{k=0}^{M-N-1}a_k&=a_0\;{}_3F_2\left[\begin{matrix}N-M,\ N-M+1,\ 2N-\ell\\ 2N-M+1,\ 2N-M-\ell+1\end{matrix}\;\middle|\;1\right]\\&=\binom{\ell-1-2N+M}{\ell-1-N}\binom{N-1}{2N-M}{}_3F_2\left[\begin{matrix}N-M,\ N-M+1,\ 2N-\ell\\ 2N-M+1,\ 2N-M-\ell+1\end{matrix}\;\middle|\;1\right].\end{aligned}$$

Since $M\geq N$, we have $N-M\leq 0$, which allows us to apply Theorem 3.3 to evaluate the series:

$$\sum_{k=0}^{M-N-1}a_k=\binom{\ell-1-2N+M}{\ell-1-N}\binom{N-1}{2N-M}\frac{(N)_{M-N}(\ell-M+1)_{M-N}}{(2N-M+1)_{M-N}(\ell-N)_{M-N}},$$

which is equal to zero if $\ell - M + 1 \leq 0$, or $M > \ell$. It is also equal to zero if $\ell - 1 - N < 0$, or $N > \ell - 1$. In both of these cases, we can check that the product $\binom{\ell-N}{\ell-M}\binom{M-1}{N}$ also vanishes, and the theorem holds. If $M \leq \ell$ and $N \leq \ell - 1$, since we also have $M > N$,

$$\begin{aligned}
\sum_{k=0}^{M-N-1} a_k &= \binom{\ell-1-2N+M}{\ell-1-N}\binom{N-1}{2N-M}\frac{(N)_{M-N}(\ell-M+1)_{M-N}}{(2N-M+1)_{M-N}(\ell-N)_{M-N}} \\
&= \frac{(\ell-N)_{M-N}}{(M-N)!}\frac{(2N-M+1)_{(M-N-1)}}{(M-N-1)!}\frac{(N)_{M-N}(\ell-M+1)_{M-N}}{(2N-M+1)_{M-N}(\ell-N)_{M-N}} \\
&= \frac{(\ell-M+1)_{M-N}}{(M-N)!}\frac{(N)_{M-N}}{N(M-N-1)!} \\
&= \binom{\ell-N}{M-N}\binom{M-1}{M-N-1} \\
&= \binom{\ell-N}{\ell-M}\binom{M-1}{N}
\end{aligned}$$

as desired. □

Corollary 4.1. Conjecture 1 and 2 are true when $r = 2$. In other words, $C^{\ell}_{M,N} \neq 0$ if and only if $0 \leq M \leq \ell$, $0 \leq N \leq \frac{\ell-1}{\ell}M$.

Proof. $C^{\ell}_{M,N} \neq 0$ only if $0 \leq N \leq M$ by Theorem 4.1. If $M > \ell$, then $\binom{\ell-N}{\ell-M} = 0$. If $N > \frac{\ell-1}{\ell}M$ and $1 \leq M \leq \ell$, then $N \geq \lceil \frac{\ell-1}{\ell}M \rceil \geq M - 1$ and $\binom{M-1}{N} = 0$. Note that if $M = 0$, then $N > \frac{\ell-1}{\ell}M$ cannot happen.

In the other direction, when $0 \leq M \leq \ell$ and $0 \leq N \leq \frac{\ell-1}{\ell}M$, the binomial $\binom{\ell-N}{\ell-M}$ is positive. If $M > 0$, then $N \leq \lfloor \frac{\ell-1}{\ell}M \rfloor = M - 1$ and $\binom{M-1}{N}$ is positive. If $M = 0$, then $N = 0$ and $\binom{M-1}{N} = 1 > 0$. □

We record in the next corollary a result that the F-polynomials for the Kronecker quiver specialize in the following way to a q-analogue of Fibonacci numbers, defined below.

Corollary 4.2. Let

$$\widetilde{F}_n(q) = \sum_{k\geq 1} q^{k-1}\binom{n-k}{k-1}.$$

Then

$$F_\ell(q, q^{-1}) = q^\ell \widetilde{F}_{2\ell+1}(q^{-1}).$$

Proof. The left hand side is as follows:

$$\begin{aligned}
F_\ell(q, q^{-1}) &= \sum_{M\geq N\geq 0}\binom{\ell-N}{\ell-M}\binom{M-1}{N}q^{M-N} \\
&= \sum_{k\geq 1}\sum_{N\geq 0}\binom{\ell-N}{\ell-k+1}\binom{\ell-k+N}{N}q^{\ell-k+1}
\end{aligned}$$

But if $\ell - k + 1 \geq 0$, we have

$$\begin{aligned}
\sum_{N\geq 0}\binom{\ell-N}{\ell-k+1}\binom{\ell-k+N}{N} &= \binom{\ell}{\ell-k+1}{}_2F_1\left[\begin{matrix}1-k, \ell-k+1 \\ -\ell\end{matrix}\,\middle|\, 1\right] \\
&= \binom{\ell}{\ell-k+1}\frac{(k-2\ell-1)_{k-1}}{(-\ell)_{k-1}} \\
&= \frac{(k)_{\ell-k+1}}{(\ell-k+1)!}\frac{(2\ell-2k+3)_{k-1}}{(\ell-k+1)_{k-1}} \\
&= \frac{(2\ell-2k+3)_{k-1}}{(k-1)!} \\
&= \binom{2\ell+1-k}{k-1}.
\end{aligned}$$

This also holds if $\ell - k + 1 < 0$, since both sides would vanish. So

$$\begin{aligned} F_\ell(q, q^{-1}) &= \sum_{k \geq 0} \binom{2\ell + 1 - k}{k - 1} q^{\ell - k + 1} \\ &= q^\ell \widetilde{F}_{2\ell+1}(q^{-1}) \end{aligned}$$

as desired. $\square$

5. Reconciling Lee's and Gupta's Formulas for the Kronecker Quiver

We will show explicitly that when $r = 2$, the coefficient of $y_1^M y_2^N$ according to Lee's formula (Theorem 1.9) equals $C^\ell_{M,N}$. Recall that Theorem 1.9 states that for the Kronecker quiver, the coefficient of $y_1^M y_2^N$ in the ℓ-th F-polynomial is

$$\sum_{(m_1,\ldots,m_\ell) \in \mathbb{Z}^\ell} \prod_{i=1}^{\ell} \left[\begin{array}{c} \ell + 1 - i - 2\sum_{j=i+1}^{\ell} (j - i) m_j \\ m_i \end{array} \right]$$

where the sum ranges over all tuples $(m_1, \ldots, m_\ell) \in \mathbb{Z}$ such that

$$\begin{aligned} &M = \sum_{i=1}^{\ell} i m_i, \\ &N = \sum_{i=2}^{\ell} (i - 1) m_i, \\ &0 \leq m_i \leq \ell + 1 - i - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \text{ for all } 1 \leq i \leq \ell - 1, \\ \text{and} \quad &(\ell - 1)M - \ell N \geq 0. \end{aligned}$$

Note that Gupta's coefficients satisfy the last inequality by Theorem 2.1.

Lemma 5.1. If $\ell+1-i-2\sum_{j=i+1}^{\ell}(j-i)m_j \geq 0$ for some $1 \leq i \leq \ell-1$ and $m_k \geq 0$ for all $i \leq k \leq \ell-1$, then $\ell + 1 - k - 2\sum_{j=k+1}^{\ell}(j - k)m_j \geq 0$ for all $k \geq i$.

Proof. This is clear if $m_k = 0$ for all $k \geq i$. Suppose not, then for all $k \geq i$,

$$\begin{aligned} &(\ell + 1 - k - 2 \sum_{j=k+1}^{\ell} (j - k) m_j) - (\ell + 1 - i - 2 \sum_{j=i+1}^{\ell} (j - i) m_j) \\ = \; &2 \sum_{j=i+1}^{k} (j - i) m_i + 2 \sum_{j=k+1}^{\ell} (k - i) m_j - (k - i) \geq 0, \end{aligned}$$

and we have $(\ell + 1 - k - 2\sum_{j=k+1}^{\ell}(j - k)m_j) \geq 0$ as desired. $\square$

Lemma 5.2. If $m_1 \leq \ell - 2N < 0$ and $M \geq N$, then $\ell - 1 - 2(2N - M + m_1) \geq 0$.

Proof. Since $\ell - 2N < 0$ and $M \geq N$, we have $2M > \ell$ and $2M - \ell - 1 \geq 0$. So

$$\ell - 1 - 2(2N - M + m_1) = 2(\ell - 2N - m_1) + (2M - \ell - 1) \geq 0.$$

$\square$

To show that Lee's coefficients agree with Gupta's, we consider two cases:

- When $\ell - 2N \geq 0$, each of $\ell + 1 - i - 2\sum_{j=i+1}^{\ell}(j - i)m_j$ for all $1 \leq i \leq \ell$ is forced to be nonnegative (Lemma 5.1). When $N \geq 0$, both $\left[\begin{array}{c} N \\ s \end{array} \right]$ and $\binom{N}{s}$ are nonzero only if $0 \leq s \leq N$. Since in this case, the same set of tuples contributes to nontrivially to Lee's coefficients as to the $C^\ell_{M,N}{}'s$, Lee's coefficients agree with Gupta's.

- When $\ell - 2N < 0$, by definition, $\begin{bmatrix} \ell - 2N \\ m_1 \end{bmatrix} \neq 0$ only if $m_1 \leq \ell - 2N$. Then by Lemma 5.2, $\ell - 1 - 2(2N - M + m_1) \geq 0$. By Lemma 5.1, $\ell + 1 - i - 2\sum_{j=i+1}^{\ell}(j-i)m_j \geq 0$ for $i \geq 2$. So Lee's coefficients are equal to
$$\sum_{m_1 \leq \ell - 2N} \binom{\ell - 2N}{m_1} C^{\ell-1}_{N,2N-M+m_1}.$$
It suffices to evaluate this sum and check that it is equal to $C^{\ell}_{M,N}$. We do so in Theorem 5.3.

Theorem 5.3. For $\ell > 0$, $\ell - 2N < 0$ and $M \geq N$,
$$\begin{aligned}\binom{\ell-N}{\ell-M}\binom{M-1}{N} &= \sum_{m_1 \leq \ell-2N} \begin{bmatrix} \ell - 2N \\ m_1 \end{bmatrix} \binom{\ell-1-(2N-M+m_1)}{\ell-1-N}\binom{N-1}{2N-M+m_1} \\ &= \sum_{m_1 \geq 0} \binom{\ell-2N}{m_1}\binom{\ell-1-(2N-M+m_1)}{\ell-1-N}\binom{N-1}{2N-M+m_1}.\end{aligned}$$

The equality between the first and third parts is Theorem 4.2. We include it here to make a comparison to the second part: they only differ in the range of m_1 and the convention of binomial coefficient. It's worth noting that this identity is a result of interesting cancellations. For example, when $\ell = 8, M = 7, N = 5$, Lee's coefficient is equal to $\begin{bmatrix} -2 \\ -2 \end{bmatrix} 60 + \begin{bmatrix} -2 \\ -3 \end{bmatrix} 21 = 18$, while Gupta's coefficient is the sum $\binom{3}{1}\binom{6}{5} = \binom{-2}{0}24 + \binom{-2}{1}3 = 18$.

Proof. Recall that $\begin{bmatrix} N \\ s \end{bmatrix} = \binom{N}{N-s}$. So we may rewrite the identity above as
$$\binom{\ell-N}{\ell-M}\binom{M-1}{N} = \sum_{m_1 \leq \ell-2N} \binom{\ell-2N}{\ell-2N-m_1}\binom{\ell-1-(2N-M+m_1)}{\ell-1-N}\binom{N-1}{2N-M+m_1}.$$
By the change of variable $\ell - 2N + m_1 = k$, the identity is equivalent to
$$\binom{\ell-N}{\ell-M}\binom{M-1}{N} = \sum_{k \geq 0}\binom{\ell-2N}{k}\binom{M+k-1}{\ell-1-N}\binom{N-1}{\ell-M-k}.$$
When $M \geq \ell$, the right hand side is zero except when $M = \ell, k = 0$. This agrees with the left hand side. When $M < \ell$, we can evaluate the right hand side, which is a hypergeometric series, by applying Theorem 3.3. Note that $M + N - \ell \geq 2N - \ell > 0$. Note also that $N \leq M < \ell$, so $\ell - 1 - N \geq 0$.
$$\begin{aligned}\text{RHS} &= \binom{M-1}{\ell-1-N}\binom{N-1}{\ell-M} {}_3F_2\left[\begin{matrix} -\ell+M,\ M,\ 2N-\ell \\ M+N-\ell+1,\ M+N-\ell \end{matrix} \,\middle|\, 1\right] \\ &= \binom{M-1}{M+N-\ell}\binom{N-1}{\ell-M}\frac{(N-\ell)_{\ell-M}(M-N)_{\ell-M}}{(M+N-\ell)_{\ell-M}(-N)_{\ell-M}} \\ &= \frac{(\ell-N)_{M+N-\ell}}{(M+N-\ell)!}\frac{(M+N-\ell)_{\ell-M}}{(\ell-M)!}\frac{(M-N+1)_{\ell-M}(M-N)_{\ell-M}}{(M+N-\ell)_{\ell-M}(M+N-\ell+1)_{\ell-M}} \\ &= \frac{(M-N+1)_{\ell-M}}{(\ell-M)!}\frac{(M-N)_N}{N!} \\ &= \binom{\ell-N}{\ell-M}\binom{M-1}{N}\end{aligned}$$
as desired. ∎

6. Combinatorial Interpretation of the Recurrence when $r = 2$

The coefficients $C^{\ell}_{M,N}$ have a combinatorial interpretation in terms of the number of ways to choose nonadjacent subsets of integers with a certain number of evens and odds. It also has an interpretation in terms of perfect matchings of a straight snake graph that is in bijection with the former. We choose to focus on the former since it is easier to state. The following theorem is written in a slightly different form in [MP06]. Let $[n] = \{1, \ldots, n\}$.

Theorem 6.1 (Theorem 3 [MP06])**.** Let $\Omega^\ell_{M,N}$ be the set of subsets $S \subset [2\ell-1]$ such that S contains $\ell - M$ odd elements, N even elements and no consecutive elements. Then

$$|\Omega^\ell_{M,N}| = \binom{\ell-N}{\ell-M}\binom{M-1}{N}.$$

Hence Theorem 4.2 is equivalent to the following:

Theorem 6.2 (Combinatorial Version of Theorem 4.2)**.**

$$|\Omega^\ell_{M,N}| = \sum_{k\geq 0}\binom{\ell-2N}{k}|\Omega^{\ell-1}_{N,2N-M+k}|.$$

This gives hope for a combinatorial proof for Theorem 4.2, which we do below. Let X be a set consisting of even numbers between 1 and $2\ell-1$ such that $|X| = N$, where $0 \leq N \leq \ell - 1$. Let $\widetilde{X} = \{n \mid 1 \leq n \leq 2\ell-1, n \text{ is even}, n \notin X\}$. Notice that $|\widetilde{X}| = \ell - 1 - N$. Let

$$O_X = |\{\text{odd numbers between 1 and } 2\ell-1 \text{ not adjacent to } X\}|$$

and

$$\begin{aligned} E_X &= |\{\text{odd numbers between 2 and } 2\ell-2 \text{ not adjacent to } \widetilde{X}\}| \\ &= |\{\text{even numbers between 1 and } 2\ell-3 \text{ not adjacent to } \widetilde{X}-1\}|, \end{aligned}$$

where $\widetilde{X} - 1 := \{n-1 \mid n \in \widetilde{X}\}$.

Lemma 6.3. $O_X - E_X = \ell - 2N$.

Proof. We use the first interpretation of E_X and proceed by induction on N. If $N = 0$, then $S = \emptyset$. So $O_X = \ell$, $E_X = 0$, and $O_X - E_X = \ell$ as desired. Suppose that $O_X - E_X = \ell - 2N$ for all X with $|X| = N$. Now consider Y, a set of even numbers between 1 and $2\ell - 1$ such that $|Y| = N + 1$. Choose a subset $X \subset Y$ such that $|X| = N$ and write $Y = X \cup \{m\}$. Then by the inductive hypothesis, $O_X - E_X = \ell - 2N$. Now we compare O_X and O_Y. There is one more even number that in Y that odd numbers counted by O_Y need to try to avoid, so $O_Y \leq O_X$. Depending on the amount of even numbers in X that the number m has as neighbor(s), the difference $O_X - O_Y$ can be $0, 1$ and 2. The three cases are as follows:
Case 1. If $m-2, m+2 \in S$, then the odd numbers not adjacent to X are exactly the odd numbers not adjacent to Y, hence $O_Y = O_X$. On the flip side, removing m from S^c, which did not contain $m-2$ and $m+2$, will result in $m-1$ and $m+1$ no longer being counted by E_Y, and $E_Y = E_X - 2$.
Case 2. If exactly one of $m-2$ and $m+2$ is in X, then after adding m to X, one of $m+1$ and $m-1$ is no longer counted by O_Y, whereas one of $m+1$ and $m-1$ is now counted by E_Y. So $O_Y = O_X - 1$ and $E_Y = E_X + 1$.
Case 3. If m is such that $m-2, m+2 \notin S$, then $O_Y = O_X - 2$ and $E_Y = E_X$.
In all three cases, $O_Y - E_Y = O_X - E_X - 2 = \ell - 2N - 2$ as desired. □

Lemma 6.4. Let X be a set consisting of even numbers between 1 and $2\ell-1$ such that $|X| = N$. Then

$$\binom{O_X}{\ell-M} = \sum_{k\geq 0}\binom{\ell-2N}{k}\binom{E_X}{2N-M+k}.$$

Proof. Using Vandermonde's identity, since

$$E_X - (O_X - \ell + M - k) = \ell - M - k - (\ell - 2N) = 2N - M - k,$$

$$\begin{aligned} \binom{O_X}{\ell-M} &= \binom{O_X}{O_X-\ell+M} \\ &= \sum_{k\geq 0}\binom{O_X-E_X}{k}\binom{E_X}{O_X-\ell+M-k} \\ &= \sum_{k\geq 0}\binom{\ell-2N}{k}\binom{E_X}{2N-M+k}. \end{aligned}$$

□

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2.

$$\begin{aligned}
|\Omega^{\ell}_{M,N}| &= \sum_{X} \binom{O_X}{\ell - M} \\
&= \sum_{k\geq 0} \binom{\ell - 2N}{k} \sum_{X} \binom{E_X}{2N - M + k} \\
&= \sum_{k\geq 0} \binom{\ell - 2N}{k} |\Omega^{\ell-1}_{N,2N-M+k}|.
\end{aligned}$$

where X ranges over all possible subsets of N even elements of $[2\ell - 1]$. □

7. Other Threads

We discovered the following phenomenon when we view the product of binomials as a function in both ℓ and L and input $L = 2\ell$ instead of $L = \ell$, as is the case in the F-polynomials.

Conjecture 3. Let $L \in \mathbb{Z}_{\geq 0}$ be even. Let

$$D_L(\ell, m_1) = \sum_{\substack{(m_1,\ldots,m_L)\in\mathbb{Z}_{\geq 0} \\ m_2+\cdots+(L-1)m_L = L}} \prod_{i=1}^{L} \binom{\ell + 1 - i - 2\sum_{j=i+1}^{L}(j-i)m_j}{m_i}.$$

Then

$$D_L(L/2, k) = (-1)^k N(L/2 - 1, k + 2L - 1, k + L),$$

where

$$N(m, n, k) = \frac{m+1}{n+1}\binom{n+1}{k}\binom{n-m-1}{k-1}$$

are the generalized Narayana numbers (`https://oeis.org/A281260/a281260.pdf`). Evidence

- $L = 4$: as k ranges from -3 to 2: -2, 15, -60, 175, -420, 882
- $L = 6$: as k ranges from -5 to -1: -3, 42, -280, 1260, -4410

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