EQUALITY IN THE EISENBUD–GOTO CONJECTURE FOR CERTAIN TORIC IDEALS

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Abstract. The Eisenbud–Goto conjecture states that for a nondegenerate graded prime ideal $\mathfrak{p}$ of $S = k[x_1, \dots, x_n]$, we have $\operatorname{reg} \mathfrak{p} \leq \deg \mathfrak{p} - \operatorname{codim} \mathfrak{p} + 1$. While this conjecture is known to be false in general, it has been proven in several special cases, and is still open when $\mathfrak{p}$ is a toric ideal. We first characterize when equality holds for toric ideals in the cases of monomial curves and complete intersections. Then we provide several necessary conditions for equality to hold when $\mathfrak{p}$ has codimension 2.

1. Introduction

Fix an algebraically closed field k of arbitrary characteristic. Let $S \coloneqq k[\mathbf{x}] = k[x_1, \dots, x_n]$ be a graded polynomial ring (with the standard grading). For a graded S-module M, we denote the graded minimal free resolution of M

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

by $F_\bullet$. Each F_i is a direct sum of twists of S, so we can write $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$ for some nonnegative integers $\beta_{i,j}$, called the *Betti numbers* of $F_\bullet$.

The *Castelnuovo–Mumford regularity* of a finitely generated graded S-module M is defined by

$$\operatorname{reg} M \coloneqq \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}.$$

It does not depend on the choice of minimal free resolution. For a projective variety $X \hookrightarrow \mathbb{P}^{n-1}$, we write $\operatorname{reg} X$ for the Castelnuovo–Mumford regularity of its vanishing ideal. We will often refer to this quantity simply as the "regularity." Regularity can be equivalently defined in terms of local cohomology or sheaf cohomology; however, the definition in terms of Betti numbers will be most suited to our purposes.

In their 1984 paper [EG84], Eisenbud and Goto made the following conjecture:

Conjecture 1.1. If $\mathfrak{p}$ is a nondegenerate graded prime ideal of S, then

$$\operatorname{reg} \mathfrak{p} \leq \deg \mathfrak{p} - \operatorname{codim} \mathfrak{p} + 1.$$

In the same paper, they proved this bound holds if $\mathfrak{p}$ is Cohen-Macaulay. In 2018, Peeva and McCullough proved that this conjecture is false in general by exhibiting an explicit counterexample ([MP18]).

While not true in full generality, the conjecture has been proven true in several special cases. For example, the results of [GLP83] imply the truth of Conjecture 1.1 in the case where $\mathfrak{p}$ defines a projective curve. [HH03] resolved the case where $S/\mathfrak{p}$ is a simplicial semigroup

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ring with isolated singularity, and [BEN12] resolved the case where $S/\mathfrak{p}$ is simplicial and seminormal.

It is still an open question whether or not Conjecture 1.1 is true for general *toric ideals*.

Definition 1.2. Let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice orthogonal to the all-1's vector $(1, 1, \ldots, 1) \in \mathbb{Z}^n$. Then $\mathcal{L}$ defines a homogeneous *lattice ideal*

$$I_{\mathcal{L}} := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u} - \mathbf{v} \in \mathcal{L} \rangle \subseteq S.$$

When the lattice $\mathcal{L}$ is saturated (that is, $\mathbb{Q}\mathcal{L} \cap \mathbb{Z}^n = \mathcal{L}$), this ideal is prime and we say that $I_{\mathcal{L}}$ is *toric*.

Fix $\mathcal{L} \subseteq \mathbb{Z}^n$, and consider the case where $\mathcal{L}$ has rank 2. Let $B \in \mathbb{Z}^{n\times 2}$ be a matrix whose columns form a basis for $\mathcal{L}$, whose entry in the (i, j) position is given by b_{ij}. The *Gale diagram* $G_{\mathcal{L}}$ of $\mathcal{L}$ is the collection of *Gale vectors*, which are the row vectors $\mathbf{b}_i := (b_{i1}, b_{i2}) \in \mathbb{Z}^2$. Note that the Gale diagram is determined up to the action of $\mathrm{GL}_2(\mathbb{Z})$. The lattice $\mathcal{L}$ is saturated if and only if the Gale vectors span $\mathbb{Z}^2$. Throughout the paper, we consider the Gale vectors to be ordered pairs $(i, \mathbf{b}_i)$, so that particular vectors $\mathbf{b}_i \in \mathbb{Z}^2$ are considered "with multiplicity," but for all other purposes we formally treat them just as vectors.

Note that the requirement that $\mathcal{L}$ be orthogonal to $(1, 1, \ldots, 1)$ implies that $\mathcal{L}$ contains no nonzero nonnegative vectors and that $I_{\mathcal{L}}$ defines a projective variety embedded in $\mathbb{P}^{n-1}$. It is known that the codimension of $I_{\mathcal{L}}$ equals the rank of $\mathcal{L}$.

In the case where $\operatorname{codim} I_{\mathcal{L}} = 2$, Conjecture 1.1 is resolved for toric ideals in [PS98] by using combinatorial objects coming from the lattice $\mathcal{L}$ (e.g. the Gale diagram) to describe the minimal free resolution of $I_{\mathcal{L}}$.

The present paper aims to answer the following question:

Question 1.3. In cases where the inequality of Conjecture 1.1 has been proven for toric ideals, when is equality achieved?

In Sections §2 and §3, we characterize when equality occurs for toric ideals $I_{\mathcal{L}}$ in the cases where $I_{\mathcal{L}}$ defines a curve or a complete intersection variety. In §4, the main section of this paper, we give combinatorial descriptions of the saturated lattices $\mathcal{L}$ that achieve equality when $\operatorname{codim} I_{\mathcal{L}} = 2$ and $I_{\mathcal{L}}$ is not Cohen-Macaulay, building on work done in [PS98]. Our main result is the following:

Theorem 1.4. *Let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a saturated lattice such that $I_{\mathcal{L}}$ is not Cohen-Macaulay and any Gale diagram of $\mathcal{L}$ contains at least 5 nonzero vectors. If the toric ideal $I_{\mathcal{L}}$ achieves equality in the Eisenbud–Goto conjecture, then there exists a Gale diagram $G_{\mathcal{L}}$ of $\mathcal{L}$ and a partition $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = G_{\mathcal{L}}$, where Q_i only contains vectors in the ith closed quadrant for all i, satisfying the following properties:*

- *the vectors in $Q_2 \cup Q_4$ all lie on a single line passing through the origin,*
- *there exist $\mathbf{b}_1 \in Q_1, \mathbf{b}_3 \in Q_3$ and a nonzero $\mathbf{u} \in \mathbb{Z}^2$ such that $\mathbf{b}_1 \cdot \mathbf{u} = -\mathbf{b}_3 \cdot \mathbf{u} = 1$ and $\mathbf{b} \cdot \mathbf{u} = 0$ for all $\mathbf{b} \in (Q_1 \cup Q_3) - \{\mathbf{b}_1, \mathbf{b}_3\}$,*
- *and up to dihedral symmetries, the set $\{\sum_{\mathbf{b}\in Q_i} \mathbf{b} : 1 \leq i \leq 4\}$ equals either*

$$\{(1, 1), (a, -b), (-1, -1), (-a, b)\} \quad \text{or} \quad \{(1, a), (1, -b), (-1, -a), (-1, b)\}$$

for some positive integers a, b.

In particular, all but two Gale vectors lie on a union of two lines, and the two exceptional vectors are "close" to one of these lines (that is, they are as close as possible in the Euclidean metric without being on the line itself). These properties of the Gale diagram are $\mathrm{GL}_2(\mathbb{Z})$-invariant and therefore hold for *any* Gale diagram of $\mathcal{L}$.

2. Monomial Curves

A nondegenerate toric variety $X \hookrightarrow \mathbb{P}^{n-1}$ of dimension 1 is (up to a reordering of coordinates) necessarily the closure of the image of a map $\mathbb{A}^1 \to \mathbb{P}^{n-1}$ of the form

$$t \mapsto [t^{a_1} : t^{a_2} : \cdots : t^{a_n}]$$

where $a_1 = 0$, $\gcd(a_2, \ldots, a_n) = 1$, and $a_1 < a_2 < \cdots < a_n$, and is therefore a monomial curve. With these assumptions, $\deg X = a_n$.

The results of [HHS10] in conjunction with the general results on curves from [GLP83] immediately resolve equality for monomial curves. We record the following corollary of their results.

Proposition 2.1. *Let $X \hookrightarrow \mathbb{P}^{n-1}$ be a toric curve of degree d. Then X achieves equality in the Eisenbud-Goto conjecture if and only if one of the following holds:*

- $d \leq n$
- $d \geq n+1$ *and, with notation as above,* $(a_1, a_2, \ldots, a_n) = (0, 1, 2, \ldots, n-3, d-1, d)$ *or* $(0, 1, d-n+3, d-n+4, \ldots, d)$.

Proof. By general theory, since X is toric, X is rational.

If $d \in \{n-1, n\}$, [GLP83, Theorem 3.1] implies that equality is achieved. So assume $d \geq n+1$. Comment 1 at the end of §2 of [GLP83] now says that a necessary condition for X to achieve equality in EG is that X be smooth. So henceforth assume that X is smooth, i.e., that $a_2 = 1$ and $a_{n-1} = d-1$.

Using the notation of [HHS10], let $\lambda(X)$ be the length of the longest gap (i.e., the largest value of $a_k - a_{k-1} - 1$) in $\mathcal{S} = \{a_1, a_2, \cdots, a_{n-1}, a_n\}$ and let $\varepsilon = \max\{i : [0, i], [d-i, d] \subseteq \mathcal{S}\}$. Observe that the sum of all gaps is $a_n - a_1 - n + 1 = d - n + 1$, so $\lambda(X) = d - n + 1$ is achieved if and only if there is only a single gap of positive length. We then see from [HHS10, Theorem 2.7] that

$$\operatorname{reg} I = \operatorname{reg}(S/I) + 1 \leq \frac{\lambda(X) - 1}{\varepsilon} + 3 \leq d - n + 3$$

with equality only if $\varepsilon = 1$ and $\lambda(X) = d - n + 1$. Since there can only be a single gap, and $\varepsilon = 1$ says that 2 and $d-2$ cannot both appear in $\mathcal{S}$, we conclude that the only possibilities for $\mathcal{S}$ are $\{0, 1, 2, \ldots, n-3, d-1, d\}$ or $\{0, 1, d-n+3, d-n+4, \ldots, d\}$ (these are the same list up to $k \mapsto d-k$).

Finally, we need to know that these cases actually give equality. This is now exactly implied by the statement of [HHS10, Theorem 3.4] with $\varepsilon = 1$ and $p = d - n + 3$. $\square$

3. Complete Intersections

Proposition 3.1. *Let $X \hookrightarrow \mathbb{P}^{n-1}$ be a nondegenerate complete intersection variety with*

$$\operatorname{reg} X = \deg X - \operatorname{codim} X + 1.$$

Then X is either a hypersurface or the intersection of two quadric hypersurfaces.

Proof. Let X be cut out by k equations of degrees $d_1, \ldots, d_k \geq 2$. The graded minimal free resolution of I is given by the Koszul complex $E_\bullet$ where

$$E_k = \bigoplus_{i_1 < \cdots < i_k} S(-d_{i_1} - \cdots - d_{i_k}).$$

From this we see directly that

$$\operatorname{reg} X = d_1 + \cdots + d_k - k + 1.$$

Since $\deg X = d_1 \cdots d_k$, we arrive at

$$d_1 \cdots d_k = d_1 + \cdots + d_k.$$

If $k = 1$, then X is a hypersurface and this is always true. If $k = 2$, then we get $d_1 d_2 = d_1 + d_2$ whose only solution is $d_1 = d_2 = 2$, forcing X to be the intersection of two quadrics.

If $k \geq 3$, there are no solutions. One way to see this is to show that

$$d_1 \cdots d_k - (d_1 + \cdots + d_k) \geq 2^k - 2k$$

by induction, which is straightforward. □

The following lemma reduces the question to a finite search:

Lemma 3.2. *Let $n > r$ and let $S := k[x_1, \ldots, x_r]$ and $S' := k[x_1, \ldots, x_n]$ be polynomial rings with the standard grading. Let I be a homogeneous ideal of S and let $I' := I \otimes_S S'$ be an ideal of S'. Then $\operatorname{reg} I' = \operatorname{reg} I$ and $\deg I' = \deg I$.*

Proof. It is easy to see that the Betti tables of I and I' are the same; indeed if $F_\bullet$ is a minimal graded resolution of I by free S-modules, then $F'_\bullet := F_\bullet \otimes_S S'$ is a graded resolution of I' by free S'-modules, with syzygies in the same degrees. Let $\mathfrak{m} = (x_1, \ldots, x_r)$ be a maximal ideal of S and $\mathfrak{m}' = (x_1, \ldots, x_n)$ be a maximal ideal of S'. By minimality, all maps $F_i \mapsto F_{i-1}$ land inside $\mathfrak{m} F_{i-1}$. It follows that all maps $F'_i \mapsto F'_{i-1}$ land inside $\mathfrak{m}' F'_{i-1}$ (for example, since the matrix entries of these maps are the same as those of $F_\bullet$). The statement on regularity now follows immediately.

We now treat the statement on degrees. By induction, it suffices to treat the case $n = r+1$. Then we have $S'/I' \cong (S/I)[x_{r+1}]$, and the grading on S'/I' is the same whether viewed as an S-module or an S'-module. In particular, the mth graded piece of S'/I' is spanned by elements of the form $f x_{r+1}^j$ where $f \in S/I$ has degree $m - j$. It follows that the Hilbert functions satisfy $h_{S'/I'}(m) = \sum_{j \leq m} h_{S/I}(j)$. In particular, for $m \gg 0$, we see that the Hilbert polynomials satisfy

$$P_{S/I}(m) = P_{S'/I'}(m) - P_{S'/I'}(m-1)$$

from which it is straightforward that $\deg I = \deg I'$. □

Corollary 3.3. *Let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice whose Gale diagram $G_{\mathcal{L}}$ has at least m instances of the zero vector. Let $\mathcal{L}' \subseteq \mathbb{Z}^{n-m}$ be the lattice whose Gale diagram is obtained from $G_{\mathcal{L}}$ by omitting any collection of m zero vectors. Then* $\operatorname{reg} I_{\mathcal{L}'} = \operatorname{reg} I_{\mathcal{L}}$ *and* $\deg I_{\mathcal{L}'} = \deg I_{\mathcal{L}}$.

Any quadric coming from a lattice has at most four variables in its support (it must be of the form $x_i x_j - x_k^2$ or $x_i x_j - x_k x_l$ up to sign). Then Proposition 3.1 implies that for any $n > 8$, there is no saturated lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ with all nonzero Gale vectors defining a complete intersection variety that achieves equality in the Eisenbud–Goto conjecture. There are therefore finitely many such lattices. Any saturated lattice giving a complete intersection variety achieving equality in the Eisenbud–Goto conjecture is equivalent to one of these finitely many lattices after removing all Gale vectors equal to zero.

4. Codimension-2 Lattice Ideals

In this section, we give some necessary conditions for a codimension-2 toric ideal that is not Cohen–Macaulay to achieve equality in the Eisenbud–Goto conjecture. The following result motivates considering the more general class of codimension-2 lattice ideals.

Theorem 4.1 ([PS98, Theorem 7.3 and Proposition 7.7])**.** *Let $I_{\mathcal{L}} \subseteq k[x_1, \dots, x_n]$ be a codimension-2 lattice ideal that is not Cohen–Macaulay. Then* $\operatorname{reg} I_{\mathcal{L}} \leq \deg I_{\mathcal{L}}$*, and the inequality is strict if $I_{\mathcal{L}}$ is toric. Furthermore, if equality holds, then any Gale diagram for $\mathcal{L}$ lies on two lines through the origin in $\mathbb{R}^2$.*

We now develop the tools that were used in [PS98] to prove the above result, restricting our attention to the case when $I_{\mathcal{L}}$ has codimension 2, equivalently, when $\mathcal{L}$ has rank 2. As usual, we assume that $\mathcal{L} \subseteq \mathbb{Z}^n$ is orthogonal to the all-1's vector $(1, 1, \dots, 1) \in \mathbb{Z}^n$.

Let $\Gamma := \mathbb{Z}^n/\mathcal{L}$ be an abelian group. Note that S, $I_{\mathcal{L}}$, and $S/I_{\mathcal{L}}$ are Γ-graded. A *fiber* is a set of all monomials of S with a fixed degree $C \in \Gamma$. Given $C \in \Gamma$ and a monomial $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} \in C$, the monomials of degree C are in bijection with the lattice points in $P_{\mathbf{a}} := \operatorname{conv}(\{\mathbf{u} \in \mathbb{Z}^2 : B\mathbf{u} \leq \mathbf{a}\})$. Note that $P_{\mathbf{a}}$ and $P_{\mathbf{a}'}$ are lattice translates if $\mathbf{a} - \mathbf{a}' \in \mathcal{L}$, so by considering polygons up to translation, we define $P_C := P_{\mathbf{a}}$ for $\mathbf{a} \in C$. We say P_C is *primitive* if it contains no lattice points other than its vertices.

If $I_{\mathcal{L}}$ is a not Cohen–Macaulay, then the projective dimension of the S-module $S/I_{\mathcal{L}}$ equals 3. If $S/I_{\mathcal{L}}$ has a minimal ith syzygy of degree C, then P_C is primitive, and is a parallelogram if $i = 3$. In this case, we say that P_C is a *syzygy quadrangle* of $I_{\mathcal{L}}$. It turns out that the free resolution of $S/I_{\mathcal{L}}$ is controlled by the syzygy quadrangles of $I_{\mathcal{L}}$, in a way that is made precise in [PS98].

Given $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^2$ with $|\det(\mathbf{v}, \mathbf{w})| = 1$, let $[\mathbf{v}, \mathbf{w}] := \operatorname{conv}\{(0,0), \mathbf{v}, \mathbf{w}, \mathbf{v}+\mathbf{w}\})$ be a primitive parallelogram. Then we have the following:

Proposition 4.2 ([PS98, Proposition 4.1 and Corollary 4.2])**.** *The parallelogram $[\mathbf{v}, \mathbf{w}]$ is a syzygy quadrangle if and only if each vertex of $[\mathbf{v}, \mathbf{w}]$ is supported by at least one vector in the Gale diagram $G_{\mathcal{L}}$. Furthermore, if $I_{\mathcal{L}}$ is not Cohen–Macaulay, then there exists a Gale diagram $G_{\mathcal{L}}$ which intersects each of the four open quadrants; in other words, there exists a Gale diagram $G_{\mathcal{L}}$ for which the unit square $[(1,0),(0,1)]$ is a syzygy quadrangle.*

We henceforth assume that $I_{\mathcal{L}}$ is not Cohen-Macaulay. A crucial insight in the proof of Theorem 4.1 is that one can reduce to the case of a curve in $\mathbb{P}^3$. We briefly outline the argument from [PS98] here. Given $\mathcal{L} \subseteq \mathbb{Z}^n$, let $G_{\mathcal{L}} \coloneqq \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a Gale diagram for which the unit square is a syzygy quadrangle attaining the regularity, so that in particular, by Proposition 4.2, the set $G_{\mathcal{L}}$ contains at least vector in each open quadrant. Let $G_{\mathcal{L}} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ be a partition of the Gale vectors so that Q_i consists only of Gale vectors in the ith closed quadrant (note that multiple such partitions are possible if there are Gale vectors on the axes). Then there is a morphism $\phi \colon k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_4]$ given by $x_j \mapsto y_i$ if $\mathbf{b}_j \in Q_i$. This morphism is surjective since each Q_i is nonempty, so $J \coloneqq \phi(I_{\mathcal{L}})$ is an ideal of $k[y_1, \ldots, y_4]$. Moreover, J has dimension 1.

It is not always true that J is a lattice ideal. However, as discussed in [PS98], the saturation $I_{\mathcal{L}'} \coloneqq (J : (y_1y_2y_3y_4)^\infty)$ is a lattice ideal corresponding to the lattice $\mathcal{L}'$ with Gale diagram $G_{\mathcal{L}'} \coloneqq \{\mathbf{b}'_1, \ldots, \mathbf{b}'_4\}$, where $\mathbf{b}'_i \coloneqq \sum_{\mathbf{b} \in Q_i} \mathbf{b}$. It is not hard to show that $\deg J = \deg I_{\mathcal{L}}$. By general properties of saturation, it follows that $\deg I_{\mathcal{L}'} \leq \deg J = \deg I_{\mathcal{L}}$.

It is easy to see that each $\mathbf{b}'_i$ intersects the ith open quadrant, so the unit square remains a syzygy quadrangle, and the monomials corresponding to its vertices retain the same total degree. It follows that $\operatorname{reg} I_{\mathcal{L}'} \geq \operatorname{reg} I_{\mathcal{L}}$. It is important to note that regularity can only *strictly* increase if the $\mathcal{L} \to \mathcal{L}'$ reduction process introduces a new syzygy quadrangle.

Thus if Theorem 4.1 holds for $n = 4$, then

$$\operatorname{reg} I_{\mathcal{L}} \leq \operatorname{reg} I_{\mathcal{L}'} \leq \deg I_{\mathcal{L}'} \leq \deg I_{\mathcal{L}}$$

and so the result holds for general r.

By inspecting this reduction process in more detail, we will be able to restrict the possible Gale diagrams of lattice ideals $I_{\mathcal{L}}$ satisfying $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$. The three key results that allow us to prove Theorem 1.4 are Corollary 4.7 and Propositions 4.9 and 4.11.

Lemma 4.3. *Suppose there exists a* 1*-dimensional associated prime* $\mathfrak{p}$ *of* J *that contains* $\langle y_1y_2y_3y_4\rangle$*. Then* $\mathfrak{p} = \langle y_i, y_j\rangle$ *for some distinct* $i, j \in \{1, 2, 3, 4\}$*.*

Proof. Since $y_1y_2y_3y_4 \in \mathfrak{p}$, we have that $y_i \in \mathfrak{p}$ for some $1 \leq i \leq 4$. Let $\mathbf{u} \in \mathbb{Z}^2$ be any vector in the ith open quadrant, and let $\mathbf{a} = B\mathbf{u} \in \mathcal{L}$, so that $\mathbf{x}^{\mathbf{a}_+} - \mathbf{x}^{\mathbf{a}_-} \in I_{\mathcal{L}}$. Note $\phi(\mathbf{x}^{\mathbf{a}_+}) - \phi(\mathbf{x}^{\mathbf{a}_-}) \in J$, and that it follows from our choice of $\mathbf{u}$ that y_i has a positive exponent in the monomial $\phi(\mathbf{x}^{\mathbf{a}_+})$ and has exponent zero in the monomial $\phi(\mathbf{x}^{\mathbf{a}_-})$. Thus, $\phi(\mathbf{x}^{\mathbf{a}_+}) \in \mathfrak{p}$, and since $\phi(\mathbf{x}^{\mathbf{a}_+}) - \phi(\mathbf{x}^{\mathbf{a}_-}) \in \mathfrak{p}$, we have $\phi(\mathbf{x}^{\mathbf{a}_-}) \in \mathfrak{p}$. But $\phi(\mathbf{x}^{\mathbf{a}_-})$ is a nonconstant monomial in which y_i has degree 0, and $\mathfrak{p}$ is prime, so $y_j \in \mathfrak{p}$ for some $j \neq i$, which means $\langle y_i, y_j\rangle \subseteq \mathfrak{p}$. Since $\dim \mathfrak{p} = 1$, we must have equality. □

Lemma 4.4. *Let* $i, j \in \{1, 2, 3, 4\}$ *be distinct. Then* $\langle y_i, y_j\rangle$ *is an associated prime of* J *if and only if for all nonzero* $\mathbf{u} \in \mathbb{Z}^2$*, there is some* $\mathbf{b} \in Q_i \cup Q_j$ *such that* $\mathbf{b} \cdot \mathbf{u} < 0$*. In particular, this can hold only if* $\{i, j\} = \{1, 3\}, \{2, 4\}$*.*

Proof. For $i \in \{1, 2, 3, 4\}$, let $P_i = \{j : \mathbf{b}_j \in Q_i\}$. Fix distinct $i, j \in \{1, 2, 3, 4\}$. Note that $\langle y_i, y_j\rangle$ is an associated prime of J if and only if $\langle y_i, y_j\rangle \supseteq J$, since if $\langle y_i, y_j\rangle \supseteq J$, then $\langle y_i, y_j\rangle$ is minimal over J. This occurs if and only if for all nonzero $\mathbf{a} \in \mathcal{L}$, we have

$$\operatorname{supp}(\mathbf{a}_+) \cap (P_i \cup P_j) \neq \emptyset \quad \text{and} \quad \operatorname{supp}(\mathbf{a}_-) \cap (P_i \cup P_j) \neq \emptyset.$$

The forward direction follows from the fact that our constraints on B make it impossible for there to exist a nonzero $\mathbf{a} \in \mathcal{L}$ such that $\phi(\mathbf{x}^{\mathbf{a}_+}) = \phi(\mathbf{x}^{\mathbf{a}_-})$. By considering the negation of $\mathbf{a}$, we see that $\langle y_i, y_j\rangle$ is an associated prime of J if and only if for all nonzero $\mathbf{a} \in \mathcal{L}$, we have $\operatorname{supp}(\mathbf{a}_-) \cap (P_i \cup P_j) \neq \emptyset$. Writing $\mathbf{a} = B\mathbf{u}$ for some nonzero $\mathbf{u} \in \mathbb{Z}^2$, this is equivalent to requiring that for all nonzero $\mathbf{u} \in \mathbb{Z}^2$, there is some $s \in P_i \cup P_j$ such that $\mathbf{b}_s \cdot \mathbf{u} < 0$. Note this cannot hold if $\{i, j\} = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$. □

Combining the two previous results, we obtain the following:

Proposition 4.5. *We have that* $\deg I_{\mathcal{L}} = \deg I_{\mathcal{L}'}$ *if and only if there are two nonzero vectors* $\mathbf{u}_{13}, \mathbf{u}_{24}$ *for which*

$$(Q_1 \cup Q_3) \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u}_{13} \geq 0\}, \qquad (Q_2 \cup Q_4) \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u}_{24} \geq 0\}.$$

Proof. Recall that $I_{\mathcal{L}'}$ is a lattice ideal that equals the saturation $(J : (y_1y_2y_3y_4)^\infty)$. It follows from [AM69, Proposition 4.9] that $\deg I_{\mathcal{L}'} = \deg J$ if and only if no 1-dimensional associated prime of J contains $\langle y_1y_2y_3y_4 \rangle$, so by Lemma 4.3, we see that $\deg I_{\mathcal{L}'} = \deg J$ if and only if J does not have an associated prime of the form $\langle y_i, y_j \rangle$ for distinct $i, j \in \{1, 2, 3, 4\}$. The desired result then follows from Lemma 4.4 and the fact that $\deg J = \deg I_{\mathcal{L}}$. □

Lemma 4.6. *If* $\deg I_{\mathcal{L}} = \deg I_{\mathcal{L}'}$*, then every syzygy quadrangle of* $I_{\mathcal{L}'}$ *(with respect to* $G_{\mathcal{L}'}$*) is a syzygy quadrangle of* $I_{\mathcal{L}}$ *(with respect to* $G_{\mathcal{L}}$*).*

Proof. Suppose that $I_{\mathcal{L}'}$ has a syzygy quadrangle which is not a syzygy quadrangle for $I_{\mathcal{L}}$. Without loss of generality, after translating we may assume that this syzygy quadrangle has vertices given by $0, \mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}$, for some vectors $\mathbf{v}, \mathbf{w}$ with nonnegative y-coordinates. Since $|\det(\mathbf{v}, \mathbf{w})| = 1$, after translating again if necessary, we may assume that $\mathbf{v}, \mathbf{w}$ both lie in the first closed quadrant or in the second closed quadrant. Suppose without loss of generality that $\det(\mathbf{v}, \mathbf{w}) = 1$. Let $[\mathbf{v}, \mathbf{w}]$ denote this syzygy quadrangle.

Assume that $\mathbf{v}, \mathbf{w}$ both lie in the first closed quadrant. The case where $\mathbf{v}, \mathbf{w}$ both lie in the second closed quadrant is similar. By Proposition 4.2, each vertex of the syzygy quadrangle is supported by some vector in $G_{\mathcal{L}'}$. Thus there exist vectors $\mathbf{b}_1', \mathbf{b}_2', \mathbf{b}_3', \mathbf{b}_4' \in G_{\mathcal{L}'}$ such that $\mathbf{b}_1' \cdot \mathbf{v}, -\mathbf{b}_2' \cdot \mathbf{v}, -\mathbf{b}_3' \cdot \mathbf{v}, \mathbf{b}_4' \cdot \mathbf{v} > 0$ and $\mathbf{b}_1' \cdot \mathbf{w}, \mathbf{b}_2' \cdot \mathbf{w}, -\mathbf{b}_3' \cdot \mathbf{w}, -\mathbf{b}_4' \cdot \mathbf{w} > 0$. Since $G_{\mathcal{L}'}$ contains exactly four vectors, one in each open quadrant, we necessarily have that $\mathbf{b}_i'$ is the unique vector of $G_{\mathcal{L}'}$ that lies in the ith open quadrant. Since $[\mathbf{v}, \mathbf{w}]$ is not a syzygy quadrangle for $I_{\mathcal{L}}$, the Gale diagram $G_{\mathcal{L}}$ does not contain an analogous collection of four vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ satisfying the same inequalities as above. Since $G_{\mathcal{L}}$ contains a vector in each open quadrant, we see that there exist $\mathbf{b}_1, \mathbf{b}_3$ satisfying the conditions. Thus there either does not exist $\mathbf{b}_2 \in G_{\mathcal{L}}$ such that $-\mathbf{b}_2 \cdot \mathbf{v}, \mathbf{b}_2 \cdot \mathbf{w} > 0$, or there does not exist $\mathbf{b}_4 \in G_{\mathcal{L}}$ such that $\mathbf{b}_4 \cdot \mathbf{v}, -\mathbf{b}_4 \cdot \mathbf{w} > 0$. Assume the first case holds; the second case is similar. This implies that each $\mathbf{b} \in Q_2$ either satisfies $\mathbf{b} \cdot \mathbf{v} \geq 0$ or $\mathbf{b} \cdot \mathbf{w} \leq 0$. Since $\sum_{\mathbf{b} \in Q_2} \mathbf{b} = \mathbf{b}_2'$, we see that there must exist some nonzero $\mathbf{c} \in Q_2$ such that $\mathbf{c} \cdot \mathbf{v} \geq 0$, and there exists some nonzero $\mathbf{d} \in Q_2$ such that $\mathbf{d} \cdot \mathbf{w} \leq 0$. But these conditions prevent the nonzero vectors $\mathbf{c}, \mathbf{d}, \mathbf{b}_4'$ from all lying in a single closed half-plane with boundary passing through the origin, contradicting Proposition 4.5, which asserts the existence of some nonzero $\mathbf{u} \in \mathbb{Z}^2$ such that $0 \leq \mathbf{u} \cdot \mathbf{c}, \mathbf{u} \cdot \mathbf{d}$ and $0 \leq \mathbf{u} \cdot \sum_{\mathbf{b} \in Q_4} \mathbf{b} = \mathbf{u} \cdot \mathbf{b}_4'$. □

We can now prove our first key result.

Corollary 4.7. *If* $\operatorname{reg} I_{\mathcal{L}} \geq \deg I_{\mathcal{L}} - 1$*, then* $\operatorname{reg} I_{\mathcal{L}'} = \operatorname{reg} I_{\mathcal{L}}$*.*

Proof. If $\operatorname{reg} I_{\mathcal{L}'} > \operatorname{reg} I_{\mathcal{L}}$, then there is a syzygy quadrangle of $I_{\mathcal{L}'}$ that is not a syzygy quadrangle of $I_{\mathcal{L}}$. Then Lemma 4.6 implies that $\deg I_{\mathcal{L}'} < \deg I_{\mathcal{L}}$. But then $\operatorname{reg} I_{\mathcal{L}} \leq \deg I_{\mathcal{L}} - 2$, a contradiction. □

The following lemma allows us to prove our next result, Proposition 4.9, which tells that if $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$ and certain nonrestrictive technical conditions are satisfied, then there exists some partition $G_{\mathcal{L}} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ for which $\operatorname{reg} I_{\mathcal{L}'} = \deg I_{\mathcal{L}'}$.

Lemma 4.8. *Let $\mathcal{L} \subseteq \mathbb{Z}^4$ and $G_{\mathcal{L}}$ a Gale diagram for which the unit square is a syzygy quadrangle attaining the regularity and* $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$*. Then $G_{\mathcal{L}}$ either lies on two lines or has the following form up to reflections over the axes:*

$$G_{\mathcal{L}} = \{(1, a), (-1, d-1), (-1, 1-a), (1, -d)\}$$

for some $a, d > 1$.

Proof. Let $C \in \mathbb{Z}^4/\mathcal{L}$ be the multidegree for which P_C is the unit square. Then $\operatorname{reg} I_{\mathcal{L}} = \deg C - 2$ since syzygy quadrangles correspond to third syzygies ([PS98], Theorem 3.4). We start in the same spirit as [PS98], Proposition 7.7. Suppose the Gale vectors are given by

$$\mathbf{a} = (a_1, a_2), \quad \mathbf{b} = (-b_1, b_2), \quad \mathbf{c} = (-c_1, -c_2), \quad \mathbf{d} = (d_1, -d_2)$$

where $a_i, b_i, c_i, d_i > 0$ for all $i = 1, 2$. By rotating 180° if necessary, we can WLOG assume that $b_2 \leq d_2$.

We now have the following chain of comparisons:

$$\begin{aligned}
\deg I_{\mathcal{L}} - 1 = \operatorname{reg} I_{\mathcal{L}} &= \deg C - 2 \\
&= a_1 + b_2 + a_2 + d_1 - 2 \\
&\leq (a_1 + d_2 - 1) + (a_2 + d_1 - 1) && (1) \\
&\leq a_1 d_2 + a_2 d_1 && (2) \\
&= |\det(\mathbf{d}, \mathbf{a})| \\
&\leq \deg I_{\mathcal{L}} && (3)
\end{aligned}$$

There are three inequalities, and hence three possibilities for where the jump by 1 can occur. We treat these case by case.

Case 1: Inequality occurs at (1). In this case we have the following:

$$d_2 = b_2 + 1, \quad (a_1 - 1)(d_2 - 1) = (a_2 - 1)(d_1 - 1) = 0, \quad |\det(\mathbf{d}, \mathbf{a})| = \deg I_{\mathcal{L}}$$

Since $d_2 = b_2 + 1$, we also have $c_2 = a_2 - 1$. If $a_2 = 1$ then $c_2 = 0$, so we conclude that $d_1 = 1$ and $a_2 > 1$. Similarly if $d_2 = 1$ then $b_2 = 0$ so we conclude that $a_1 = 1$ and $b_2 > 1$. Since $b_1 + c_1 = a_1 + d_1 = 2$, we conclude that $b_1 = c_1 = 1$. Thus the Gale vectors have the form

$$\mathbf{a} = (1, a_2), \quad \mathbf{b} = (-1, d_2 - 1), \quad \mathbf{c} = (-1, 1 - a_2), \quad \mathbf{d} = (1, -d_2).$$

Case 2: Inequality occurs at (2). In this case we have the following:

$$d_2 = b_2, \quad (a_1 - 1)(d_2 - 1) + (a_2 - 1)(d_1 - 1) = 1, \quad |\det(\mathbf{d}, \mathbf{a})| = \deg I_{\mathcal{L}}$$

Now we have $c_2 = a_2$.

- *Case 2.1:* $a_1 = 1$ and $a_2 = d_1 = 2$. Then the Gale vectors have the form
$$\mathbf{a} = (1, 2), \quad \mathbf{b} = (-b_1, d_2), \quad \mathbf{c} = (-c_1, -2), \quad \mathbf{d} = (2, -d_2)$$
We then also have
$$4 + c_1 d_2 = |\det(\mathbf{c}, \mathbf{d})| \leq \deg I_{\mathcal{L}} = |\det(\mathbf{a}, \mathbf{d})| = 4 + d_2$$
forcing $c_1 = 1$. We conclude that $b_1 = 2$, so $\mathbf{b} + \mathbf{d} = \mathbf{a} + \mathbf{c} = 0$.
- *Case 2.2:* $d_2 = 1$ and $a_2 = d_1 = 2$. Then the Gale vectors have the form
$$\mathbf{a} = (a_1, 2), \quad \mathbf{b} = (-b_1, 1), \quad \mathbf{c} = (-c_1, -2), \quad \mathbf{d} = (2, -1)$$
Then we again have
$$4 + c_1 = |\det(\mathbf{c}, \mathbf{d})| \leq \deg I_{\mathcal{L}} = |\det(\mathbf{a}, \mathbf{d})| = 4 + a_1$$
so $c_1 \leq a_1$. By the same inequality with $|\det(\mathbf{a}, \mathbf{b})|$, we obtain $b_1 \leq 2$. Then
$$2 + a_1 = b_1 + c_1 \leq 2 + c_1$$
so $a_1 \leq c_1$. We conclude that $a_1 = c_1$. So $b_1 = 2$ and again $\mathbf{b} + \mathbf{d} = \mathbf{a} + \mathbf{c}$.
- *Case 2.3:* $a_1 = d_2 = 2$ and $a_2 = 1$. Proceed as in Case 2.1 to conclude that $G_{\mathcal{L}}$ lies on two lines.
- *Case 2.4:* $a_1 = d_2 = 2$ and $d_1 = 1$. Proceed as in Case 2.2 to conclude that $G_{\mathcal{L}}$ lies on two lines.

Case 3: Inequality occurs at (3). In this case we have the following:
$$d_2 = b_2, \quad (a_1 - 1)(d_2 - 1) = (a_2 - 1)(d_1 - 1) = 0, \quad |\det(\mathbf{d}, \mathbf{a})| = \deg I_{\mathcal{L}} - 1$$
We have $a_2 = c_2$ once again.

- *Case 3.1:* $a_1 = a_2 = 1$. Then the Gale vectors have the form
$$\mathbf{a} = (1, 1), \quad \mathbf{b} = (-b_1, d_2), \quad \mathbf{c} = (-c_1, -1), \quad \mathbf{d} = (d_1, -d_2)$$
We now have
$$c_1 d_1 + d_2 = |\det(\mathbf{c}, \mathbf{d})| \leq \deg I_{\mathcal{L}} = |\det(\mathbf{d}, \mathbf{a})| + 1 = d_1 + d_2 + 1$$
Then either $c_1 = 1$ or $(c_1, d_1) = (2, 1)$. In the former case, $d_1 = b_1$ and the Gale diagram lies on two lines. In the latter case, $b_1 = 2$ the Gale diagram must be
$$\mathbf{a} = (1, 1), \quad \mathbf{b} = (-2, d_2), \quad \mathbf{c} = (-2, -1), \quad \mathbf{d} = (1, -d_2)$$
But now we see that
$$2d_2 + 2 = |\det(\mathbf{b}, \mathbf{c})| \leq \deg I_{\mathcal{L}} = |\det(\mathbf{d}, \mathbf{a})| + 1 = d_2 + 2$$
which is absurd, so $(c_1, d_1) = (2, 1)$ does not occur.
- *Case 3.2:* $a_1 = d_1 = 1$. Since $b_1 + c_1 = a_1 + d_1 = 2$, we conclude that $b_1 = c_1 = 1$. Then the Gale vectors have the form
$$\mathbf{a} = (1, a_2), \quad \mathbf{b} = (-1, d_2), \quad \mathbf{c} = (-1, -a_2), \quad \mathbf{d} = (1, -d_2)$$
and so lie on two lines.
- *Case 3.3:* $a_2 = d_2 = 1$. Proceed as in Case 3.2 to conclude that $G_{\mathcal{L}}$ lies on two lines.
- *Case 3.4:* $d_1 = d_2 = 1$. Proceed as in Case 3.1 to conclude that $G_{\mathcal{L}}$ lies on two lines.

□

We now prove our second key result.

Proposition 4.9. *Let $n \geq 5$ and let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice satisfying the following conditions:*

- $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$,
- *the Gale diagram $G_{\mathcal{L}}$ consists of nonzero vectors,*
- *the Gale diagram $G_{\mathcal{L}}$ is not contained on two lines.*

Suppose there is a partition $G_{\mathcal{L}} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ giving rise to a lattice $\mathcal{L}' \subseteq \mathbb{Z}^4$ satisfying $\operatorname{reg} I_{\mathcal{L}'} = \deg I_{\mathcal{L}'} - 1$. *Then there is a different choice of partition $G_{\mathcal{L}} = R_1 \cup R_2 \cup R_3 \cup R_4$ giving rise to a lattice $\mathcal{L}'' \subseteq \mathbb{Z}^4$ satisfying* $\operatorname{reg} I_{\mathcal{L}''} = \deg I_{\mathcal{L}''}$.

Proof. First, observe that by construction, the unit square remains a syzygy quadrangle of $G_{\mathcal{L}'}$. Since $\operatorname{reg} I_{\mathcal{L}'} = \operatorname{reg} I_{\mathcal{L}}$ by Corollary 4.7, it follows that the unit square still attains the regularity. We therefore know that $G_{\mathcal{L}'}$ is of one of the forms described in Lemma 4.8. Write $G_{\mathcal{L}'} = \{\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3, \mathbf{b}'_4\}$ where $\mathbf{b}'_i = \sum_{\mathbf{b} \in Q_i} \mathbf{b}$. Suppose that $G_{\mathcal{L}'}$ lies on two lines $\ell_1 \supseteq \{\mathbf{b}'_1, \mathbf{b}'_3\}$ and $\ell_2 \supseteq \{\mathbf{b}'_2, \mathbf{b}'_4\}$ passing through the origin. Since $\operatorname{reg} I_{\mathcal{L}'} = \operatorname{reg} I_{\mathcal{L}}$, we see that $\deg I_{\mathcal{L}} = \deg I_{\mathcal{L}'}$, so Proposition 4.5 implies that there is a closed half-plane $\mathcal{H}_{13}$ containing $Q_1 \cup Q_3$. Since $\mathcal{H}_{13}$ is closed under addition, this implies that $\mathbf{b}'_1, \mathbf{b}'_3 \in \mathcal{H}_{13}$, so we conclude that $\partial \mathcal{H}_{13} = \ell_1$. But then every vector in Q_1 must lie on ℓ_1, otherwise $\mathbf{b}'_1 \in \mathcal{H}_{13} \setminus \ell_1$. Likewise, $Q_3 \subseteq \ell_1$. Analogously, $Q_2, Q_4 \subseteq \ell_2$. This contradicts that $G_{\mathcal{L}}$ is not contained in two lines. So henceforth suppose $G_{\mathcal{L}'}$ is not contained in two lines.

Let ℓ^+ and ℓ^- denote the positive and negative y-axis, respectively. If $G_{\mathcal{L}} \cap \ell^+$ and $G_{\mathcal{L}} \cap \ell^-$ are both nonempty, Proposition 4.5 implies there is a partition of $G_{\mathcal{L}}$ leading to a lattice $\mathcal{L}''$ with $\deg I_{\mathcal{L}''} < \deg I_{\mathcal{L}}$ (choose any partition $R_1 \cup R_2 \cup R_3 \cup R_4$ so that R_1 contains vectors from $G_{\mathcal{L}} \cap \ell^+$ and R_3 contains vectors from $G_{\mathcal{L}} \cap \ell^-$). So henceforth suppose WLOG that $G_{\mathcal{L}} \cap \ell^- = \emptyset$.

Lemma 4.8 implies that up to reflections over the axes, we have

$$G_{\mathcal{L}'} = \{(1, a), (-1, d-1), (-1, 1-a), (1, -d)\}$$

for some $a, d > 1$. Combining this with $G_{\mathcal{L}} \cap \ell^- = \emptyset$, we have

$$G_{\mathcal{L}} = \{(1, a'), (-1, b'), (-1, 1-a), (1, -d)\} \cup Q_1^{x=0} \cup Q_2^{x=0}$$

where $Q_i^{x=0}$ is the intersection of Q_i with y-axis and $0 < a' \leq a$ and $0 < b' \leq d-1$. Suppose that $Q_2^{x=0}$ is nonempty, that is, there is some $(0, \varepsilon) \in Q_2$. Since $\deg I_{\mathcal{L}} = \deg I_{\mathcal{L}'}$, we can choose a half-plane $\mathcal{H}_{24} = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u} \geq 0\}$ containing $Q_2 \cup Q_4$. If $\mathbf{u} = (u_1, u_2)$, this now implies

$$-u_1 + u_2 b' \geq 0, \quad u_1 - u_2 d \geq 0, \quad u_2 \geq 0$$

which quickly implies $u_1 = u_2 = 0$, a contradiction. So $Q_2^{x=0} = \emptyset$ and $b' = d-1$. If $a' < a-1$, the same argument shows that $Q_1^{x=0} = \emptyset$. But this is impossible, since $n \geq 5$. So $a' \geq a-1$. Since $Q_1^{x=0}$ contains some nonzero vector, we also see that $a' \leq a-1$. So $a' = a-1$ and $Q_1^{x=0} = \{(0,1)\}$ (and also, evidently, $n = 5$). To finish the argument, consider the partition $\{R_i\}_{i=1}^4$ obtained by taking $\{Q_i\}_{i=1}^4$ and moving $(0,1)$ from Q_1 to Q_2. The resulting lattice $\mathcal{L}'' \subseteq \mathbb{Z}^4$ has Gale diagram

$$G_{\mathcal{L}''} = \{(1, a-1), (-1, d), (-1, 1-a), (1, -d)\}$$

and so lies on two lines. Conclude as in the first paragraph that $\deg I_{\mathcal{L}''} \neq \deg I_{\mathcal{L}}$. $\square$

Now, we begin to prove Proposition 4.11.

Lemma 4.10. *Suppose $\langle y_i, y_j \rangle$ is an associated prime of J for distinct $i, j \in \{1,2,3,4\}$. If $J \not\subseteq \langle y_i, y_j^2 \rangle$, then there exist $\mathbf{b}_j \in Q_j$ and a nonzero $\mathbf{u} \in \mathbb{Z}^2$ such that $Q_i \cup (Q_j - \{\mathbf{b}_j\}) \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u} \geq 0\}$ and $\mathbf{b}_j \cdot \mathbf{u} = -1$.*

Proof. Suppose that $J \not\subseteq \langle y_i, y_j^2 \rangle$. Then there exists a nonzero $\mathbf{a} \in \mathcal{L}$ such that y_i and y_j^2 do not divide $\phi(\mathbf{x}^{\mathbf{a}_-})$. Letting $\mathbf{u} \in \mathbb{Z}^2$ such that $B\mathbf{u} = \mathbf{a}$, we see that $Q_i \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u} \geq 0\}$, and that there is at most one vector $\mathbf{b}_j \in Q_j$ such that $\mathbf{b}_j \cdot \mathbf{u} < 0$, and moreover if such a $\mathbf{b}_j$ exists, then $\mathbf{b}_j \cdot \mathbf{u} = -1$. By Lemma 4.4, since $\mathbf{u}$ is nonzero, we see that such a $\mathbf{b}_j$ indeed exists, so we are done. ☐

Proposition 4.11. *Suppose that $\mathcal{L} \subseteq \mathbb{Z}^n$ is chosen such that $\deg(I_{\mathcal{L}'}) = \deg(I_{\mathcal{L}}) - 1$. Suppose furthermore that the Gale diagram $G_{\mathcal{L}}$, chosen such that the unit square is a syzygy quadrangle attaining the regularity, satisfies $\sum_{\mathbf{b} \in Q_1 \cup Q_3} \mathbf{b} = \sum_{\mathbf{b} \in Q_2 \cup Q_4} \mathbf{b} = 0$. Then for either $\{i, j\} = \{1, 3\}$ or $\{i, j\} = \{2, 4\}$, the vectors in $G_{\mathcal{L}} - (Q_i \cup Q_j)$ all lie on a single line passing through the origin. Furthermore, there exist vectors $\mathbf{b}_i \in Q_i, \mathbf{b}_j \in Q_j$ and a nonzero vector $\mathbf{u} \in \mathbb{Z}^2$ such that $\mathbf{b}_i \cdot \mathbf{u} = 1$, $\mathbf{b}_j \cdot \mathbf{u} = -1$, and $\mathbf{b} \cdot \mathbf{u} = 0$ for all $\mathbf{b} \in (Q_i \cup Q_j) - \{\mathbf{b}_i, \mathbf{b}_j\}$.*

Proof. Since $\deg(I_{\mathcal{L}'}) = \deg(I_{\mathcal{L}}) - 1 = \deg(J) - 1$, it follows from [AM69, Proposition 4.9] and Lemmas 4.3 and 4.4 that exactly one of $\langle y_1, y_3 \rangle$ and $\langle y_2, y_4 \rangle$ is an associated prime of J. Let $\langle y_i, y_j \rangle$ be this prime, so that $\{i, j\} = \{1, 3\}$ or $\{i, j\} = \{2, 4\}$. Let $\{1,2,3,4\} - \{i, j\} = \{i', j'\}$. Since $\langle y_{i'}, y_{j'} \rangle$ is not an associated prime of J, it follows from Lemma 4.4 and $\sum_{\mathbf{b} \in Q_{i'} \cup Q_{j'}} \mathbf{b} = 0$ that all the vectors in $Q_{i'} \cup Q_{j'}$ lie on a line passing through the origin.

Note that the ideal $\langle y_i, y_j^2 \rangle$ has degree 2. If $J \subseteq \langle y_i, y_j^2 \rangle$, then it follows that $\deg(I_{\mathcal{L}'}) \leq \deg(J) - 2$, a contradiction. Thus, $J \not\subseteq \langle y_i, y_j^2 \rangle$, and similarly $J \not\subseteq \langle y_i^2, y_j \rangle$. So, there exist vectors $\mathbf{b}_i \in Q_i, \mathbf{b}_j \in Q_j$ and nonzero $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^2$ such that $Q_i \cup (Q_j - \{\mathbf{b}_j\}) \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u}_1 \geq 0\}$, $\mathbf{b}_j \cdot \mathbf{u}_1 = -1$, $(Q_i - \{\mathbf{b}_i\}) \cup Q_j \subseteq \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{u}_2 \geq 0\}$, and $\mathbf{b}_i \cdot \mathbf{u}_2 = -1$.

Recall that by definition, $\mathbf{b}'_i = \sum_{\mathbf{b} \in Q_i} \mathbf{b}$ and $\mathbf{b}'_j = \sum_{\mathbf{b} \in Q_j} \mathbf{b}$. By assumption, $\mathbf{b}'_j = -\mathbf{b}'_i$. Note that $\mathbf{b}'_i \cdot \mathbf{u}_1 \geq 0$, $\mathbf{b}'_j \cdot \mathbf{u}_1 \geq -1$, $\mathbf{b}'_i \cdot \mathbf{u}_2 \geq -1$, and $\mathbf{b}'_j \cdot \mathbf{u}_2 \geq 0$. Thus, $0 \leq \mathbf{b}'_i \cdot \mathbf{u}_1 \leq 1$ and $-1 \leq \mathbf{b}'_i \cdot \mathbf{u}_2 \leq 0$.

Suppose that $\mathbf{u}_2 = -\alpha \mathbf{u}_1$ for some positive real number α. If $\mathbf{b}'_i \cdot \mathbf{u}_1 = 0$, then $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_i$, but this contradicts that $\mathbf{b}_i \cdot \mathbf{u}_2 = -1$. Thus, we have that $\mathbf{b}'_i \cdot \mathbf{u}_1 = 1$, so $\mathbf{b}'_i \cdot \mathbf{u}_2 = -\alpha$, and $\alpha = 1$. This implies that $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_i - \{\mathbf{b}_i\}$. We also have $\mathbf{b}'_j \cdot \mathbf{u}_1 = -1$, so that $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_j - \{\mathbf{b}_j\}$. So, we are done by taking $\mathbf{u} = \mathbf{u}_1$.

Henceforth, suppose that $\mathbf{u}_2 \neq -\alpha \mathbf{u}_1$ for any positive real number α. Clearly we also have $\mathbf{u}_2 \neq -\alpha \mathbf{u}_1$ for any nonpositive real number α, so $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent over $\mathbb{R}$. For convenience, in the remaining cases, we will take advantage of the linear independence of $\mathbf{u}_1, \mathbf{u}_2$ by occasionally referring to a vector $\mathbf{v} \in \mathbb{R}^2$ as an ordered pair $[\mathbf{v} \cdot \mathbf{u}_1, \mathbf{v} \cdot \mathbf{u}_2]$ of real numbers enclosed in brackets. Note that each ordered pair uniquely determines the corresponding member of $\mathbb{R}^2$.

Suppose $\mathbf{b}'_i \cdot \mathbf{u}_1 = 0$. Then $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_i$. This implies that $\mathbf{b}_i = [0, -1]$ must be the only nonzero vector in Q_i. Also, $\mathbf{b}'_j \cdot \mathbf{u}_1 = 0$, so there exists $\mathbf{v}_j \in Q_j$ different from $\mathbf{b}_j$ such that $\mathbf{v}_j \cdot \mathbf{u}_1 = 1$, and moreover $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_j - \{\mathbf{b}_j, \mathbf{v}_j\}$. Since $\mathbf{b}'_i \neq 0$, we have $\mathbf{b}'_i \cdot \mathbf{u}_2 = -1$, hence $\mathbf{b}'_j \cdot \mathbf{u}_2 = 1$. This implies there exists $\mathbf{v}'_j \in Q_j$ such that $\mathbf{v}'_j \cdot \mathbf{u}_2 = 1$ and $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_j - \{\mathbf{v}'_j\}$. Note that for all $\mathbf{b} \in Q_j - \{\mathbf{b}_j, \mathbf{v}_j, \mathbf{v}'_j\}$, we have

$\mathbf{b} \cdot \mathbf{u}_1 = \mathbf{b} \cdot \mathbf{u}_2 = 0$, hence $\mathbf{b} = 0$; thus, the only nonzero vectors in Q_j are $\mathbf{b}_j, \mathbf{v}_j, \mathbf{v}'_j$. If $\mathbf{v}'_j$ is different from both $\mathbf{b}_j$ and $\mathbf{v}_j$, then $\mathbf{b}_j \cdot \mathbf{u}_2 = \mathbf{v}_j \cdot \mathbf{u}_2 = 0$, which implies that $\mathbf{b}_j = -\mathbf{v}_j$, a contradiction. Thus, either $\mathbf{v}'_j = \mathbf{b}_j$ or $\mathbf{v}'_j = \mathbf{v}_j$. In the first case, the only nonzero members of Q_j are $[-1, 1], [1, 0]$, and in the second case, they are $[-1, 0], [1, 1]$. In either case, we are done by taking $\mathbf{u} = -\mathbf{u}_2$ and relabeling vectors if necessary.

Now suppose that $\mathbf{b}'_i \cdot \mathbf{u}_1 = 1$. Then there exists $\mathbf{v}_i \in Q_i$ such that $\mathbf{v}_i \cdot \mathbf{u}_1 = 1$, and $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_i - \{\mathbf{v}_i\}$. Also, we have that $\mathbf{b}'_j \cdot \mathbf{u}_1 = -1$, so $\mathbf{b} \cdot \mathbf{u}_1 = 0$ for all $\mathbf{b} \in Q_j - \{\mathbf{b}_j\}$. We now break into cases:

- *Case 1:* $\mathbf{b}'_i \cdot \mathbf{u}_2 = -1$. Thus, $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_i - \{\mathbf{b}_i\}$. For all $\mathbf{b} \in Q_i - \{\mathbf{b}_i, \mathbf{v}_i\}$, we have $\mathbf{b} \cdot \mathbf{u}_1 = \mathbf{b} \cdot \mathbf{u}_2 = 0$, so that $\mathbf{b} = 0$. This implies the only nonzero vectors in Q_i are $\mathbf{b}_i, \mathbf{v}_i$. If $\mathbf{b}_i = \mathbf{v}_i$, then Q_i contains exactly one nonzero vector, namely $[1, -1]$ and if $\mathbf{b}_i \neq \mathbf{v}_i$, then Q_i contains exactly two distinct nonzero vectors, namely $\mathbf{b}_i = [0, -1]$ and $\mathbf{v}_i = [1, 0]$.

  We also have $\mathbf{b}'_j \cdot \mathbf{u}_2 = 1$. Then there exists $\mathbf{v}_j \in Q_j$ such that $\mathbf{v}_j \cdot \mathbf{u}_2 = 1$, and $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_j - \{\mathbf{v}_j\}$. It follows that the only nonzero vectors in Q_j are $\mathbf{b}_j, \mathbf{v}_j$. If $\mathbf{b}_j = \mathbf{v}_j$, then the only nonzero vector in Q_j is $[-1, 1]$, and if $\mathbf{b}_j \neq \mathbf{v}_j$, then the only nonzero vectors in Q_j are $\mathbf{b}_j = [-1, 0]$ and $\mathbf{v}_j = [0, 1]$. It is easy to see that for all four possibilities, the desired vectors exist.
- *Case 2:* $\mathbf{b}'_i \cdot \mathbf{u}_2 = 0$. So, there exists $\mathbf{v}'_i \in Q_i$ different from $\mathbf{b}_i$ such that $\mathbf{v}'_i \cdot \mathbf{u}_2 = 1$, and $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_i - \{\mathbf{b}_i, \mathbf{v}'_i\}$. So $\mathbf{b}_i, \mathbf{v}_i, \mathbf{v}'_i$ are the only nonzero vectors in Q_i. If $\mathbf{v}_i$ is not equal to both $\mathbf{b}_i$ and $\mathbf{v}'_i$, then $\mathbf{b}_i \cdot \mathbf{u}_1 = \mathbf{v}'_i \cdot \mathbf{u}_1 = 0$. This implies $\mathbf{b}_i = -\mathbf{v}'_i$, a contradiction. Thus $\mathbf{v}_i = \mathbf{b}_i$ or $\mathbf{v}_i = \mathbf{v}'_i$. In the first case, the only nonzero vectors in Q_i are $[1, -1], [0, 1]$, and in the second case, they are $[0, -1], [1, 1]$.

  Also, $\mathbf{b}'_j \cdot \mathbf{u}_2 = 0$, so that $\mathbf{b} \cdot \mathbf{u}_2 = 0$ for all $\mathbf{b} \in Q_j$. This implies that $\mathbf{b}_j = [-1, 0]$ is the only nonzero vector in Q_j. It is easy to check that for either possibility for Q_i, the desired vectors exist.

Having exhausted all cases, we are done. □

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose that $I_{\mathcal{L}}$ is a codimension-2 toric ideal that is not Cohen-Macaulay and achieves equality in the Eisenbud–Goto conjecture. Then $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$. Choose a Gale diagram $G_{\mathcal{L}}$ for which the unit square is a syzygy quadrangle achieving the regularity. Since $I_{\mathcal{L}}$ is not a complete intersection, [PS98, Proposition 7.10] implies that $G_{\mathcal{L}}$ does not lie on two lines. Applying Proposition 4.9, we conclude there is a reduction $\mathcal{L}' \subseteq \mathbb{Z}^4$ for which $\operatorname{reg} I_{\mathcal{L}'} = \deg I_{\mathcal{L}'}$. Then Theorem 4.1 and [PS98, Remark 7.9] imply that $G_{\mathcal{L}'}$ lies on two lines, and in particular is of one of the following two forms:

$$\{(1, 1), (a, -b), (-1, -1), (-a, b)\} \quad \text{or} \quad \{(1, a), (1, -b), (-1, -a), (-1, b)\}.$$

This implies that the sum of the vectors in $Q_1 \cup Q_3$ (resp. $Q_2 \cup Q_4$) is 0. Now Proposition 4.11 gives the remaining properties of $G_{\mathcal{L}}$, after reflecting across an axis if necessary. □

5. Future Directions

5.1. **Overview of simplicial affine semigroup rings.** Since [Nit14] proved the Eisenbud–Goto conjecture for certain simplicial affine semigroup rings, it makes sense to see when equality should occur in these cases as well. We spend this section recalling the setup of [Nit14] and discussing starting points for analyzing when equality occurs. The following definitions can be found in [Nit14]. We largely follow the notation there and as such, conventions may be inconsistent with those in previous sections.

Let $\alpha \in \mathbb{N}$ and let $e_i = (0, 0, \dots, \alpha, \dots, 0) \in \mathbb{N}^d$ be the vector with an α in the ith coordinate and 0's elsewhere. Let $a_1, \dots, a_c \in \mathbb{N}^d$ be a set of vectors, with $a_i = (a_{i1}, \dots, a_{id})$, and assume that $\gcd(a_{ij})_{i,j} = 1$. Let B be the submonoid of $(\mathbb{N}^d, +)$ generated by the set $\{e_1, \dots, e_d, a_1, \dots, a_c\}$. Then $c = \operatorname{codim} k[B]$.

Let A be the submonoid of B generated by $\{e_1, \dots, e_d\}$, and let $\mathbb{Z}A$ and $\mathbb{Z}B$ be the groups generated by A and B respectively. For $x = (x_1, \dots, x_d) \in \mathbb{N}^d$ we set $\deg(x) = \alpha^{-1} \sum_{i=1}^d x_i$. Now let

$$B_A \coloneqq \{x \in B : x - a \notin B \text{ for all } a \in A\}.$$

Let $\sim$ denote the equivalence relation on $\mathbb{Z}B$ such that $x \sim y$ when $x - y \in \mathbb{Z}A = \alpha\mathbb{Z}^d$. There are finitely many equivalence classes which we denote $\Gamma_1, \dots, \Gamma_f$. For $1 \le t \le f$, we define $h_t \in \mathbb{Z}^d$ via

$$h_t \coloneqq (\min\{m_1 : m \in \Gamma_t\}, \dots, \min\{m_d : m \in \Gamma_t\}).$$

Let $T \coloneqq k[\mathbf{y}] = k[y_1, \dots, y_d]$ be a polynomial ring with the standard grading. Let

$$\widetilde{\Gamma}_t \coloneqq \{\mathbf{y}^{\frac{x-h_t}{\alpha}} : x \in \Gamma_t\}$$

and let $I_t = \widetilde{\Gamma}_t T$. We get from [Nit14] that

$$\operatorname{reg} k[B] = \max\{\deg I_t + \deg h_t : 1 \le t \le f\}.$$

For some $x \in B$ and $b_1, \dots, b_n \in B$ we say $\lambda = (b_1, \dots, b_n)$ has the *$*$-property of x* if $b_i \in \{e_1, \dots, e_d, a_1, \dots, a_c\}$ and $x - \sum_{k=1}^i b_k \in B$ for all $1 \le i \le n$. Let $x(\lambda, i)$ denote $x - \sum_{k=1}^i b_k$ and let Λ_x denote all sequence of length $\deg(x)$ with the $*$-property of x. Finally let $x, y \in B \setminus \{0\}$, $\lambda \in \Lambda_x$ and $\nu \in \Lambda_y$. We define

- $\Delta(\lambda, \nu) = \{(i, j) \in \mathbb{N}^2 : 0 \le i \le \deg(x),\ 0 \le j \le \deg(y), x(\lambda, i) \sim y(\nu, j)\}$,
- $\delta(\lambda, \nu) = |\Delta(\lambda, \nu)| - 2$,
- $\delta(x, y) = \min\{\delta(\lambda', \nu') : \lambda' \in \Lambda_x, \nu' \in \Delta_y\}$.

We are prepared to state the conditions in which equality is obtained in the results of [Nit14, §3].

Proposition 5.1. *Suppose there exists a t such that $\operatorname{reg} k[B] = \operatorname{reg} I_t + \deg h_t$ and $|\Gamma_t| = 2$. Let $\Gamma_t = \{x, x'\}$, and let $\lambda \in \Lambda_x$ and $\nu \in \Lambda_{x'}$. Then $\operatorname{reg} k[B] = \deg k[B] - \operatorname{codim} k[B]$ if and only if the following 3 conditions are satisfied:*

- *$\deg k[B] = \deg x + \deg x' - |\Delta(\lambda, \nu)| + c$,*
- *$(\deg x - 1, \deg x' - 1) \notin \Delta(\lambda, \nu)$,*
- *$\delta(\lambda, \nu) = \deg h_t - 1$.*

Proof. See [Nit14, Theorem 3.2]. ∎

Thus a natural starting point for describing when $k[B]$ achieves equality (in the cases where the hypotheses of Proposition 5.1 are met) is to analyze when the various conditions in Proposition 5.1 can hold.

5.2. **Other directions for further study.** There are a few natural directions in which one could try to extend the results of §4:

- Addressing the Cohen-Macaulay case of codimension-2 toric ideals. This would involve looking at syzygy triangles as opposed to quadrangles.
- Proving partial converses to the main results of §4.
- Analyzing the case of generic lattice ideals to see if the statements of §4 can be made stronger under this assumption.

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