Whittaker coefficients and crystals

Aidan Kelley and Siki Wang
joint with Swapnil Garg and Frank Lu
Mentor: Prof. Ben Brubaker, TAs: Emily Tibor, Kayla Wright, Meagan Kenney

August 4, 2021

Root system Φ

Root system Φ

- Vectors in $\mathbb{R}^{n+1}$, like $\alpha_1 = (1, -1, 0, 0, ..)$.

Root system Φ

- Vectors in $\mathbb{R}^{n+1}$, like $\alpha_1 = (1, -1, 0, 0, ..)$.
- Contain simple roots, positive roots, etc.

Root system Φ

- Vectors in $\mathbb{R}^{n+1}$, like $\alpha_1 = (1, -1, 0, 0, ..)$.
- Contain simple roots, positive roots, etc.
- **Matrix parameterization**. E.g., for A_2, $\begin{bmatrix} * & \alpha_1 & \alpha_1 + \alpha_2 \\ & * & \alpha_2 \\ & & * \end{bmatrix}$

Root system Φ

- Vectors in $\mathbb{R}^{n+1}$, like $\alpha_1 = (1, -1, 0, 0, ..)$.
- Contain simple roots, positive roots, etc.
- **Matrix parameterization**. E.g., for A_2, $\begin{bmatrix} * & \alpha_1 & \alpha_1 + \alpha_2 \\ & * & \alpha_2 \\ & & * \end{bmatrix}$
- **Dynkin diagram** for the associated Weyl group. E.g,

$$A_5: \quad \alpha_1 \text{ --- } \alpha_2 \text{ --- } \alpha_3 \text{ --- } \alpha_4 \text{ --- } \alpha_5$$

Dirichlet L-series

Dirichlet L-series

- $\mathcal{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.

Dirichlet L-series

- $\mathcal{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.
- E.g., $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$ (Riemann zeta-function)

Combinatorics of Dirichlet Series

Combinatorics of Dirichlet Series

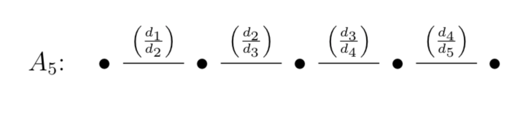
- (Hoffstein) There is a way to associate a multiple Dirichlet series to a Dynkin diagram.

Combinatorics of Dirichlet Series

- (Hoffstein) There is a way to associate a multiple Dirichlet series to a Dynkin diagram.

Example

Dynkin diagram for the Weyl group of A_5.



Associating each simple root $\alpha_i \in \Phi^+$ with a complex variable s_i, we get the corresponding multiple Dirichlet series

$$\sum_{d_1,\ldots,d_5=1}^{\infty} \frac{\left(\frac{d_1}{d_2}\right)\left(\frac{d_2}{d_3}\right)\left(\frac{d_3}{d_4}\right)\left(\frac{d_4}{d_5}\right)}{d_1^{s_1} d_2^{s_2} d_3^{s_3} d_4^{s_4} d_5^{s_5}}$$

Whittaker coefficient

Whittaker coefficient

- (Brubaker-Friedberg) We can compute the Whittaker coefficient for a maximal parabolic Eisenstein series (subgroups of GL_n):

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times} H(d_1,\dots,d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Whittaker coefficient

- (Brubaker-Friedberg) We can compute the Whittaker coefficient for a maximal parabolic Eisenstein series (subgroups of GL_n):

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times} H(d_1, ..., d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

- We want to compute the Whittaker coefficient for A_5. Why?

Whittaker coefficient

- (Brubaker-Friedberg) We can compute the Whittaker coefficient for a maximal parabolic Eisenstein series (subgroups of GL_n):

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_s/\mathfrak{o}_s^\times} H(d_1,\ldots,d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

- We want to compute the Whittaker coefficient for A_5. Why?
- **Conjecture of Bump, 1996**: A multiple Dirichlet series (Chinta) coincide with the H-part (exponential sums) of the Whittaker coefficient.

Chinta Series

Chinta Series

- (Chinta) A multiple Dirichlet series related to A_5:

$$\sum_{I} \frac{\chi_{I_2}(\hat{I}_1)\chi_{I_2}(\hat{I}_3)\chi_{I_4}(\hat{I}_3)\chi_{I_5}(\hat{I}_5)}{|I_1|^{S_1}|I_2|^{S_2}\ldots|I_5|^{S_5}} \cdot g(I_1, \ldots, I_5)$$

Chinta Series

- (Chinta) A multiple Dirichlet series related to A_5:

$$\sum_{I} \frac{\chi_{I_2}(\hat{I_1})\chi_{I_2}(\hat{I_3})\chi_{I_4}(\hat{I_3})\chi_{I_5}(\hat{I_5})}{|I_1|^{S_1}|I_2|^{S_2}\ldots|I_5|^{S_5}} \cdot g(I_1, \ldots, I_5)$$

- With a change of variable, we get $g(I_1, \ldots, I_5) = H(x, y, z, w, v)$ a polynomial of 366 terms:

$$1 - vw - xy + vwxy - wz + vwz + pv^2w^2z - \ldots + p^7v^4w^7x^4y^7z^8.$$

Chinta Series

- (Chinta) A multiple Dirichlet series related to A_5:

$$\sum_I \frac{\chi_{I_2}(\hat{I_1})\chi_{I_2}(\hat{I_3})\chi_{I_4}(\hat{I_3})\chi_{I_5}(\hat{I_5})}{|I_1|^{S_1}|I_2|^{S_2}\ldots|I_5|^{S_5}} \cdot g(I_1,\ldots,I_5)$$

- With a change of variable, we get $g(I_1,\ldots,I_5) = H(x,y,z,w,v)$ a polynomial of 366 terms:

$$1 - vw - xy + vwxy - wz + vwz + pv^2w^2z - \ldots + p^7v^4w^7x^4y^7z^8.$$

- We suspect that the Chinta series comes from the Whittaker coefficient. Reason: Both have nice functional equations that generate a group isomorphic to the Weyl group of A_5.

Our Goal (REU Problem 4)

1. Compute Whittaker coefficients using data from A_5.
2. Understand the support of $H(d_1, ..., d_N)$. (Does it form a polytope in the Euclidean space? It is infinite?)

Questions we ask:

- How do we simplify $H(d_1, ..., d_N)$ and when is it nonzero?
- How does the polynomial from Whittaker compare with the Chinta series?

Our Strategy for removing roots

Our Strategy for removing roots

- Recall, a maximal parabolic corresponds to the choice of removing a simple root.

Our Strategy for removing roots

- Recall, a maximal parabolic corresponds to the choice of removing a simple root.
- Heuristic:

$$\sum_{d_i=1}^{\infty} \frac{\left(\frac{d_1}{d_2}\right)\left(\frac{d_2}{d_3}\right)\left(\frac{d_3}{d_4}\right)\left(\frac{d_4}{d_5}\right)}{d_1^{s_1} d_2^{s_2} d_3^{s_3} d_4^{s_4} d_5^{s_5}} = \sum_{d_2,d_4=1}^{\infty} \frac{\mathcal{L}(s_1, \chi_{d_2})\mathcal{L}(s_3, \chi_{d_2 d_4})\mathcal{L}(s_5, \chi_{d_4})}{d_2^{s_2} d_4^{s_4}}$$

For computation, removing α_2 and α_4 could give us a nicer polynomial to compare.

Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_s/\mathfrak{o}_s^\times} H(d_1, \ldots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) \Psi(\mathfrak{D}) \zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Some results:

Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_s/\mathfrak{o}_s^\times} H(d_1, \ldots, d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Some results:

- $\delta_P^{s+1/2}(\mathfrak{D}) = (d_1d_2d_3d_4d_5d_6d_7d_8)^{-3s-3/2}$

Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_s/\mathfrak{o}_s^\times} H(d_1, ..., d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Some results:

- $\delta_P^{s+1/2}(\mathfrak{D}) = (d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8)^{-3s-3/2}$
- $H(d_1, ..., d_N) := \sum_{c_i \mathrm{mod}\, D_j} \prod_{k=1}^N \left(\frac{c_k}{d_k}\right) e^{2\pi i \sum_j v_j}$: Gauss sums calculated from removing α_2 (will be explained in detail)

Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_s/\mathfrak{o}_s^\times} H(d_1, ..., d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Some results:

- $\delta_P^{s+1/2}(\mathfrak{D}) = (d_1d_2d_3d_4d_5d_6d_7d_8)^{-3s-3/2}$
- $H(d_1, ..., d_N) := \sum_{c_i \text{mod } D_j} \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right) e^{2\pi i \sum_j v_j}$: Gauss sums calculated from removing α_2 (will be explained in detail)
- $\zeta_D = (d_4d_3d_2d_1, d_5)_S(d_4d_3d_2, d_6)_S(d_4d_3, d_7)_S(d_4, d_8)_S.$

Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times} H(d_1, ..., d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D})$$

Some results:

- $\delta_P^{s+1/2}(\mathfrak{D}) = (d_1d_2d_3d_4d_5d_6d_7d_8)^{-3s-3/2}$
- $H(d_1, ..., d_N) := \sum_{c_i \text{mod } D_j} \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right) e^{2\pi i \sum_j v_j}$: Gauss sums calculated from removing α_2 (will be explained in detail)
- $\zeta_D = (d_4d_3d_2d_1, d_5)_S(d_4d_3d_2, d_6)_S(d_4d_3, d_7)_S(d_4, d_8)_S$.
- $c^{\psi}_{f_1,f_2}(\mathfrak{D})$: The inductive step for further removing roots from $A_1 \times A_3$

Root System A_n

- A root system $\Phi \subset \mathbb{R}^{n+1}$ is a finite collection of vectors ("roots") under some axioms
- There is a method of enumerating the positive roots

Example

Below is one possible enumeration of roots for the A_3 case

$$\begin{bmatrix} * & \beta_3 & \beta_2 & \beta_1 \\ & * & \beta_5 & \beta_4 \\ & & * & \beta_6 \\ & & & * \end{bmatrix}$$

Removing the second root from A_5

- We want to split up A_5 into $A_1 \times A_3$ and “what’s left”

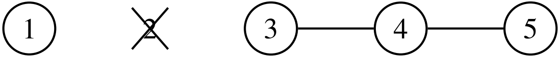
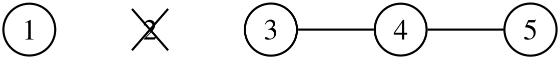


Figure: The Dynkin diagram corresponding to removing the second node

Removing the second root from A_5



- In the below diagram, the asterisks represent the A_1 and A_3 root systems
- We can rig the enumeration to do the $A_1 \times A_3$ roots first and $\gamma_1, \gamma_2, \ldots, \gamma_8$ last.

$$\begin{bmatrix} * & * & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ * & * & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \end{bmatrix}$$

- We'll compute the asterisk $A_1 \times A_3$ part inductively

Gauss Sums – A Prototype for the Exponential Sum

Definition

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t e^{2\pi i \frac{mc}{d}}$$

Gauss Sums – A Prototype for the Exponential Sum

Definition

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t e^{2\pi i \frac{mc}{d}}$$

Example

$$g_1(1,p^2) = \sum_{c \bmod p^2} \left(\frac{c}{p^2}\right) e^{2\pi i \frac{c}{p^2}}$$

Gauss Sums – A Prototype for the Exponential Sum

Definition

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t e^{2\pi i \frac{mc}{d}}$$

Example

$$g_1(1,p^2) = \sum_{c \bmod p^2} \left(\frac{c}{p^2}\right) e^{2\pi i \frac{c}{p^2}}$$

Reindex to $c = x + py$ with $x, y \bmod p$.

$$g_1(1,p^2) = \sum_{x,y \bmod p} \left(\frac{x}{p^2}\right) e^{2\pi i \frac{x}{p^2} + \frac{y}{p}} = \sum_{x \bmod p} \left(\frac{x}{p^2}\right) e^{2\pi i \frac{x}{p^2}} \sum_{y \bmod p} e^{2\pi i \frac{y}{p}}$$

$$= \sum_{x \bmod p} \left(\frac{x}{p^2}\right) e^{2\pi i \frac{x}{p^2}} \cdot 0 = 0$$

Gauss Sums – A Prototype for the Exponential Sum

Definition

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t e^{2\pi i \frac{mc}{d}}$$

Example

$$g_1(p,p) = \sum_{c \bmod p} \left(\frac{c}{p}\right) e^{2\pi i \frac{cp}{p}}$$

Gauss Sums – A Prototype for the Exponential Sum

Definition

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)^t e^{2\pi i \frac{mc}{d}}$$

Example

$$g_1(p,p) = \sum_{c \bmod p} \left(\frac{c}{p}\right) e^{2\pi i \frac{cp}{p}}$$

For $c \bmod p$, $(c,p) = 1$, half of c are squares and half are not, so

$$g_1(p,p) = \sum_{c \bmod p} \left(\frac{c}{p}\right) = 0.$$

Defining the Exponential Sum

- We associate an exponential sum to removing a certain root from a root system

Definition (Brubaker-Friedberg)

For $\mathbf{d} = (d_1, d_2, \ldots, d_N)$ with $d_i = p^{l_i}$ for some prime

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i \left(\sum_i v_i\right)\right) \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)$$

Defining the Exponential Sum

Definition (Brubaker-Friedberg)

For $\mathbf{d} = (d_1, d_2, \ldots, d_N)$ with $d_i = p^{l_i}$ for some prime

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i \left(\sum_i v_i\right)\right) \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)$$

We define $v_j = \frac{c_N}{d_N}$ when j is the removed root and is otherwise

$$\sum_{(k,k')\in S_j} (-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l^\vee \rangle} \prod_{k'<l<k} (d_l^{-1})^{i' \langle \gamma_k', \gamma_l^\vee \rangle}$$

Defining the Exponential Sum

Definition (Brubaker-Friedberg)

For $\mathbf{d} = (d_1, d_2, \ldots, d_N)$ with $d_i = p^{l_i}$ for some prime

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i \left(\sum_i v_i\right)\right) \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)$$

We define $v_j = \frac{c_N}{d_N}$ when j is the removed root and is otherwise

$$\sum_{(k,k')\in S_j} (-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l\geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l^\vee \rangle} \prod_{k'<l<k} (d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee \rangle}$$

D_j are defined in terms of d_js as follows:

$$D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle}$$

Removing the second root from A_5

- We compute $H(\mathbf{d})$ in the A_5 case with second node of the Dynkin diagram removed.

Removing the second root from A_5

- We compute $H(\mathbf{d})$ in the A_5 case with second node of the Dynkin diagram removed.

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4}\right.\right.$$

$$\left.\left. + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right) \prod_{k=1}^{8} \left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4}\right.\right.$$

$$\left.\left. + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right) \prod_{k=1}^{8}\left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

Proposition (S. Garg-K.-F. Lu-W.)

Put the d_js in a matrix corresponding to the position of γ_j. Then,

$$D_j = d_j \times d_k\text{s below } d_j \text{ in the same column}$$
$$\times\, d_k\text{s to the left of } d_j \text{ in the same row}$$

Recall the original definition: $D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle}$

Removing the second root from A_5

Proposition (S. Garg-K.-F. Lu-W.)

Put the d_js in a matrix corresponding to the position of γ_j. Then,

$$D_j = d_j \times d_k s \text{ below } d_j \text{ in the same column}$$
$$\times\, d_k s \text{ to the left of } d_j \text{ in the same row}$$

Recall the original definition: $D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle}$

Example

Here, the matrix is

d_4	d_3	d_2	d_1
d_8	d_7	d_6	d_5

We then have

$$D_3 = d_3 d_7 d_4, \quad D_4 = d_4 d_8$$

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4} + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right)\prod_{k=1}^{8}\left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

Proposition (GKLW)

Each term in the exponent other than $\frac{c_8}{d_8}$ is of the form

$$\pm\frac{b_ic_jD_i}{D_j}$$

Recall the original definition of a term:

$$(-1)^{i+i'}\eta_{i,i',k,-k'}(b_kd_k^{-1})^i(c_{k'}d_{k'}^{-1})^{i'}\prod_{l\geq k}(d_l^{-1})^{\langle\alpha_j,\gamma_l^\vee\rangle}\prod_{k'<l<k}(d_l^{-1})^{i'\langle\gamma_k',\gamma_l^\vee\rangle}$$

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4}\right.\right.$$

$$\left.\left.+\frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right)\prod_{k=1}^{8}\left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

Proposition (GKLW)

Each term in the exponent other than $\frac{c_8}{d_8}$ is of the form

$$\pm\frac{b_ic_jD_i}{D_j}$$

- We can check this for b_4, c_3 with $D_3 = d_3d_7d_4$, $D_4 = d_4d_8$.

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4}\right.\right.$$

$$\left.\left. + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right)\prod_{k=1}^{8}\left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

- To better understand the sum, we draw a "dependency graph"

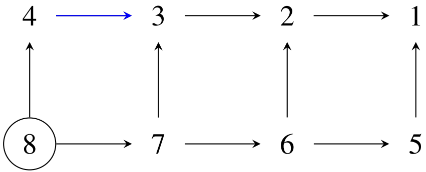


Figure: There is a b_ic_j term in the sum $\iff$ there is an edge $i \to j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{c_i \bmod D_i} \exp\left(2\pi i\left(-\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4} + \frac{c_8}{d_8} + \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7} + \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6} + \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}\right)\right) \prod_{k=1}^{8} \left(\frac{c_k}{d_k}\right),$$

where loosely we define $b_i \equiv c_i^{-1} \bmod d_i$

- To better understand the sum, we draw a "dependency graph"

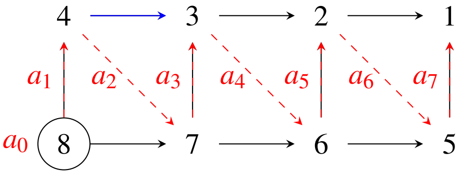


Figure: There is a b_ic_j term in the sum $\iff$ there is an edge $i \to j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term

Removing the second root from A_5

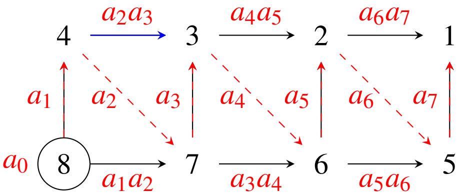


Figure: There is a b_ic_j term in the sum $\iff$ there is an edge $i \to j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term

- We can follow paths to compute what the other edges are in terms of the a_js.

Example

Since $b_i = c_i^{-1}$, we have

$$b_4c_3 = b_4c_7b_7c_3 = a_2a_3$$

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{a_i} \exp\Bigg(2\pi i\Bigg(-\frac{a_7 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{a_5 d_7 d_8}{d_2 d_3 d_4} - \frac{a_3 d_8}{d_3 d_4} - \frac{a_1}{d_4}$$

$$+\frac{a_0}{d_8} + \frac{a_2 a_3 d_8}{d_3 d_7} + \frac{a_1 a_2}{d_7} + \frac{a_4 a_5 d_7}{d_2 d_6} + \frac{a_3 a_4}{d_6} + \frac{a_6 a_7 d_6}{d_1 d_5} + \frac{a_5 a_6}{d_5}\Bigg)\Bigg) \prod_{k=1}^{8} \left(\frac{a_k}{\dots}\right),$$

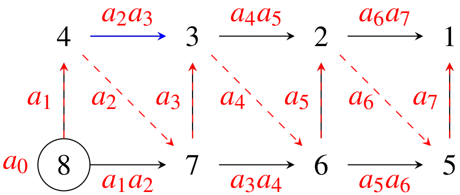


Figure: There is a $b_i c_j$ term in the sum $\iff$ there is an edge $i \to j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term

Removing the second root from A_5

$$H(\mathbf{d}) = \sum_{a_i} \exp\left(2\pi i\left(-\frac{a_7 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{a_5 d_7 d_8}{d_2 d_3 d_4} - \frac{a_3 d_8}{d_3 d_4} - \frac{a_1}{d_4}\right.\right.$$

$$\left.\left. + \frac{a_0}{d_8} + \frac{a_2 a_3 d_8}{d_3 d_7} + \frac{a_1 a_2}{d_7} + \frac{a_4 a_5 d_7}{d_2 d_6} + \frac{a_3 a_4}{d_6} + \frac{a_6 a_7 d_6}{d_1 d_5} + \frac{a_5 a_6}{d_5}\right)\right) \prod_{k=1}^{8} \left(\frac{a_k}{\dots}\right),$$

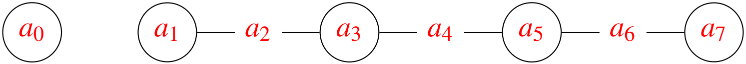


Figure: A visualization of the dependencies in the re-indexed sum

Progress Summary

- We compute a Dirichlet Series from a Dynkin Diagram
- We show how to interpret relevant quantities in terms of the geometry of the γ_js
- We model the exponential sum as a graph and use it to facilitate re-indexing to "nicer" coordinates

this gives us...

- An understanding of where the $H(d,t)$'s are supported: Finite cases (most exponents ≤ 1) and a few infinite cases.

Future Directions

- Change of variables from the Whittaker coefficient to the Chinta polynomial
- Understand the 15 zeta functions that got pulled out from the Chinta series, and how it coincide with the normalizing zeta factor of the Whittaker function
- Another description of the same polynomial is through ”string data” defined in Littelmann. We bounded a polytope but it currently has 12624 vertices...

Acknowledgements

Acknowledgements

- Thanks Ben for patiently explaining this problem to us again and again (until one of us finally starts to understand).

Acknowledgements

- Thanks Ben for patiently explaining this problem to us again and again (until one of us finally starts to understand).
- Thanks Kayla, Emily and Megan for all the TA sessions, answering endless discord Q&As' and spending 2.5 hours yesterday to help us polish the talk.

Acknowledgements

- Thanks Ben for patiently explaining this problem to us again and again (until one of us finally starts to understand).
- Thanks Kayla, Emily and Megan for all the TA sessions, answering endless discord Q&As' and spending 2.5 hours yesterday to help us polish the talk.
- Thanks all staff & participants of the REU for providing invaluable intellectual and emotional support for 2 months.

Acknowledgements

- Thanks Ben for patiently explaining this problem to us again and again (until one of us finally starts to understand).
- Thanks Kayla, Emily and Megan for all the TA sessions, answering endless discord Q&As' and spending 2.5 hours yesterday to help us polish the talk.
- Thanks all staff & participants of the REU for providing invaluable intellectual and emotional support for 2 months.
- The End!