Alcove walks and Iwahori triple intersections

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August 7, 2020

Abstract

The alcove walk model gives a combinatorial method for computing intersections of subsets of the affine flag variety. Work by Beazley-Brubaker [4] expanding upon work of Parkinson-Ram-Schwer [1] has shown that we can define the Whittaker function in terms of the alcove walk model. In this paper we prove results on folding in the alcove walk model: namely how crossings change sign after folding, the independence of the existence of a folded walk on the reduced expression we choose, and the maximum number of allowed folds in the alcove model for any particular simple Lie algebra. Lastly we describe the triple intersections on which the Whittaker function is supported given a weight.

Contents

1 Introduction

Given a field $\mathcal{K}$ and a simple complex Lie algebra $\mathfrak{g}$, we can construct a Chevalley group $G(\mathcal{K})$. For any field, we can define subgroups B, U^+, U^-, K, I of G, and G decomposes into double cosets for pairs of these subgroups [1]:

$$G = \bigsqcup_{w \in W} BwB \qquad K = \bigsqcup_{w \in W} IwI$$

$$G = \bigsqcup_{w \in \widetilde{W}} IwI \qquad G = \bigsqcup_{\lambda \in Q_+^\vee} Kt_\lambda K$$

$$G = \bigsqcup_{\mu \in Q^\vee} U^\pm t_\mu K \qquad G = \bigsqcup_{v \in \widetilde{W}} U^\pm vI$$

We are particularly interested in double intersections of the form $U^{-}t_{\lambda}K \cap U^{+}t_{\mu}K$ which further decompose into triple intersections of the form $U^{-}t_{\lambda}wI \cap IvI \cap U^{+}t_{\mu}w'I$. Labelled folded alcove walks are the combinatorial tools we use to enumerate the points of this triple intersection in the affine flag variety G/I. We demonstrate how steps in a walk change after folding which depends only on the action of the finite Weyl group. We also prove that the existence of a labelled folded walk of type v ending in w is independent of the reduced expression that we choose for v. Finally, for any simple Lie algebra, we show that the maximum number of folds in a given walk is equal to the length of the long element w_0 in the finite Weyl group.

These alcove walks can also be used to compute the Whittaker function of an antidominant weight $\lambda \in Q^{\vee}$ in a way that is independent of Cartan type. We comment on the triple intersections which gives non-trivial contribution to the Whittaker function for any particular weight for the Lie algebra $\mathfrak{sl}_3$.

2 Background

2.1 Root systems and Weyl groups

Here we introduce some introductory theory of finite root systems and finite Weyl groups. Let $\mathfrak{h}^*_{\mathbb{R}}$ be a finite dimensional vector space over $\mathbb{R}$ with $(\cdot,\cdot)$ the inner product induced by the Killing form. If $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$, let H_α denote the hyperplane perpendicular to α:

$$H_\alpha := \langle \beta \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \beta, \alpha \rangle = 0 \rangle.$$

If α and β are elements of $\mathfrak{h}^*_{\mathbb{R}}$, then define

$$\langle \beta, \alpha \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$$

Then

$$s_\alpha(\beta) := \beta - \langle \beta, \alpha \rangle \alpha$$

is the reflection over H_α. In particular, $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*_{\mathbb{R}})$ is diagonalizable with exactly one eigenvalue not equal to 1. A root system R is a finite set of nonzero elements of $\mathfrak{h}^*_{\mathbb{R}}$ such that

1. $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in R$.
2. $s_\alpha(R) = R$ for all $\alpha \in R$.
3. $\langle \beta, \alpha \rangle$ is integral for all $\alpha, \beta \in R$.

We will assume R is irreducible; that is no subset of R is also a root system. Then there is a subset $\Delta \subset R$ called simple roots which form an integral basis for R. Moreover, if

$$\alpha = \sum_{\alpha_i \in \Delta} k_i \alpha_i$$

with $k_i \in \mathbb{Z}$ for all i then $k_i \geq 0$ or $k_i \leq 0$ for all i. We say α is positive or negative respectively. There is a disjoint decomposition

$$R = R^+ \cup R^-.$$

where R^+ is the set of positive roots and R^- is the set of negative roots. The Weyl group W is defined by

$$W := \langle s_i \mid \alpha_i \in \Delta \rangle$$

where $s_i = s_{\alpha_i}$.

Remark 2.1. Property (2) of a root system is often phrased by saying that R is W-invariant.

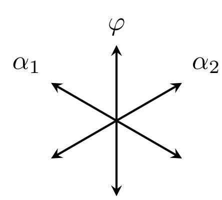
If α is a root, then we say
$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$$
is the coroot of α. The set of coroots $R^\vee$ form a root system called the dual root system of R, but we will not need this fact. Instead, Let $Q^\vee$ be the $\mathbb{Z}$-linear span of $R^\vee$ called the weight lattice. Since R has integral basis Δ , we have a decomposition
$$Q^\vee = \sum_{\alpha_i \in \Delta} \mathbb{Z}\alpha_i^\vee.$$

2.2 The affine Weyl group

The Weyl groups W can be extended to an infinite group W_{aff} by introducing translates of hyperplanes. We will see that the affine Weyl group W_{aff} induces a tessellation of $\mathfrak{h}^*_{\mathbb{R}}$ into alcoves. Let W be a Weyl group and R its associated root system. Let $\delta : \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$ be the (non-linear) functional with $\delta(\lambda) = 1$ for all $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. The affine root system is $R_{\text{aff}} = R + \mathbb{Z}\delta$. The affine hyperplane $H_{\alpha+j\delta}$ is defined by
$$H_{\alpha+j\delta} := \{\beta \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \beta, \alpha\rangle = j\}.$$
If we normalize the length of the roots so that the short roots have length 1, then $H_{\alpha+j\delta}$ is the hyperplane H_α shifted by j in the direction of α. The affine reflection across $H_{\alpha+j\delta}$ is
$$s_{\alpha+j\delta}(\beta) := \beta - (\langle \beta, \alpha\rangle + j)\alpha^\vee.$$
We let $\alpha_0 = -\varphi + \delta$ where φ is the highest root. The affine Weyl group W_{aff} is defined by
$$W_{\text{aff}} := \langle s_i \mid 0 \leq i \leq n\rangle$$
where $n = |\Delta|$ is the number of simple roots of R so that W embeds in W_{aff} as the subgroup $\langle s_i \mid 1 \leq i \leq n\rangle$. If $t_\mu : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ is the translation by μ:
$$t_{\mu^\vee}(\lambda) := \lambda + \mu^\vee,$$
then $s_{\alpha+k\delta} = t_{-\kappa\alpha^\vee} s_\alpha$ and we have an isomorphism $W_{\text{aff}} \cong W \ltimes Q^\vee$.

2.3 The $\mathfrak{sl}_3$ case

We will from now most examples will come from $\mathfrak{sl}_3$. However, most of our results for alcove walks hold for general simple Lie algebras. The finite root system for $\mathfrak{sl}_3$ is type A_2 so α_1, α_2 are the simple roots and the root system takes the form



The finite Weyl group W is isomorphic to the symmetric group S_3 and the affine Weyl group W_{aff} is isomorphic to the affine symmetric group $\widetilde{S}_3$. Write
$$W_{\text{aff}} = \langle s_0, s_1, s_2 \mid s_i^2 = (s_i s_j)^3 = 1 \text{ for } 0 \leq i < j \leq 2\rangle$$
so that $W \subset W_{\text{aff}}$ embeds as the subgroup generated by s_1 and s_2. The coroot lattice is
$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee.$$

3 The Alcove Walk Model

3.1 The alcove model and actions on alcoves

The alcoves are the open connected components of

$$\mathfrak{h}_{\mathbb{R}}^* \setminus \left(\bigcup_{-\alpha_i + j\delta \in \widetilde{R}_{\mathrm{re}}^I} H_{-\alpha_i + j\delta} \right) \quad \text{where} \quad H_{-\alpha_i+j\delta} = \{x^\vee \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x^\vee, \alpha_i \rangle = j\}$$

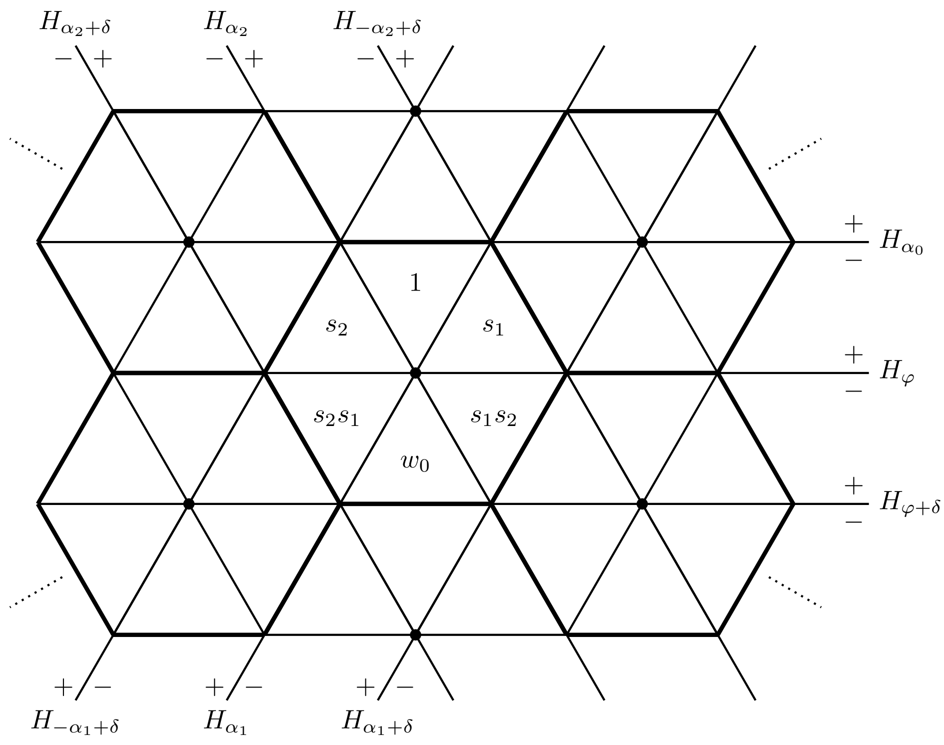
There is a natural left action of W_{aff} on alcoves defined on generators where s_i acts by reflecting an alcove A across the hyperplane H_{α_i}. Each alcove is a fundamental region for action of W_{aff} and this action is simply transitive (see [1] (5.16) and (5.17)). We identify $1 \in W_{\mathrm{aff}}$ with the fundamental alcove

$$\mathbf{1} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \beta, \alpha_i \rangle > 0 \text{ for all } 0 \le i \le 2\},$$

so there is a bijection

$$W_{\mathrm{aff}} \longleftrightarrow \{\text{alcoves}\} \tag{1}$$

Under (1), we will write w both for elements of W_{aff} and alcoves. In our setting, the alcove model is the infinite diagram



where we have suppressed some of the alcove labels and hyperplanes for simplicity . The centres of hexagons are in bijective correspondence with elements of the coroot lattice $Q^\vee$, so if we identify W with the corresponding alcoves in (1),

$$\bigcup_{\lambda^\vee \in Q^\vee} t_{\lambda^\vee} W$$

is a mutually disjoint open cover for the alcoves. Hence to each alcove (or walk) w there corresponds a unique $\lambda^\vee$ such that $w \in t_{\lambda^\vee}(W)$, and we say the alcove (or walk) w belongs to $\lambda^\vee$. We also assign an orientation to the hyperplanes such that

1. 1 lies on the positive side of H_α for every $\alpha \in R^+$.

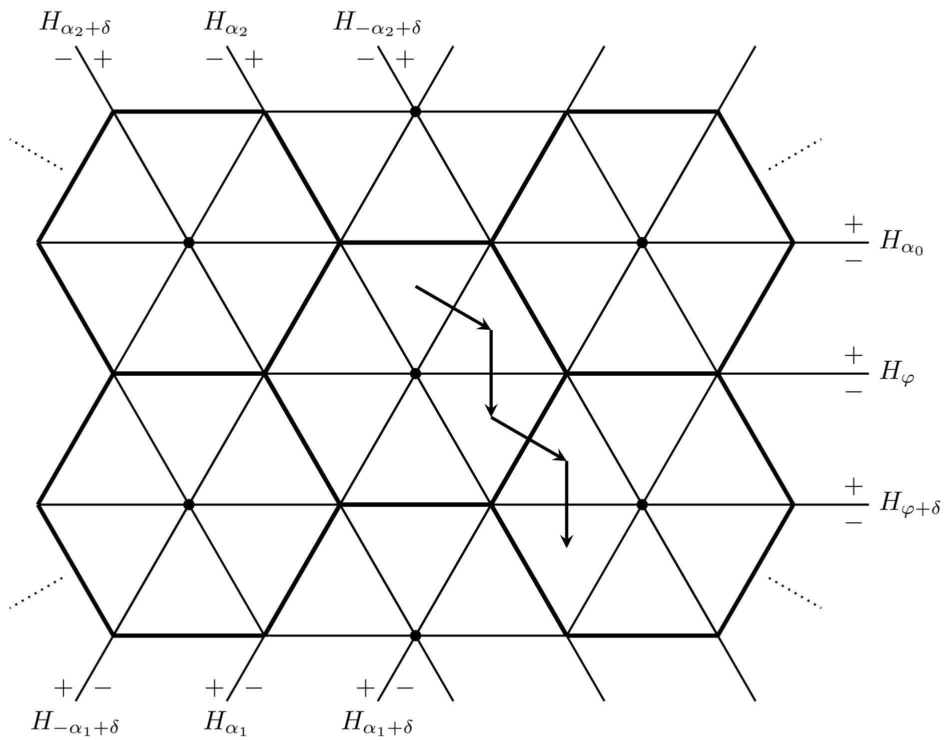
2. $H_{\alpha_i+j\delta}$ and H_{α_i} have parallel orientations for $0 \leq i \leq 2$,

and we say the hyperplanes have parallel periodic orientation (see [1]) in accordance with property (2). Observe that the chambers of $\mathfrak{h}_\mathbb{R}$ correspond to the alcoves of W with the dominant chamber corresponding to the fundamental alcove and the antidominant chamber corresponding to w_0.

Words w in the generators s_i can be viewed as walks starting from 1 to w and we will refer to elements of $w \in W_{\text{aff}}$ as walks in the alcove model. When we do so, we will always assume w is a minimal (or reduced) expression unless otherwise specified. In particular, folded walks will often not be reduced.

Remark 3.1. Elements $w \in W_{\text{aff}}$ have three meanings. They are either elements of W_{aff}, alcoves in the alcove model, or walks in the alcove model from 1 to w . In the third case, we additionally assume w is minimal. From context it will always be clear which meaning we are referring to.

Example 3.1. Consider the reduced expression $w = s_1 s_2 s_0 s_1$.



The walk w is the path given by moving across the sequence of alcoves 1, s_1, $s_1 s_2$, $s_1 s_2 s_0$, and then into w. On the other hand, the alcove w is obtained from the action of W_{aff} on 1 by applying the reflections $s_1 s_2 s_0 s_1$ from right to left.

In general, if

$$w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}},$$

then we say the action of $s_{i_1} \cdots s_{i_{k-1}}$ on s_{i_k} is step k in the walk. Geometrically, the k-th step of the walk can be viewed as the segment of the path from the alcove $s_{i_1} \cdots s_{i_{k-1}}$ to $s_{i_1} \cdots s_{i_k}$:

$$s_{i_1}\cdots s_{i_{k-1}} \xrightarrow{\;H_{s_{i_k}\cdots s_{i_1}}\;} s_{i_1}\cdots s_{i_k} \tag{2}$$

The head of w at step k is the subwalk consisting of steps $i_1,\ldots,i_{k-1}$, and the tail of the walk at step k is the subwalk consisting of steps $i_{k+1},\ldots,i_{\ell(w)}$ starting at the alcove $s_{i_1}\cdots s_{i_k}$.

The hyperplanes are defined such that each step in a walk w moves further away from the fundamental alcove with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}_{\mathbb{R}}$. We define the inversion set $R(w) := \{\beta_1,\ldots,\beta_\ell\}$ to be the ordered set of hyperplanes crossed by the walk w . Since each step of the walk crosses a hyperplane, the k-th element of the inversion set is precisely the hyperplane crossed in the walk at step k. Unfortunately, it's not obvious how to compute $R(w)$ from the expression for w in terms of the s_i. We will introduce another action on alcoves from which it is clear how to obtain w by walking across a sequence of hyperplanes. We say that a hyperplane

$$H_{\pm\alpha_i+j\delta}$$

is of type i. In particular every hyperplane is of type 0, 1, or 2 ($\ell(w_0)$ types) . The closure of every alcove w intersects a unique hyperplane of type i on its boundary for $0\le i\le 2$. Accordingly, we say w is bounded by hyperplanes of type $0\le i\le 2$. Define actions σ_i for $0\le i\le 2$ on the fundamental alcove 1 such that σ_i takes 1 to the alcove obtained by step the hyperplane of type i that bounds it. Define

$$V := \langle\sigma_0,\sigma_1,\sigma_2\rangle$$

be the group generated by these actions and extend this action to V by composition. Observe V has relations similar to W_{aff} including modified braid relations:

$$\sigma_0^2=1,\quad \sigma_1^2=1,\quad \sigma_2^2=1,$$
$$\sigma_2\sigma_0\sigma_1=\sigma_1\sigma_0\sigma_2,\quad \sigma_1\sigma_2\sigma_0=\sigma_0\sigma_2\sigma_1,\quad \sigma_0\sigma_1\sigma_2=\sigma_2\sigma_1\sigma_0.$$

V acts simply transitively on the alcoves and there are bijections

$$W_{\text{aff}}\longleftrightarrow\{\text{alcoves}\}\longleftrightarrow V,$$

but V is not isomorphic to W_{aff} since the σ_i are not reflections over fixed hyperplanes. However, V enjoys the property that the alcove $\sigma=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ is obtained by walking across the sequence of hyperplanes of type $i_1,i_2,\ldots,i_k$, read from left to right, starting from the fundamental alcove 1.

Example 3.2. Consider again the walk $w=s_1s_2s_0s_1$. In the σ_i, w corresponds to $\sigma_1\sigma_1\sigma_1\sigma_0$. Indeed, w in Example 3.1 is obtained by walking across the sequence of hyperplanes of type $1,0,1,0$.

There is an explicit bijection between W_{aff} and V.

Proposition 3.1. The bijection $\psi: W_{\text{aff}}\to V$ is given on W by

$$\begin{aligned}\psi(1)&:=1,\\ \psi(s_1)&:=\sigma_1,\\ \psi(s_2)&:=\sigma_2,\\ \psi(s_1s_2)&:=\sigma_1\sigma_0,\\ \psi(s_2s_1)&:=\sigma_2\sigma_0,\\ \psi(w_0)&=\sigma_1\sigma_0\sigma_2,\end{aligned}$$

and extends to W_{aff} by setting

$$\begin{aligned}\psi(s_0) &:= \sigma_0,\\ \psi(s_1s_0) &:= \sigma_1\sigma_2,\\ \psi(s_2s_0) &:= \sigma_2\sigma_1,\\ \psi(s_1s_2s_0) &:= \sigma_1\sigma_0\sigma_1,\\ \psi(s_2s_1s_0) &:= \sigma_2\sigma_0\sigma_2,\\ \psi(w_0s_0) &:= \sigma_1\sigma_0\sigma_2\sigma_0.\end{aligned}$$

Proof. Since every alcove can be viewed as a product of elements in W and Ws_0, ψ is surjective. If $\psi(w_1) = \psi(w_2)$, then $\psi(w_1)$ and $\psi(w_2)$ are the same word in the σ_i up to the braid relations

$$\sigma_0\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_0, \quad \sigma_1\sigma_2\sigma_0 = \sigma_0\sigma_2\sigma_1, \quad \text{and} \quad \sigma_2\sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_2.$$

As $\psi(w_0) = \sigma_1\sigma_0\sigma_2$, ψ takes braid relations to braid relations so w_1 and w_2 differ up to braid relations in the s_i. Hence w_1 and w_2 represent the same element in W_{aff}. The bijectivity of ψ follows. □

3.2 Positive and negative folding

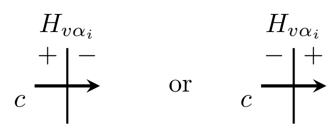
As in [1] , the elements of

$$IwI = \{x_{i_1}(c_1)n_{i_1}^{-1}x_{i_2}(c_2)n_{i_2}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}I \mid c_1,\ldots,c_\ell \in \mathbb{F}_q\}$$

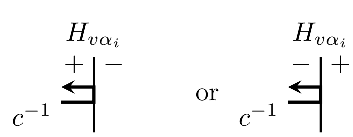
are in bijection with walks to w with edges labeled by $c_1,\ldots,c_\ell$ in $\mathbb{F}_q$. Positively and negatively folded walks to v_1 and v_2 with edges labels in $\mathbb{F}_q$ are in bijection with elements of U^-v_1I and U^+v_2I respectively. Therefore to describe Iwahori double and triple intersections it is necessary to understand how folding a labeled walk w in the alcove model alters the walk. Our main tool is a rewritten group relation which we will refer to as the main folding law:

$$x_\alpha(c)n_\alpha^{-1} = x_{-\alpha}(c^{-1})x_\alpha(-c)h_{\alpha^\vee}(c). \tag{3}$$

which allows us to rewrite elements of IwI as elements in U^-v_1I or U^+v_2I (see [1] equation (7.6)). Geometrically, the main folding law takes steps of the form



and replaces them with folded steps:

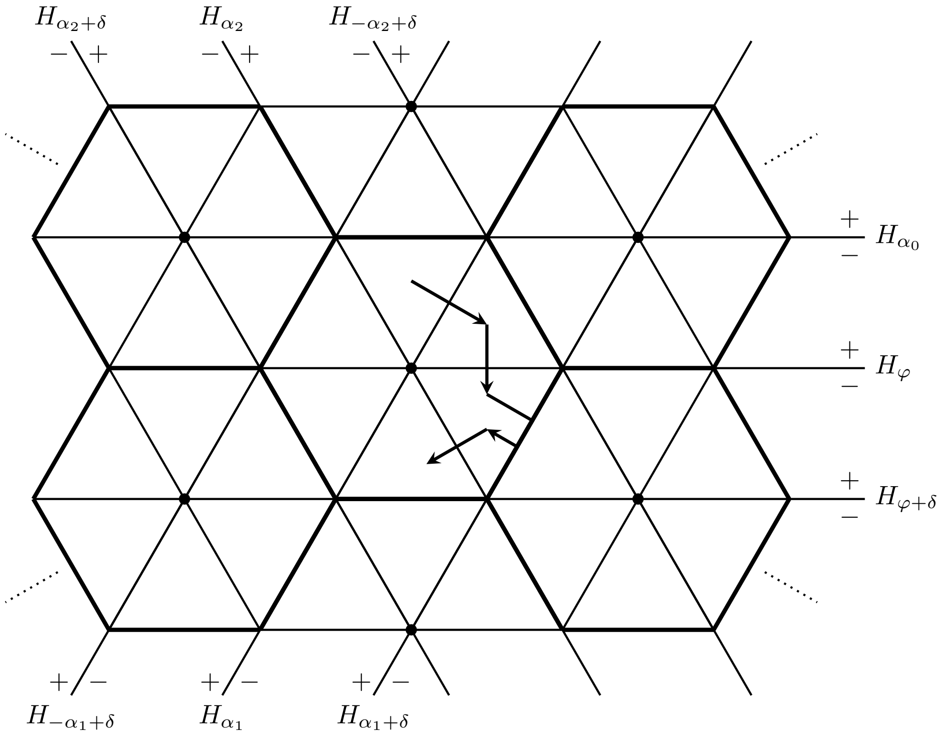


(4)

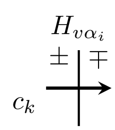
In the first case, we say the fold is positive and in the second we say the fold is negative . If w is a labeled walk that folds positively (resp. negatively) at step k then the folded walk w' as a word is obtained from w by removing the letter s_{i_k}. Unless otherwise specified, we will assume we always fold positively or always fold negatively and it will be clear to which case we are referring. The positively folded walk w_+ (resp. negatively folded walk w_-) corresponding to w is the walk obtained by folding at every possible step . Since w_+ and w_- are obtained by removing letters from w it is not true that w_+ and w_- are always minimal expressions. Accordingly, when we refer to folded walks we mean walks with steps of the forms in (4).

Remark 3.2. It is necessary to assume that the label of the step is nonzero in order for c^{-1} to be defined. That is, folding cannot occur at steps with edge label 0.

Example 3.3. If the walk $w = s_1 s_2 s_0 s_1$ has labels $(0, 0, c, 0)$ with $c \in \mathbb{F}_q^*$. Then the only folding allowed occurs at step 3 and it is a positive fold. Thus $w = w_-$ and $w_+ = s_1 s_2 s_1$. The negatively folded walk w_+ is



We will be slightly terse and consider walks w that fold either positively or negatively at step k. Here it is implicit that the walk is labeled and step k in the walk is of the form



with c_k nonzero. In this sense, we can avoid explicitly mentioning the labels.

We say a walk w is of positive (resp. negative) type v if $w_+ = v$ (resp. $w_- = v$) in W_{aff} . Then from [1] (see Theorem (7.1)) we have

$$\mathcal{P}(w)_v := \left\{ \substack{\text{labeled walks to } w \\ \text{of positive type } v} \right\} \longleftrightarrow U^- vI \cap IwI. \tag{5}$$

Since positive and negative folding are the same up to a change in sign, by following the same reasoning as in [1] there is an analogous result for negative type:

$$\mathcal{N}(w)_v := \left\{ \substack{\text{labeled walks to } w \\ \text{of negative type } v} \right\} \longleftrightarrow U^+ vI \cap IwI. \tag{6}$$

How the sets $\mathcal{P}(w)_{v_1}$ and $\mathcal{N}(w)_{v_2}$ interact is critical to understanding the Iwahori triple intersections.

3.3 Combinatorics of hyperplanes and walks

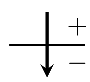
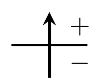
The parallel periodic orientation of the hyperplanes encodes combinatorial information about walks in the alcove model, and quite often heavily restricts the possible types of folded walks that can occur. We start with a lemma that tells us how folding affects the tail of a walk.

Lemma 3.1. If w is a walk with a positive (resp. negative) fold at step k across the hyperplane $H_{\pm\alpha_i+j\delta}$, then the folded walk w_f is obtained from w by introducing a folded step at step k and reflecting the tail across $H_{\pm\alpha_i+j\delta}$ where $w = us_{i_k}v$.

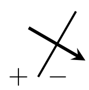
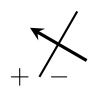
Proof. Write $w = us_{i_k}v$ so that v is the tail. By applying u^{-1} to the left-hand side of w, the claim is equivalent to showing that v and s_iv are reflections of each other across H_{α_i}. The claim follows since s_iv is obtained from v by reflecting over H_{α_i}. □

We can partition the set of hyperplanes by parallelism. Furthermore, each set of this partition is indexed by a positive root α, which we will call the "type" of the hyperplane. Observe that a minimal alcove walk crosses hyperplanes of type α in exactly one orientation: either from the negative side to the positive side, or vice versa. Thus we can identify each step in the walk $\overrightarrow{w}$ with a root: if step k in $\overrightarrow{w}$ crosses a hyperplane of type α from the negative side to the positive side, then we say it has "shape" α, and if it crosses the hyperplane in the other orientation, then we say it has shape $-\alpha$. Positive folding only occurs at steps whose shape is a negative root since these are exactly the steps that cross a hyperplane from the positive side to the negative side. Similarly, negative folding occurs at steps whose shape is a positive root.

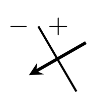
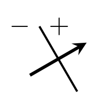
For example in $\mathfrak{sl}_3$

 and 

these are steps of shape $\mp\varphi$. Similarly, the steps of shape $\mp\alpha_1$ are

 and 

and the steps of shape $\mp\alpha_2$ are

 and 

Now we know that folding a minimal walk $\overrightarrow{v}$ at step k across a hyperplane $H_{\pm\alpha+j\delta}$ reflects the steps in the tail of our walk over this hyperplane. Therefore, we can compute the new shapes of the steps in the tail: we simply apply the reflection s_α to these steps (which correspond to roots) in order to get their shapes after folding.

We now have the machinery to show that the number of folds which can occur in a walk is heavily restricted by W for all simple Lie algebras $\mathfrak{g}$.

Lemma 3.2. If a minimal walk $\overrightarrow{v}$ ends in a chamber $w \cdot C$ where $w \in W$ and C is the dominant chamber, then the steps of $\overrightarrow{v}$ have shape $w \cdot R^+$.

Proof. Minimal walks $\overrightarrow{v}$ that end in C have steps of shape R^+ by our definition of shapes and our orientation of the hyperplanes. Thus a minimal walk ending in $w \cdot C$ has steps of shape $w \cdot R^+$. □

Theorem 3.1. Suppose $\overrightarrow{v}$ is a labeled walk ending in the chamber $w \cdot C$. Then the maximum number of folds that can occur in the positive folded walk is the length of a reduced expression for w in terms of the simple reflections of W. Similarly the maximum number of folds in a negatively folded walk of type v is the length of a reduced expression for w_0w.

Proof. The argument is analogous for negatively folded walks, so we prove the theorem for positively folded walks.

By the previous lemma, we know that the steps of $\overrightarrow{v}$ have shapes lying in the set $w \cdot R^+$. By [2] Proposition 5.6, we know that the number of negative roots contained in the set $w \cdot R^+$ is equal to the length of a reduced expression for w in terms of the simple reflections. Furthermore, we can only positively fold at steps whose shape is a negative root, and we can compute the shapes of steps after a fold at a hyperplane of type β by applying s_β to the steps in the tail. In the tail, the steps of the walk have shape $s_\beta w \cdot R^+$.

Thus we have proven the theorem once we show that $s_\beta w$ has length less than w where $-\beta$ is a negative root in $w \cdot R^+$. We can explicitly write down the negative roots of $w \cdot R^+$ which are:

$$\begin{aligned}
&s_{i_1} \cdots s_{i_{n-1}}(-\alpha_{i_n}) \\
&s_{i_1} \cdots s_{i_{n-2}}(-\alpha_{i_{n-1}}) \\
&\quad \vdots \\
&-\alpha_{i_1}
\end{aligned}$$

Say $-\beta = s_{i_1} \cdots s_{i_{k-1}}(-\alpha_{i_k})$. Then

$$s_\beta = (s_{i_1} \cdots s_{i_{k-1}}) s_{i_k} (s_{i_1} \cdots s_{i_{k-1}})^{-1}$$

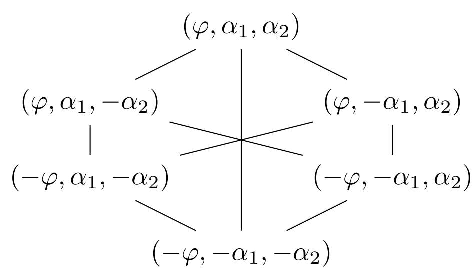
since this is a reflection that takes β to $-\beta$. Finally observe that

$$s_\beta w = s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_n}$$

Therefore the maximum number of folds in a positively folded walk is $l(w)$. $\square$

Corollary 3.1. The maximum number of folds (positive or negative) in a particular alcove model is equal to the length of the long element of the Weyl group for that Lie algebra.

We can put a partial ordering on the sets $\{\pm\varphi, \pm\alpha_1, \pm\alpha_2\}$ given by the action of the s_i:



The ordering is such that positive folding corresponds to moving up the ordering and negative folding corresponds to moving down. This embodies the fact that no positive folding can occur in $(\varphi, \alpha_1, \alpha_2)$ and no negative folding can occur in $(-\varphi, -\alpha_1, -\alpha_2)$. Theorem 3.1 then loosely says there are at most $\ell(w_0)$ elements in any saturated chain of this ordering.

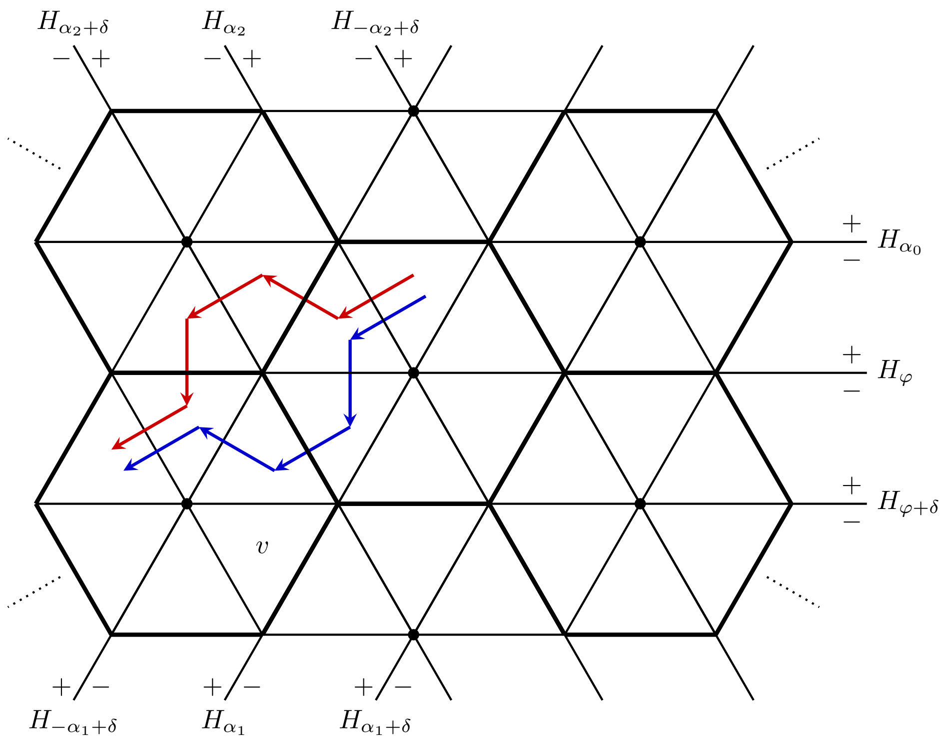
Since walks in the alcove model are not in bijection with elements of W_{aff} it is natural to wonder if two different reduced expressions for the same word fold to different alcoves. This is possible if we are given the freedom to choose labels.

Proposition 3.2. If w is a labeled walk that positively (resp. negatively) folds to some alcove, then any other reduced expression w' positively (resp. negatively) folds using the same number of folds to the same alcove for a possibly different labeling.

Proof. The claim is analogous for negative folds so assume all folds are positive. We will prove by strong induction on $\ell(w)$. This is allowed since the length of a word is independent of its reduced expression. If $\ell(w) = 1$, then the reduced expression is unique and the claim follows. So suppose that the claim holds for all walks of length at most n and all labelings.

If $\ell(w) = n + 1$. Write $w = vs_i$ and $w' = v's_j$. By the induction hypothesis choose a labeling for v' such that v and v' both positively fold using the same number of folds to the same alcove. If $i = j$, then we are done. If $i \neq j$, then $w = us_is_js_i$ and $w' = u's_js_is_j$ where $us_is_j = v$ and $u's_js_i = v'$ because w and w' have equivalent expressions up to braid relations . By the induction hypothesis choose labels for u' such that u and u' both fold to the same alcove using the same number of folds. If we do not fold in the tail of u then we are done since the braids $s_is_js_i$ and $s_js_is_j$ are walks to the same alcove. If we fold in the tail of u, then since the induction is strong we can choose labels for the tail of u' such that both tails fold to the same alcove . In all cases we can find labels for w' such that w' and w fold to the same alcove. $\square$

Example 3.4. Consider the two walks $w = s_2s_0s_1s_0s_2$ and $w' = s_2s_1s_0s_1s_2$. They are walks to the same alcove since they differ by the braid $s_0s_1s_0 = s_1s_0s_1$. We have colored w and w' by red and blue respectively:



Let the labels for w be $(c_1, c_2, 0, 0, c_5)$ with $c_1, c_2, c_5 \in \mathbb{F}_q^*$. Consider the negatively folded walk. Then the only fold in w occurs at step 2 and folding reflects the red braid to the blue braid with an addition step into v. Choosing the c_4 label for w' nonzero means we fold at step 4 and indeed folding here reflects the blue tail of w' to the additional red step into v. Thus w' folds to the same alcove as w with labels $(c_1, c_2, 0, 0, c_5)$ given $c_4 \neq 0$.

3.4 Iwahori triple intersections and double Iwasawa cells

Throughout, we have assumed that our loop group G has two Iwahori decompositions

$$G = \bigsqcup_{v \in \widetilde{W}} U^- vI \quad \text{and} \quad G = \bigsqcup_{v \in \widetilde{W}} U^+ vI$$

where the Iwahori subgroup I we have chosen is the same for both decompositions. Here we prove that we can actually do so.

Proposition 3.3. Let $\mathcal{K} = \mathbb{C}((t))$, let $\mathcal{O} = \mathbb{C}((t))$ be the ring of integers in $\mathcal{K}$, and let k be the residue field $\mathcal{O}/\mathfrak{p}$. Let $B(k) \leq G(k)$ be the Borel subgroup of upper-triangular matrices . Let $K = G(\mathcal{O})$, and let f be the map from K to $G(k)$ given by sending t to 0, and let $I \leq K$ be the inverse image of $B(k)$ under this map.

Then $I = U^+(\mathcal{O})T(\mathcal{O})U^-(\mathfrak{p})$. Now let $g \in G$. Then because we have the Iwasawa decomposition

$$G = \bigsqcup_{t^\lambda \in Q^\vee} U^+ t^\lambda K$$

we can write

$$g = t^\lambda ua \quad \text{where} \quad u \in U^+ \quad \text{and} \quad a \in K = \bigsqcup_{w \in W} IwI$$

Then we can write a as

$$a = (u_+ t u_-)wi = u_+ wi'$$

which gives us the Iwahori decomposition since moving a torus element t past the elements of U^+ leaves them in U^+

$$G = \bigsqcup_{v \in \widetilde{W}} U^+ vI$$

Recall the main folding law

$$x_\alpha(c) n_\alpha^{-1} = x_{-\alpha}(c^{-1}) x_\alpha(-c) h_{\alpha^\vee}(c) \quad \text{where} \quad c \in \mathcal{O}^\times$$

which can be rewritten as

$$n_\alpha^{-1} x_{-\alpha}(c) = x_\alpha(\pm c) n_\alpha^{-1} = x_{-\alpha}(\pm c^{-1}) x_\alpha(\mp c) h_{\alpha^\vee}(\pm c).$$

Let $g \in G$. Because we have the Iwasawa decomposition of G with respect to U^-, we can write

$$g = t^\lambda ua \quad \text{where} \quad u \in U^- \quad \text{and} \quad a \in K = \bigsqcup_{w \in W} IwI$$

and we can write $a = u_+ wi'$ as before. We have finished the proof once we can show that g is in $U^- vI$ for some $v \in \widetilde{W}$.

Now by [3] Lemma 18 Corollary 2, every element of U^+ can be written uniquely as a product of elements of the one parameter subgroups $\mathcal{X}_\alpha$ for α a positive root, once we pick an ordering on the positive roots. If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression in terms of the simple reflections, then $n_{i_1}^{-1} \cdots n_{i_k}^{-1}$ is a representative of w in $G(\mathcal{K})$. We wish to move these $n_{i_j}^{-1}$ past the x_α's in u_+. However,

$$x_\alpha(c) n_\beta^{-1} = n_\beta^{-1} x_{s_\beta \alpha}(c)$$

and it is possible that $x_{s_\beta \alpha}(c)$ does not lie in I. Thus we apply the following algorithm: the first step consists of ordering R^+ so that α_{i_1} comes first. Then start moving $n_{i_1}^{-1}$ past the elements x_α in u_+, which leaves them in I. If the product u_+ contains an element from $\mathcal{X}_{i_1}$, then apply the main folding law, so that $x_{-\alpha}(c^{-1})$ lies in U^- and move the torus element $h_{\alpha_{i_1}^\vee}(c)$ to the right since it lies in I. Otherwise we do not have to apply the main folding law, and we proceed with the next step.

Assume we have done this successfully for m steps, so we have

$$g = t^\lambda u' (n_{j_1}^{-1} \cdots n_{j_\ell}^{-1}) u'_+ (n_{i_{m+1}}^{-1} \cdots n_{i_k}^{-1}) i''$$

Now for the $m+1$ step, order R^+ so that $\alpha_{i_{m+1}}$ comes first, and write u'_+ as a product of elements in $\mathcal{X}_\alpha$ with this ordering. Start moving $n_{i_{m+1}}^{-1}$ past the x's as before. If we do need to apply the main folding law, then we need to move $x_{-\alpha_{i_{m+1}}}(c)$ past the n_j^{-1}'s on the left. Now it is possible that as we move $x_{-\alpha_{i_{m+1}}}(c)$ to the left, that at some point, we have the situation $n_\beta^{-1}x_{-\beta}(c')$ for β a positive root. Since $U^- = \prod_{\alpha\in R^+}\mathcal{X}_{-\alpha}$, we need to apply the main folding law again. Now we have gotten rid of n_β^{-1} and can proceed with moving $x_{-\beta}(c')$ to the left. Move the other terms from this application of the main folding law to the right, and if need be repeatedly apply the main folding law. Proceeding in this way, we can eventually write g as an element of U^-vI for some $v \in \widetilde{W}$, and so we finally have the Iwahori decomposition

$$G = \bigsqcup_{v\in\widetilde{W}} U^-vI$$

The reason for going through all of that trouble is the following. For two weights $\lambda, \mu \in Q^\vee$, we can define the *double Iwasawa cell*

$$C_{\lambda\mu} = U^-t^\lambda K \cap U^+t^\mu K$$

and write it as a *triple intersection*

$$C_{\lambda\mu} = U^-t^\lambda(\sqcup_{w\in W} IwI) \cap U^+t^\mu(\sqcup_{w\in W} IwI) = \bigcup_{\substack{w,w'\in W\\ v\in\widetilde{W}}} U^-t^\lambda wI \cap IvI \cap U^+t^\mu w'I$$

Proposition 3.4. Let w be a walk that belongs to $\lambda^\vee = k_1\alpha_1^\vee + k_2\alpha_2^\vee$. Then w_+ (resp. w_-) belongs to some weight $\mu^\vee = m_1\alpha_1^\vee + m_2\alpha_2^\vee$ such that $|m_1| \le |k_1|$ and $|m_2| \le |k_2|$.

Proof. We will prove the slightly stronger claim where w is assumed to be a walk starting at any alcove in W , and we will prove by strong multidimensional induction on k_1 and k_2. If $k_1 = k_2 = 0$ then $w \in W$ and the claim follows since W is closed under multiplication by s_1 and s_2 and folding corresponds to removing a letter s_1 or s_2. Suppose that $\lambda^\vee = (n+1)\alpha_1^\vee + (n+1)\alpha_2^\vee$ and the claim holds for all walks belonging to $\nu^\vee = n\alpha_1^\vee + n\alpha_2^\vee$. Write $w = v_{i_1}\cdots v_{i_l}$ (with $l \le \ell(w_0)$ by Theorem 3.1) where the last step of each v_{i_j} is a fold. For each fold, applying a sufficient translation sends the start of the tail to an alcove in W and then the claim holds by the strong induction hypothesis . Thus w_+ belongs to $\mu^\vee = m_1\alpha_1^\vee + m_2\alpha_2^\vee$ such that $|m_1| \le |k_1|$ and $|m_2| \le |k_2|$. The induction for negative k_1 and k_2 is analogous finishing the proof. ☐

Since there are finitely many weights with coefficients bounded in absolute value so that by Proposition 3.4, finitely many translates $t_{\lambda^\vee}(W)$ may give contributions to the Iwahori triple intersection.

Proposition 3.5. Let λ, μ be weights $Q^\vee$ with λ antidominant. Then $U^- t^\lambda K \cap U^+ t^\mu K$ is nonempty when $\lambda \succeq \mu$ with respect to the usual ordering on weights. There is a stratification on the intersection above given by

$$U^- t^\lambda K \cap U^+ t^\mu K = \bigcup_{v \in \widetilde{W}} \left(\bigcup_{w, w' \in W} U^- t^\lambda w I \cap I v I \cap U^+ t^\mu w' I \right).$$

In the alcove walk model for $\mathfrak{g}$, the fundamental domains for the translation subgroup $T \leq \widetilde{W}$ are the hexagons whose centers correspond bijectively to elements of T. Thus it suffices to show that given λ and μ, there exists a labelled positively folded walk of type v to an alcove in t^λ which is also a labelled negatively folded walk of type v to an alcove in t^μ.

Informally, if $\lambda \succeq \mu$, then there is a sequence of affine reflections that take t^λ to t^μ. Folding a walk at step k corresponds to reflecting the remaining steps of the walk (which we will call the tail) across the hyperplane at which we have folded. Thus a walk to an alcove in t^μ can be folded positively to an alcove in t^λ where the folds kill off the coefficients k_1, k_2 in $k_1\alpha_1^\vee + k_2\alpha_2^\vee = \lambda - \mu$.

Now suppose that μ_1, μ_2 are two weights for $\mathfrak{g}$ in $Q^\vee$ such that $\mu_1 - \mu_2 = k\alpha_i^\vee$. Then there are $2k$ hyperplanes of type i between μ_1 and μ_2, thought of as centers of hexagons in the alcove walk model, including the one containing μ_2. This is because there are 2 hyperplanes of type i between 0 and $\alpha_i^\vee$. Also notice that the image of t^{μ_1} under reflection across the kth hyperplane of type i between the two weights is t^{μ_2}.

Proof. We are now ready to prove the claim. Assume that λ is antidominant and $\lambda - \mu$ equals $k_1\alpha_1^\vee$ or $k_2\alpha_2^\vee$ where k_i is a positive integer in each case. Let v be a minimal walk to an alcove in t^μ, and forget about the labels for now. Notice that by our assumptions, we cross hyperplanes of type i from the positive side to the negative side, which means that we are able to fold positively at steps of shape $-\alpha_i$. Since λ is antidominant and $\lambda \succeq \mu$, we actually cross all $2k$ hyperplanes of type i separating the two weights except possibly the one containing μ, so if we positively fold v at the step crossing the kth hyperplane, then the tail of v lies in an alcove of t^λ.

Now suppose that $\lambda - \mu = k_1\alpha_1 + k_2\alpha_2$ where both coefficients are positive integers. μ lies in a chamber where we can either fold at steps of shape $-\alpha_1$ or $-\alpha_2$. Without loss of generality, suppose that μ lies in the s_1s_2 chamber or the antidominant chamber so that we are able to postively fold at steps of shape $-\alpha_1$. Then if $\mu' = \mu + k_1\alpha_1^\vee$, then there are $2k_1$ hyperplanes of type 1 between μ and μ', and we can fold v at the step that crosses the k_1th hyperplane between the two weights so that the tail of v lands in an alcove in $t^{\mu'}$. Now $\lambda - \mu' = k_2\alpha_2^\vee$, and we have a walk to $\widetilde{w} \in t^{\mu'}$ (not necessarily minimal) whose tail (steps after the fold) has steps of shape $\alpha_1, -\alpha_2, \varphi$. The tail crosses all $2k_2$ hyperplanes of type 2 between λ and μ' (because the angle between α_1 and α_2 lies in $[\pi/2, \pi)$ and our assumptions about the ordering $\lambda \succeq \mu' \succeq \mu$) . Thus we can fold the tail of $\widetilde{w}$ at the step where it crosses the k_2th hyperplane of type 2 between λ and μ' to obtain a positively folded path of type v to an alcove in t^λ. We now address the labels. The only constraints in the negatively folded walk to v come from positive crossings. In the above example, these positive crossings (if any) come at steps of shape α_2. Steps of shape R^+ don't have any constraints on the labels in the positively folded walk, so we can make them 0 so that the labellings are compatible with the negatively folded walk. Because $s_1(\alpha_2) = \varphi$ is a positive crossing, there is no problem after the first fold at the hyperplane of type 1. For the rest of the labels, use the contraints forced by the positively folded walk. ☐

Proposition 3.6. Suppose that we have a minimal walk to an alcove w in t^μ. Then any positively folded walk of type w ends in an alcove v contained in t^ν such that $\nu \succeq \mu$.

Proof. We proceed by induction on the number of folds. If we don't fold at all, then w ends in w contained in t^μ and $\mu \succeq \mu$. Now suppose that the conclusion holds for positively folded walk of type w where we have folded k times for $0 \leq k \leq n-1$.

Say we have a positively folded walk of type w ending in v where we have folded n times. Let w' be the same folded walk of type w without the nth fold. Then w' lies in a hexagon $t^{\mu'}$ and $\mu' \succeq \mu$. v lies in a hexagon t^{ν}, and all that's left to show is that $\nu \succeq \mu'$. We know that v is got from w' by folding at a step k after the $n-1$th fold, and that this step k looks like $-\alpha_i$ for some $\alpha_i \in R^+$. Because folding at this step k corresponds to reflecting the tail of w' across a hyperplane of type i, we actually know that $\nu - \mu' = m\alpha_i^\vee$ for some positive integer m. Thus $\nu \succeq \mu$. $\square$

By a similar argument, we know that any negatively folded walk of type $v \in t^{\nu}$ ending in an alcove $w \in t^{\mu}$ satisfies $\nu \succeq \mu$. Thus if there exists an alcove $v \in t^{\nu}$ such that v positively folds to $t^{\lambda}w$ and negatively folds to $t^{\mu}w'$, then we have $\lambda \succeq \nu \succeq \mu$. Therefore the double Iwasawa cell $C_{\lambda\mu} := U^- t^{\lambda} K \cap U^+ t^{\mu} K$ is nonempty if and only if $\lambda \succeq \mu$ for $\mathfrak{g} = \mathfrak{sl}_3$.

4 Whittaker Functions

4.1 Definitions

Let $\mathfrak{g}$ be a simple Lie algebra and λ an antidominant weight for $\mathfrak{g}$. Let G be the loop group given by the field $\mathbb{F}_q((t))$ and Lie algebra $\mathfrak{g}$. Then the Whittaker coefficient $W(t^{\lambda})$ can be defined as in [4] by

$$W(t^{\lambda}) = \int_{U^-} v_K(ut^{\lambda})\psi(u)\,du$$

This integral is evaluated over the double Iwasawa cells $C_{\lambda\mu}$ defined earlier, and each

$$ut^{\lambda} = u't^{\mu}k \in U^+AK$$

The stratification of the double Iwasawa cells into triple intersections allows us to reinterpret the Whittaker coefficient formula in terms of alcove walks. We normalize the volume form so that I has integral 1, which allows us to write

$$W(t^{\lambda}) = \frac{1}{\mathrm{vol}(K)} \sum_{\substack{\mu \in Q^\vee \\ w,w' \in W \\ v \in \widetilde{W}}} \chi(t^{\mu}) \int_{U^- t^{\lambda} w I \cap I v I \cap U^+ t^{\mu} w' I} \psi(u)\,du$$

Here we slightly abuse notation: we consider the points in the triple intersection as points in the affine flag variety G/I and take ψ to mean the function that takes the unipotent part of an element and then evaluates the character.

In order to compute the Whittaker coefficient in terms of alcove walks, follow this procedure:

1. First find all labelled walks v that can positively fold to an alcove in $\lambda' = w_0 \cdot \lambda$. (In the alcove model, the fundamental domain for the translation subgroup $T \leq \widetilde{W}$ is the closure of the union of the alcoves for W. Thus λ' can be identified with the alcoves $t^{\lambda'} \cdot W$.)

2. Then for each labelled walk v from the first step, find all the alcoves $t^{\mu}w'$ that v can negatively fold to.

3. Let $(v, l, t^{\lambda'}w, t^{\mu}w')$ be a tuple that denotes a labelled walk to v positively folding to $t^{\lambda'}w$ and negatively folding to $t^{\mu}w'$. Evaluate the character ψ on the unipotent part of this tuple which corresponds to a point in G/I, and then sum over the possible labellings, i.e. the tuples which have the same first, third, and fourth entries . This sum is the integral in the formula written above.

4. $w_0 \cdot \mu$ is an integer combination of the simple coroots, and $\chi(t^\mu)$ is equal to the product of indeterminates z_i where the power of z_i in the product is the coefficient of $\alpha_i^\vee$. Multiply the integral from the previous step by $\chi(t^\mu)$.

5. Finally, divide the expression by

$$\mathrm{vol}(K) = \sum_{w \in W} q^{l(w)}$$

For $G = SL_2(\mathbb{F}_q((t)))$, Beazley and Brubaker proved that this computation gives Tokuyama's formula bijectively (see [4]). However, Tokuyama's formula is only valid for Cartan types A_n. There are other similar formulas for the other Cartan types, but the advantage of the alcove walk formulation is that it is independent of Cartan type and of any ordering on the positive roots.

4.2 Support on $U^- v_1 I \cap IwI \cap U^+ v_2 I$

Since the alcove walk formulation requires us to sum over all v in the affine Weyl group, it may appear that our sum is infinite. However in practice, taking the character and integrating over all l in the tuples $(v, l, t^{\lambda'} w, t^\mu w')$ makes the integral 0 for all but finitely many v.

Explicitly ψ can be taken to be the following. If $q = p^n$ for p prime, then let ψ_0 be the character on $\mathbb{F}_q((t))$ which sends $\mathbb{F}_q[[t]]$ to 1, and sends at^{-k} where $a = c_1 + c_2\alpha_1 + \cdots + c_n\alpha_{n-1}$ to $\zeta^{c_1} \cdots \zeta^{c_n}$ where $k > 0$ and ζ is a pth root of unity. Then define the character ψ on U^- by

$$\psi(A) = \psi_0 \left(\sum_{i=1}^{n-1} a_{i+1,i} \right)$$

Now if the positively folded walk of type v positively crosses a hyperplane corresponding to a simple root α with label $\alpha + k\delta$ for $k > 0$, and the allowed labels on this crossing are all elements of $\mathbb{F}_q$, then the integral for this walk is 0, since the sum of all pth roots of unity is 0. What this means pictorially is that v cannot lie in a weight ν that is "too far" from λ, since this increases the likelihood of such a positive crossing mentioned previously.

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