F-Polynomial Ratios in the r-Kronecker

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r-Kronecker Setup

- We investigate the r-Kronecker, the cluster algebra corresponding to exchange matrix $B = \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}$. (You may black box this)
- There is a distinguished set of generators $\{x_i\}_{i \in \mathbb{N}}$ of the r-Kronecker called *cluster variables*.
- Cluster variables are rational functions in the variables x_1, x_2, y_1, y_2.
- The F-polynomial F_i is a polynomial in y_1, y_2 obtained from x_i by setting x_1, x_2 to 1.
- Ultimately, we want to analyze certain ratios of F-polynomials.

F-polynomial recurrence

Definition

Let the sequence $\{a_i\}$ be defined by $a_1 = 0, a_2 = 1, a_n = ra_{n-1} - a_{n-2}$ for $n \geq 3$.

Then we have the following recurrence:

Proposition

For F-polynomials of the r-Kronecker, we have $F_1 = F_2 = 1$,

$$F_{k-1}F_{k+1} = F_k^r + y_1^{a_k} y_2^{a_{k-1}}.$$

By cluster algebra magic, F_{k-1} divides $F_k^r + y_1^{a_k} y_2^{a_{k-1}}$. This lets us calculate F-polynomials recursively.

F-polynomial examples

Using $F_{k-1}F_{k+1} = F_k^r + y_1^{a_k} y_2^{a_{k-1}}$, we can calculate the first few F-polynomials.

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 1 + y_1$$

$$F_4 = (1 + y_1)^r + y_1^r y_2$$

$$F_5 = \frac{((1 + y_1)^r + y_1^r y_2)^r + y_1^{r^2-1} y_2^r}{1 + y_1}$$

REU Problem 3

We fix a positive integer $r \geq 2$, and let Q_i be a sequence of nonnegative integers such that $Q_i = rQ_{i-1} - Q_{i-2}$ for $i \geq 2$. Our goal is to find

$$_rG_{Q_1,Q_2}(y_1, y_2) := \lim_{i \to \infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}},$$

due to the fact that it appears to converge.

Reading's results

Nathan Reading found that for $r = 2$, we have the equality

$$
\begin{aligned}
{}_2G_{1,1} &= \lim_{i\to\infty} \frac{F_i}{F_{i-1}} \\
&= \frac{1 + y_1 + y_1y_2 + \sqrt{(1 + y_1 + y_1y_2)^2 - 4y_1y_2}}{2} \\
&= 1 + y_1 \sum_{i,j\geq 0} (-1)^{i+j} \mathrm{Nar}(i,j) y_1^i y_2^j,
\end{aligned}
$$

where

$$
\mathrm{Nar}(i,j) = \begin{cases} 1 & i = j = 0 \\ 0 & \text{otherwise if } ij = 0 \\ \frac{1}{i}\binom{i}{j}\binom{i}{j-1} & i \geq 1 \text{ and } j \geq 1 \end{cases}.
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- Note that the first equality is as a function in positive y_1, y_2, and the second is as a power series.

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Our results

Our results include:

- reproving Reading's result without introducing scattering diagrams.
- finding ${}_rG_{Q_1,Q_2}$ for $r = 2$ and general Q_1, Q_2.
- finding a functional equation for ${}_rG_{Q_1,Q_2}$ for general r and general Q_1, Q_2.

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Infinite product

Our results use the following infinite product:

Proposition [C.–Chin–Davis–G.]

For all $i \geq 2$, we have the equality

$$\frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{k=2}^{i} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}.$$

Furthermore, the limit is well-defined as a formal power series, i.e.

$$\lim_{i\to\infty} \frac{F_{i+1}^{Q_i}}{F_i^{Q_{i+1}}} = \prod_{k=2}^{\infty} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r}\right)^{Q_k}.$$

This follows from the recurrence $F_{k-1}F_{k+1} = F_k^r + y_1^{a_k} y_2^{a_{k-1}}$.

Finding a functional equation

- To investigate the infinite product, we emulate Reading's strategy by finding a functional equation.
- The substitution below transforms each term in the product into the next term in the product.

Proposition [C.–Chin–Davis–G.]

For $y_1' = (\frac{y_1}{1+y_1})^r y_2, y_2' = \frac{1}{y_1}$, we have

$$\frac{(y_1')^{a_k}(y_2')^{a_{k-1}}}{F_k(y_1', y_2')^r} = \frac{y_1^{a_{k+1}} y_2^{a_k}}{F_{k+1}(y_1, y_2)^r}.$$

Corollary

With y_1', y_2' as above, we have

$$\prod_{k=2}^{\infty} \left(1 + \frac{(y_1')^{a_k}(y_2')^{a_{k-1}}}{F_k(y_1', y_2')^r}\right)^{Q_k} = \prod_{k=3}^{\infty} \left(1 + \frac{y_1^{a_k} y_2^{a_{k-1}}}{F_k^r}\right)^{Q_{k-1}}.$$

2-Kronecker functional equation

When $r = 2$ and $Q_i = 1$, we can obtain a functional equation directly from this corollary.

Corollary (restated)

For $y_1' = (\frac{y_1}{1+y_1})^r y_2, y_2' = \frac{1}{y_1}$, as before, we have

$$\prod_{k=3}^{\infty}\left(1+\frac{y_1^{a_k}y_2^{a_{k-1}}}{F_k^r}\right)^{Q_{k-1}} = \prod_{k=2}^{\infty}\left(1+\frac{(y_1')^{a_k}(y_2')^{a_{k-1}}}{F_k(y_1', y_2')^r}\right)^{Q_k}.$$

The RHS is exactly ${}_2G_{1,1}(y_1', y_2')$, while the LHS differs from ${}_2G_{1,1}(y_1, y_2)$ by a factor of $1 + y_1$. This gives the functional equation

$${}_2G_{1,1}(y_1, y_2) := \lim_{i\to\infty}\frac{F_{i+1}}{F_i} = {}_2G_{1,1}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right)\cdot(1+y_1).$$

r-Kronecker functional equation

Similar substitutions can be made to find a functional equation for general r.

Theorem [C.–Chin–Davis–G.]

Fix r and a sequence $\{Q_i\}$. Then with $y_1' = (\frac{y_1}{1+y_1})^r y_2, y_2' = \frac{1}{y_1}$ and $y_1'' = (\frac{y_1'}{1+y_1'})^r y_2', y_2'' = \frac{1}{y_1'}$, we have

$$ {}_rG_{Q_1,Q_2}(y_1, y_2) = \frac{{}_rG_{Q_1,Q_2}(y_1', y_2')^r}{{}_rG_{Q_1,Q_2}(y_1'', y_2'')} \cdot (1+y_1)^{Q_2} \cdot \left(1 + \frac{y_1^r y_2}{(1+y_1)^r}\right)^{Q_3 - rQ_2}. $$

Functional equation derivation

Set $r = 3$, $Q_1 = 0$, $Q_2 = 1$ and $y_1' = (\frac{y_1}{1+y_1})^3 y_2, y_2' = \frac{1}{y_1}$. Then

$$
\begin{aligned}
{}_3G_{0,1}(y_1, y_2) &= (1+y_1)\left(1+\frac{y_1^3 y_2}{F_3^3}\right)^3\left(1+\frac{y_1^8 y_2^3}{F_4^3}\right)^8 \cdots \\
{}_3G_{0,1}(y_1', y_2') &= \left(1+\frac{y_1^3 y_2}{F_3^3}\right)\left(1+\frac{y_1^8 y_2^3}{F_4^3}\right)^3 \cdots \\
{}_3G_{0,1}(y_1'', y_2'') &= \left(1+\frac{y_1^8 y_2^3}{F_4^3}\right) \cdots
\end{aligned}
$$

yielding the functional equation

$$
{}_3G_{-1,1}(y_1, y_2) = \frac{{}_3G_{-1,1}(y_1', y_2')^3}{{}_3G_{-1,1}(y_1'', y_2'')}(1+y_1).
$$

2-Kronecker solution

It turns out the 2-Kronecker case is fully analytically solvable.

Theorem [C.–Chin–Davis–G.]

In the 2-Kronecker case we have

$$
\begin{aligned}
{}_2G_{0,1} &= \lim_{i\to\infty} \frac{F_{i+1}^{i-1}}{F_i^i} \\
&= \frac{1}{2y_1}\Bigg((1+y_1+y_1y_2)^2 - 4y_1y_2 \\
&\qquad + (-1+y_1+y_1y_2)\sqrt{(1+y_1+y_1y_2)^2-4y_1y_2}\Bigg).
\end{aligned}
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- We have ${}_2G_{Q_1,Q_2} = {}_2G_{1,1}^{Q_1} \cdot {}_2G_{0,1}^{Q_2-Q_1}$.

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Proof details

- Recall that Reading proved

$$ {}_2G_{1,1} = \lim_{i\to\infty} \frac{F_i}{F_{i-1}} = \frac{1 + y_1 + y_1y_2 + \sqrt{(1 + y_1 + y_1y_2)^2 - 4y_1y_2}}{2}. $$

- Reading's method was to find the functional equation

$$ {}_2G_{1,1}\left(\frac{y_1}{(1+y_2)^2}, y_2\right)(1+y_2) = {}_2G_{1,1}\left(\frac{y_2}{(1+y_1)^2}, y_1\right)(1+y_1), $$

  and then use it to find the coefficients of the function as a power series.

- We can use the same strategy using our functional equation!

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Proof details (continued)

- As an intermediate step, we can reprove Reading's result using our functional equation

$$_2G_{1,1}(y_1, y_2) = {_2G_{1,1}}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) \cdot (1+y_1).$$

- We also have the functional equation

$$_2G_{0,1}(y_1, y_2) = \frac{_2G_{0,1}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right)}{_2G_{1,1}(y_1, y_2)}.$$

Proof details (continued)

- As an intermediate step, we can reprove Reading's result using our functional equation

$$ {}_2G_{1,1}(y_1, y_2) = {}_2G_{1,1}\left(\frac{y_1^2 y_2}{(1+y_1)^2}, \frac{1}{y_1}\right) \cdot (1 + y_1). $$

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Future work

- investigate the subset of $\mathbb{C}^2$ on which ${}_rG_{Q_1,Q_2}$ converges
- see whether we can find a closed form for coefficients of the power series ${}_rG_{Q_1,Q_2}$
  - in particular, find coefficients of ${}_rG_{1,\alpha}$ for α a root of $x^2 = rx - 1$

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