%-Immanants and Kazhdan-Lusztig Immanants

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Abstract

In this paper, we investigate a certain class of nice immamants, which we call %-immanants, and their relationship to Temperley-Lieb immanants, first introduced by Rhoades and Skandera [8]. In particular, we classify when a Temperley-Lieb immanant can be written as a linear combination of %-immanants. As a byproduct, we also arrive an explicit formula for computing the Temperley-Lieb immanant coming from a 321-, 1324-avoiding permutation w which contains the pattern 2143. We also make some partial progress on extending these results to Kazhdan-Lusztig immanants, a generalization of Temperley-Lieb immanants.

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1 Introduction

Immanants are functions defined on square matrix which are generalizations of the determinant. For a function $f : \mathfrak{S}_n \to \mathbb{C}$, we define the immanant associated to f, $\mathrm{Imm}_f : M_{n\times n} \to \mathbb{C}$, as

$$\mathrm{Imm}_f(X) = \sum_{\sigma\in\mathfrak{S}_n} f(\sigma)x_{1,\sigma(1)}x_{2,\sigma(2)}\cdots x_{n,\sigma(n)}.$$

We are interested in the following three families of immanants.

Definition 1 (Kazhdan-Lusztig immanants)**.** Let $w \in \mathfrak{S}_n$. The Kazhdan-Lusztig immanant $\mathrm{Imm}_w : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$ is given by

$$\mathrm{Imm}_w^{KL}(M) := \sum_{u\in\mathfrak{S}_n} (-1)^{l(u)-l(w)} P_{w_0u,w_0w}(1) m_{1,u_1}m_{2,u_2}\cdots m_{n,u_n}$$

where $P_{x,y}(q)$ is the Kazhdan-Lusztig polynomial associated to $x, y \in \mathfrak{s}_n$ and w_0 is the longest word in $\mathfrak{S}_n$. See [1, Section 5.5] for definitions of Kazhdan-Lusztig polynomials.

Another class of immanants, Temperley-Lieb immanants, were introduced using a different algebra, known as the Temperley-Lieb algebra, in an earlier paper by Rhoades and Skandera, [8]. It was later proven in [9] that these immanants are the same as Kazhdan-Lusztig immanants, in the case when w avoids the pattern 321. These immanants, unlike Kazhdan-Lusztig immanants, have coefficients that can be defined non-recursively within the Temperley-Lieb algebra.

We then introduce a new class of immanants which we call %-immanants.

Definition 2 (%-immanants)**.** Suppose λ/μ is a skew tableau. The %-immanant associated to this permutation is defined by

$$\mathrm{Imm}_{\lambda/\mu}^{\%}(X) = \sum_{\sigma\in A} \mathrm{sgn}(\sigma)x_{1,\sigma(1)}x_{2,\sigma(2)}\cdots x_{n,\sigma(n)},$$

where $\sigma \in A$ iff $\forall i$, $(i, \sigma(i)) \in \lambda/\mu$. This is related to a 'Skew Ferrers Matrix' from [10].

Recently, using the connections between Kazhdan-Lusztig immanants and Schubert varities, Chepuri and Sherman-Bennett [4] proved that for a 2143 and 1324-avoiding permutation w, the corresponding Kazhdan-Lusztig immanent $\mathrm{Imm}_w(M)$ is the same as some %-immanant (up to signs) which are significantly easier to compute. Motivated by their result, out paper seeks to answer the following question.

Question 1.1. Which Kazhdan-Lusztig immanants are linear combinations of %-immanants?

In this paper, we give an complete answer to the above question in the case of Temperley-Lieb immanants, and obtain partial results for Kazdhan-Lusztig immanants in general. In particular, our proofs for the Temperley-Lieb immanant case will be purley combinatorial, based on the relationship between Temperley-Lieb immanants, non-crossing matchings, and colorings. Our main result is as follows.

Theorem. *Let w be a 321-avoiding permutation. The following statements are equivalent:*

1. *The Temperley-Lieb immanant Imm_w is a linear combination of %-immanants;*

2. *The signed Temperley-Lieb immanant* $\mathrm{sgn}(w)\,\mathrm{Imm}_w$ *is a sum of at most two %-immanants;*

3. *The permutation* w *avoids the patterns* $24153, 31524, 231564,$ *and* 312645, *in addition to avoiding* 321.

We furthermore conjecture the following for Kazhdan-Lusztig immanants in general.

Conjecture. *The Kazhdan-Lusztig immanant for a permutation* $w \in \mathfrak{S}_n$, Imm_w, *can be expressed as a linear combination of %-immanants if and only if* w *avoids the patterns*

$$1324, 24153, 31524, 231564, 312645, 426153.$$

The plan of the paper is as follows. In section 2, we first go through the important concepts and introduce the important objects that we'll be studying, and review some preliminary results about them. Then, in section 3, we prove that the Temperley-Lieb immanant of a 321-avoiding permutation w is a %-immanant if and only if w avoids the patterns 2143 and 1324, using more combinatorial methods. Although one direction was already known from the work of Chepuri and Sherman-Bennett in [4], the setup here will evoke the flavor of the method of proof we will use in section 4.

Following this, in section 4 we prove that the Temperley-Lieb immanant of a 321-avoiding permutation is a sum of two %-immanants, and more generally a linear combination of %-immanants, if and only if it avoids the patterns $1324, 21453, 31524, 231564, 312645$. Over the course of this proof, we will arrive at a relatively simple combinatorial formula for the Temperley-Lieb immanant for a 321- and 1324-avoiding permutation w that contains the pattern 2143.

In section 5, we turn to Kazhdan-Lusztig immanants in general and show that certain patterns need to be avoided for the associated Kazhdan-Lusztig immanant to be a linear combination of %-immanants. We also conclude that certain %-immanants are a sum of a Temperley-Lieb immanant and a Kazhdan-Lusztig immanant. In section 6, we briefly discuss how this result allows us to make statements about some notion of positivity of these %-immanants.

2 Preliminaries

In this section, we go through the main preliminary concepts that we will be extensively using throughout this paper. We start with some discussion of permutations, before introducing our main objects of study, the %-immanant and the Temperley-Lieb immanant. We conclude with a discussion about complementary minors and their relation to Temperley-Lieb immanants.

2.1 Permutations

We begin by introducing some notation about permutations, as well as reviewing through the Bruhat order.

For the purpose of notation, we denote $\{1, 2, \ldots, n\}$ as $[n]$. Given a set of indices $I \subset [n]$, let $\overline{I} = [n] - I$, and for integers $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$. Furthermore, we write $[a, b : c, d]$ to be $[a, b] \cup [c, d]$, and similarly if we have a longer expression $[a_1, a_2 : a_3, a_4 : a_5, a_6 : \ldots : a_{2n-1}, a_{2n}]$. Let $s(A)$ be the sum of the elements of a set $A \subset \mathbb{R}$.

We will use $v \cdot w$ to denote multiplication in $\mathfrak{S}_n$: $v \cdot w(i) = v(w(i))$.

We will use (i, j) to denote the transposition swapping i and j. It will be clear by context whether (i, j) denotes a transposition or an ordered pair.

Furthermore, given a permutation $v \in \mathfrak{S}_n$, we denote x_v as the monomial $\prod_{i=1}^{n} x_{i,v(i)}$.

We will also be referring to the block structure of a given permutation.

Definition 3. In particular, we say that a **block** is a string of ascending consecutive integers. Given two disjoint blocks $[a]$ and $[b]$, we say that $[a] < [b]$ if the largest element in a is smaller than the smallest element of b.

Then, given a permutation $w \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_n$, where $m \leq n$, the block structure $[v(1)][v(2)] \dots [v(m)]$ is used to notate a permutation whose one-line notation consists of m blocks, where $[1] < [2] < \dots < [m]$.

For instance, the permutation with one-line notation 56123784 has block form [3][1][2][4].

We now recall the Bruhat order, which is defined in [1].

Definition 4. The **Bruhat order** on $\mathfrak{S}_n$ is defined as follows: $u \leq v$ if some reduced word for v contains a subword equal to u. The length of a permutation $\ell(u)$ is the length of a reduced word for u.

Equivalently, the Bruhat order is the transitive closure of the following relation: $u \cdot (i, j)$ covers u if $\ell(u \cdot (i, j)) = \ell(u) + 1$. It follows that the Bruhat order is graded by the length function: if $u \leq v$, then $\ell(u) \leq \ell(v)$.

Equivalently, $u \leq v$ in the Bruhat order if for all $i, j \leq n$, we have $|u([1, i]) \cap [1, j]| \geq |v([1, i]) \cap [1, j]|$.

An alternate definition is given as Theorem 2.1.5 in [1].

In a few places, we wish to compare two permutations that differ on a prescribed set of indices. The following lemma makes this possible.

Definition 5. Given a permutation $w \in \mathfrak{S}$ and a set of indices $I \subset [n]$, **the restricted permutation** $w|_I$ is the permutation on $\mathfrak{S}_{|I|}$ defined as follows: $w|_I(i) = j$ if the i-th smallest element of I is mapped to the j-th smallest element of $w(I)$ under w.

Lemma 2.1. *Let $v, w \in \mathfrak{S}$. If $I = \{i \mid v(i) \neq w(i)\}$ and $w|_I \leq v|_I$, then $w \leq v$.*

Proof. We employ the third equivalent definition of the Bruhat order. Our goal is to show that $|u([1, i]) \cap [1, j]| \geq |v([1, i]) \cap [1, j]|$ for positive integers i, j so $i, j \leq n$. Clearly $u([1, i] \backslash I) \cap [1, j] = v([1, i] \backslash I) \cap [1, j]$, and $|u([1, i] \cap I) \cap [1, j]| \geq |v([1, i] \cap I) \cap [1, j]|$ follows by applying this third definition of Bruhat order to $u|_I$ and $v|_I$. $\square$

The following is an immediate consequence of Lemma 2.1:

Corollary 2.2. *If (i, j) is an inversion of w, then $w \cdot (i, j) \leq w$.*

Remark. We are tempted to conclude that $w\tau$ has exactly one less inversion than w (i.e. that w covers $w\tau$), but this is not true, e.g. consider $(1, 3)$ an inversion of 321. However, if w is 321-avoiding, we can easily show w must in fact cover $w\tau$.

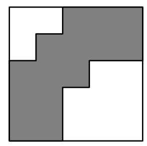
Finally, we say that a permutation that avoids 321 and 1324 is called nice. We will see why such permutations are nice in the next two sections.

2.2 %-immanants

We now define one of the main types of immanant that we will be studying in this paper, which we refer to as %-immanants.

Definition 6. For a skew tableau λ/μ, define $\mathrm{Imm}^{\%}_{\lambda/\mu} = \sum_{\sigma \in A} \mathrm{sgn}(\sigma) x_\sigma$, where $\sigma \in A$ iff $\forall i$, $(i, \sigma(i)) \in \lambda/\mu$. We will refer to these as **%-immanants**. (See [10], and how this is related to a 'Skew Ferrers Matrix')

For instance, the following is a %-immanant associated to the skew-tableau $(5,5,3,2,2)/(2,1)$.



For $\mathrm{Imm}^{\%}_{\lambda/\mu}$ to be nonzero, the skew tableau has to enclose the antidiagonal.

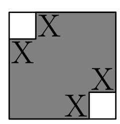
Lemma 2.3. *If* $\mathrm{Imm}^{\%}_{\lambda/\mu} \neq 0$*, then* $(i, n+1-i) \in \lambda/\mu$ *for all* $1 \leq i \leq n$*.*

Proof. We show the contrapositive assertion. If $(i, n+1-i) \notin \lambda/\mu$ for some i, then either $(a,b) \notin \lambda/\mu$ for all $a \leq i, b \leq n+1-i$ or $(a,b) \notin \lambda/\mu$ for all $a \geq i, b \geq n+1-i$. In the first case, for any permutation w, there must exist $j \leq i$ such that $w(j) \leq n+1-i$. Thus, $(j, w(j)) \notin \lambda/\mu$, and so $\mathrm{Imm}^{\%}_{\lambda/\mu} = 0$. The second case is analogous. $\square$

Certain %-immanants are naturally associated to a permutation w, as follows.

Definition 7. Suppose $w \in \mathfrak{S}_n$. Let $m_w(i) = \min\{w(1), w(2), \ldots, w(i)\}$ and let $M_w(i) = \max\{w(i), w(i+1), \ldots, w(n)\}$. Consider the skew shape λ/μ where the entries in the ith row that are included lie between $m_w(i)$ and $M_w(i)$, inclusive. Then, we define $\mathrm{Imm}^{\%}_w = \mathrm{Imm}^{\%}_{\lambda/\mu}$.

For instance, the following is a %-immanant associated to the permuation 2143, where the positions of $(i, w(i))$ are marked with an "X," and the areas within the skew shape are shaded.



2.3 The Temperley-Lieb algebra

We will also be referring to the group algebra $\mathbb{C}[\mathfrak{S}_n]$, which is the algebra with basis elements corresponding to elements in $\mathfrak{S}_n$. We will refer to some special elements in this group algebra, which we denote as $z_{i,j}$.

Definition 8. Define $z_{i,j} \in \mathbb{C}[\mathfrak{S}_n]$ to be $\sum_v v$, over all permutations v that fix elements not in the set $[i,j]$.

For instance, $z_{1,3} = (1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)$. This definition will come into play when we deal with positivity in Section 5.

The main algebra that we will be working with in this paper, however, will be the following.

Definition 9. (See [6, §2] and [8, §3]) The **Temperley-Lieb algebra**, which we denote as $TL_n(2)$, is the $\mathbb{C}-$algebra generated by elements $t_1, t_2, \ldots, t_{n-1}$ with the relations $t_i^2 = 2t_i$ for $i \in [n]$, $t_it_j = t_jt_i$ for $i, j \in [n]$ where $|i-j| \geq 2$, and $t_it_jt_i = t_i$ for $i, j \in [n]$ where $|i-j| = 1$.

Notice that the Temperley-Lieb algebra in general depends on a parameter, but throughout this paper we will be specializing to the case when the parameter equals 2.

Proposition 2.4. *$\theta : \mathbb{C}(\mathfrak{S}_n) \to TL_n(2)$ where $\theta(s_i) = t_i - 1$ is a ring homomorphism. β : $\mathbb{C}$-span of 321-avoiding permutations $\to TL_n(2)$ where $\beta(s_{i_1}s_{i_2}\cdots s_{i_k}) = t_{i_1}t_{i_2}\cdots t_{i_k}$ for any reduced word $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a vector space isomorphism (See, for instance, the beginning of section 3 of Rhoades and Skandera's paper on Temperley-Lieb immanants, [8]).*

Definition 10. For $w, u \in \mathfrak{S}_n$ where w is 321-avoiding, let $f_w(u)$ be the coefficient of $\beta(w)$ in $\theta(u)$.

It turns out a lot of the coefficients $f_w(u)$ are zero. To characterize when this happens, we can use the Bruhat order.

The following lemma appears as Proposition 3.7 in [8]; we include a proof for completeness.

Lemma 2.5. *Let $w, u \in \mathfrak{S}_n$, with w being 321-avoiding.*

1. *If $u \not\geq w$, then $f_w(u) = 0$.*
2. *$f_w(w) = 1$.*

Proof. Let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ and $u = s_{j_1}s_{j_2}\cdots s_{j_m}$ be reduced words. Consider expanding $\theta(u) = (t_{j_1} - 1)(t_{j_2} - 1)\cdots(t_{j_m} - 1)$ and simplifying each monomial using the Temperley-Lieb algebra relations. The resulting monomial will be a subword of the original monomial, which turn is a subword of $t_{j_1}t_{j_2}\cdots t_{j_m}$. Thus, we can only have a $t_{i_1}t_{i_2}\cdots t_{i_k}$ term in the expansion if $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a subword of $s_{j_1}s_{j_2}\cdots s_{j_m}$. Thus, if $f_w(u) \neq 0$, then $u \geq w$, proving part 1 of the Lemma. For part 2, we take $w = u$, and observe that when expanding $(t_{i_1} - 1)(t_{i_2} - 1)\cdots(t_{i_k} - 1)$, the only way to get a monomial of length k is to take the first term of each binomial, and thus the coefficient of $t_{i_1}t_{i_2}\cdots t_{i_k}$ is one. $\square$

Definition 11. For a 321-avoiding permutation w, $\mathrm{Imm}_w = \sum_{u \in \mathfrak{S}_n} f_w(u)x_u$, where $f_w(u)$ is the coefficient of $\beta(w)$ in $\theta(u)$.

We will abuse notation slightly, and denote by the pairings of a 321-avoiding permutation w as the pairings that occur in the corresponding non-crossing matching, per the bijection above.

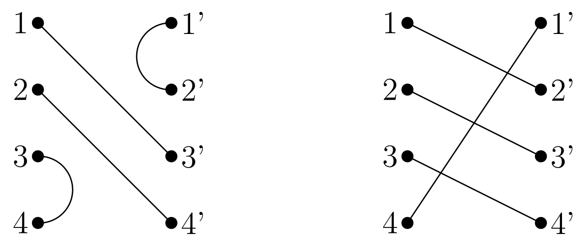
Definition 12. (See [6, §2]) Consider two columns of vertices, with the vertices on the left column labelled 1 to n (going down the column), and vertices labelled $2n$ to $n+1$ on the right column, going down the column. A **non-crossing matching** of these $2n$ vertices is a perfect matching so that there doesn't exist pairs (a, c) and (b, d) that are matched, where $a < b < c < d$.

For the purposes of this paper, we denote $n + i$ as $(n + 1 - i)'$, for $i = 1, 2, \ldots n$. Also, let $[a, b]' = \{a', (a+1)', \ldots, b'\}$.

Proposition 2.6. *321-avoiding permutations of $[n]$ are in bijection with non-crossing matchings of $2n$ vertices. (See, for instance, the beginning of section 3 of [8] to get from s_i to t_i representation, and then Kauffman's paper [5] to get to the non-crossing matching).*

Remark. Any crossing in the wiring diagram of a 321-avoiding permutation corresponds to a s_i, which will be mapped to a t_i by the map β. This finite process will produce a non-crossing permutation that is well-defined. Furthermore, as noted in [8], the correspondence given by replacing s_i with t_i is a bijection between a basis of $TL_n(2)$ and the set of 321-avoiding permutations. This process bijectively maps reduced decompositions to each other.

For instance, the following is a non-crossing matching corresponding to the 321-avoiding permutation 2341.

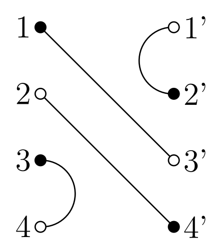


2.4 Complementary Minors and Colorings

The Temperley-Lieb immanants are closely related to complementary minors, which in turn can be represented with a coloring. We first describe colorings and how they are associated with complementary minors.

Definition 13. A black or white coloring of vertices $1, 2, ..., n, 1', 2', ..., n'$ is **compatible** with a non-crossing matching if every black vertex is matched with a white vertex.

In the above matching, for instance, the following is a compatible coloring.



We now define some notation for a minor of a matrix.

Definition 14. We denote $\Delta_{I,J}$ as the determinant of the minor with rows indexed by I and columns indexed by J.

These compatible colorings can be associated to complementary minors, with $\Delta_{I,J}\Delta_{\bar{I},\bar{J}}$ being associated to the coloring where I is the set of labels of unprimed black vertices and J is the set of labels of primed white vertices (after the primes are removed).

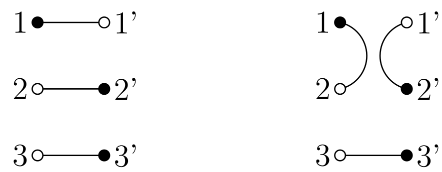
Proposition 2.7. *[8, Proposition 4.3] Let $I, J \subseteq [n]$. Color $I, \bar{J}$ black and $J', \bar{I}'$ white, then*

$$\Delta_{I,J}\Delta_{\bar{I},\bar{J}} = \sum_{w \text{ compatible with coloring}} \mathrm{Imm}_w \tag{1}$$

Remark. Using the proof of Proposition 4.7 in [8], if we treat the equations in (1) for all $I, J \subset [n]$ as an (over-determined) linear system in the variables Imm_w, we see that the system has a unique solution.

Example. For $I = \{1\}$ and $J = \{1\}$, we have:

$$x_{11} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} = \mathrm{Imm}_{123} + \mathrm{Imm}_{213}$$



In this paper, we also make the distinction between products of complementary minors as standalone determinants, and products of complementary minors that are "embedded" in the matrix.

Definition 15. Define $\mathrm{CM}_{I,J}$ to be the determinant of the matrix (y_{ij}) given by

$$y_{ij} = \begin{cases} x_{ij}, & i \in I, j \in J \text{ or } i \notin I, j \notin J, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.8. *We have* $\mathrm{CM}_{I,J} = \Delta_{I,J}\Delta_{\overline{I},\overline{J}}(-1)^{s(I)+s(J)}$.

Proof. This is clearly true if $I = J = \{1, 2, \cdots, k\}$ for some k. In the general case, notice that we can reduce to this case by performing $\sum_{i\in I}(i-1) = s(I) - |I|$ row swaps and $\sum_{j\in J}(j-1) = s(J) - |J|$ column swaps, and $|I| = |J|$, since each swap changes the sign of the complementary minor. □

As an example, consider the product of complementary minors

$$\begin{vmatrix} x_{14} \end{vmatrix} \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{vmatrix},$$

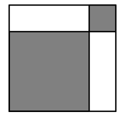
where $I = \{1\}, J = \{4\}$. Observe that $x_{1,4}x_{2,3}x_{3,2}x_{4,1}$ has negative coefficient in the product of complementary of minors it is negative. However, if we instead considered these complementary minors by zeroing out entries (being "embedded" in the determinant), we would instead have the following:

$$\begin{vmatrix} 0 & 0 & 0 & x_{14} \\ x_{21} & x_{22} & x_{23} & 0 \\ x_{31} & x_{32} & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & 0 \end{vmatrix}.$$

Notice here that the coefficient of $x_{1,4}x_{2,3}x_{3,2}x_{4,1}$ is positive, and agrees with what we'd expect if we instead took the normal determinant.

As such, when we represent the complementary minors with pictures, we will be referring to this latter picture, of considering the minors as being within the larger matrix, rather than standing alone as minors. Furthermore, the sign in front will signify whether a permutation v lying in the complementary minors will contribute $\mathrm{sgn}(v)$ or $-\mathrm{sgn}(v)$ to $x_{1,v(1)}x_{2,v(2)} \cdots x_{n,v(n)}$.

The pictoral representation we will be using for this complementary minor will be the following.



In this case we denote by the shaded region the places that we take for our complementary minor, and the white areas as the regions that are zeroed out. This convention will be followed throughout the rest of the paper.

We also note that the matching diagram admits certain symmetries. Specifically, reflecting a non-crossing matching about either axis of the enclosing rectangle will also give a non-crossing matching.

Lemma 2.9. *If $w, u \in \mathfrak{S}_n$ with w 321-avoiding, then $f_w(u) = f_{w^{-1}}(u^{-1}) = f_{w_0ww_0}(w_0uw_0)$.*

Proof. From Proposition 2.7, we have for all $I, J \subset [n]$,

$$\operatorname{sgn}(u)\mathbb{1}_{u(I)=J} = \sum_{w \text{ compatible with } I,J} f_w(u). \tag{2}$$

We will leverage the fact (see Remark after Proposition 2.7) that for a given u, this linear system of equations has a unique solution in the $\{f_w(u)\}$. Note that w is compatible with (I, J) iff w^{-1} is compatible with (J, I), so

$$\operatorname{sgn}(u^{-1})\mathbb{1}_{u^{-1}(J)=I} = \sum_{w \text{ compatible with } J,I} f_{w^{-1}}(u^{-1}).$$

But this holds for all J, I, so by uniqueness of solution to (2) with u^{-1}, we have $f_w(u) = f_{w^{-1}}(u^{-1})$.

Define $I^* = \{n + 1 - i \mid i \in I\}$. Note that w is compatible with (I, J) iff w_0ww_0 is compatible with (I^*, J^*), so

$$\operatorname{sgn}(w_0uw_0)\mathbb{1}_{w_0uw_0(I^*)=J^*} = \sum_{w \text{ compatible with } I^*,J^*} f_{w_0ww_0}(w_0uw_0).$$

But this holds for all I^*, J^*, so by uniqueness of solution to (2) with w_0uw_0, we have $f_w(u) = f_{w_0ww_0}(w_0uw_0)$.

The interested reader is encouraged to find an alternate proof of Lemma 2.9 directly using Definition 11. $\square$

3 Temperley-Lieb Immanants as One %-Immanant

In this section, we present the classification of Temperley-Lieb Immanants that is a % Immanant up to a sign.

Theorem 3.1. *Let w be a 321-avoiding permutation. Then Imm_w is a %-immanant up to sign if and only if w avoids both 1324 and 2143. In that case, $\operatorname{Imm}_w = \operatorname{sgn}(w) \operatorname{Imm}^{\%}_w$.*

3.1 2143-, 1324-Avoiding Implies $\mathrm{Imm}_w = \mathrm{sgn}(w)\,\mathrm{Imm}_w^{\%}$

This subsection will be devoted to proving the if direction of Theorem 3.1. This is a special case of [4, Corollary 3.6] which drops the assumption that w is 321-avoiding, and the proof is based on results on Schubert varieties. Here, we opt for a more combinatorial proof, using the relationship between the Temperley-Lieb immanants, non-crossing matchings, and complementary minors described in [8]. We'll also use some of the ideas in this proof in the next section.

Theorem 3.2. *[4, Corollary 3.6] If w is a permutation that avoids the patterns 321, 1324, and 2413, then Imm_w is a %-immanant up to sign (specifically, $\mathrm{Imm}_w = \mathrm{sgn}(w)\,\mathrm{Imm}_w^{\%}$).*

Before we prove this theorem, we present a general lemma, which will be useful for us.

Lemma 3.3. *Let w be a 321-avoiding permutation. Define a coloring on $[n], [n]'$ as follows: for each i, color i black and $w(i)'$ white if $w(i) \geq i$, and color i white and $w(i)'$ black if $w(i) < i$. Then every white vertex i or i' is paired with a black vertex j or j' with $j < i$. In particular, the non-crossing matching corresponding to w is compatible with the coloring.*

Proof. We first argue that if (i, j) are an inversion, then i, j are different colors by our coloring. Suppose for the sake of contradiction that $i > j$ but they are the same color. If they are both white, it follows that $i > j \geq w(j) > w(i)$. Now, consider $w([1, j])$. Notice that, since $i > w(j)$, it follows that $[1, i]$ has more elements than $[1, w(j)]$. This means that there is some element k so that $k < j < i$ but $w(k) > w(j) > w(i)$, which forms a 321-pattern, contradiction. A similar situation holds when i, j are both black.

We now show that each white vertex gets paired with a black vertex in the corresponding non-crossing matching. To show this, we consider what happens when we walk along one pairing of the non-crossing matching. We can interpret this geometrically as follows: embed the vertices into the $xy-$plane, giving the vertex i coordinates $(-1, n-i)$ and the vertex i' coordinates $(1, n-i)$. From here, represent the wiring diagram of the permutation w by drawing a straight line between i and $w(i)'$.

Furthermore, suppose that $i = w(i)$ for some i. We claim that there are no inversions involving i. Otherwise, suppose that (i, j) is an inversion, with $i < j$ but $w(i) > w(j)$. Then, notice that $[1, i-1]$ cannot map onto $[1, w(i)-1]$, meaning there is some $k < i < j$ so $w(k) > w(i) > w(j)$, again contradicting the 321-pattern avoidance.

To get a non-crossing matching, draw a sufficiently small rectangle with sides parallel to the coordinate axes around each intersection point, so that no two boxes overlap, and so that no line segment contains any of our boundary vertices. Resolve each crossing within their respective box: for instance, replace the portion of the two intersecting lines within the box with short vertical lines. This yields a non-crossing matching. Each pairing can then be thought of as a piecewise-linear path in the $xy-$plane.

Suppose we start from a white vertex, and walk along the curve representing the pairing involving this white vertex. Then, it follows that we start with increasing along the $y-$direction. Furthermore, every time we change line segments, the fact that the $y-$direction is increasing doesn't change. In particular, this means that along the final line segment, we are increasing in $y-$value. But then it follows that, if we started from this vertex, we would begin by decreasing in $y-$value. But this means that this final vertex is black.

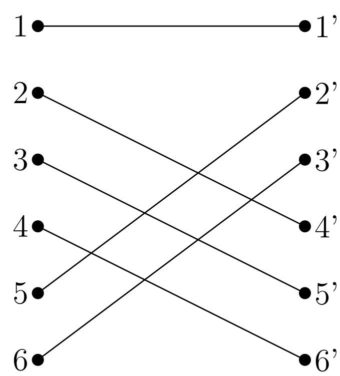
We thus see that every white vertex is paired by a black vertex below it, which finishes the claim. $\square$

As a corollary, we obtain the following lemma, which will also be generally useful:

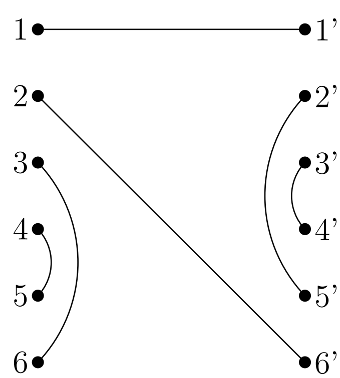
Lemma 3.4. *Suppose we have a permutation σ on $[n]$ that fixes all the elements, other than those in the set of integers between a and b, inclusive. Furthermore, suppose that for some x such that $a < x \leq b$ and that $\sigma(x) < \sigma(x+1) < \ldots < \sigma(b) < \sigma(a) < \ldots < \sigma(x-1)$ are consecutive integers.*

Then, $x+i$ is paired with $x-i-1$, and $(\sigma(b)-i)'$ is paired with $(\sigma(b)+i+1)'$, for integers i so that $0 \leq i \leq \min(x-a-1, b-x)$. Of the remaining elements within $[a,b]$, the ith smallest unpaired element that is unprimed is paired with the ith smallest unpaired element that is primed. For $i \notin [a,b]$, i is paired with i'.

As an example, we can see a permutation where we have the permutation that has one line notation 145623. Here, we can draw the permutation as follows.



We apply Lemma 3.4 with $a = 2, x = 5, b = 6$. This permutation has the following corresponding perfect matching.



Proof. We may assume $a = 1, b = n$. Construct the coloring in Lemma 3.3: then $[1, x-1], [1, \sigma(b)]'$ are colored black and the remaining vertices are colored white. Now, for convenience, we re-index the $2n$ vertices starting from $x-1$ and go clockwise. That is,

- Vertex $i \in [1, x-1]$ has new index $x - i$
- Vertex $i \in [x, n]$ has new index $2n + x - i$

- Vertex $i' \in [1,n]'$ has new index $x-1+i$

Then $1,2,\cdots,n$ are colored black and $n+1,n+2,\cdots,2n$ are colored white under this new indexing. There is a unique non-crossing matching compatible with this coloring, that matches i and $2n+1-i$. Converting back to the old indexing, we recover the desired result.

$\square$

The main method of the proof is as follows. First, we show that our %-immanant is equal to some nice sum of complementary minors. From there, we then argue that this sum of complementary minors is, up to sign, the desired Temperley-Lieb immanant. The exact sign can then be extracted by comparing the coefficients of x_w.

In order to do this, we first proceed by classifying the permutations that avoid the three patterns.

We first have the following proposition:

Proposition 3.5. *If $w \in \mathfrak{S}_n$, $w \neq \mathrm{Id}$ so that $w(1)=1$ and w is* 321-, 2143- *and* 1324-*avoiding, then*

$$\mathrm{Imm}_w^{\%} = \sum_{|I|=n-w(n), I\subset[1,w^{-1}(n)]} \mathrm{CM}_{I,[w(n)+1,n]}\,.$$

Otherwise, suppose that $w(n)=n$, and w is 2143-, 1324-*avoiding. Then*

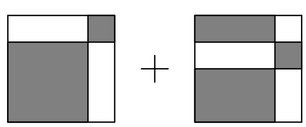
$$\mathrm{Imm}_w^{\%} = \sum_{|I|=w(1)-1, I\subset[w^{-1}(1),n]} \mathrm{CM}_{I,[1,w(1)-1]}\,.$$

Proof. We may assume $w(1)=1$, as the case $w(n)=n$ is analogous. We first show $w(n)\neq n$. Notice that $w(n)=n$ and $w(1)=1$ implies $w(i)=i$ for all i. This is because otherwise, there exists $i<j$ with $w(i)>w(j)$, then $(1,i,j,n)$ forms a 1324 pattern. Assuming $w\neq \mathrm{Id}$ and $w(1)=1$, we must have that $w(n)\neq n$.

Our goal is to show

$$\mathrm{Imm}_w^{\%} = \sum_{|I|=n-w(n), I\subset[1,w^{-1}(n)]} \mathrm{CM}_{I,[w(n)+1,n]}\,.$$

For instance, we can see Imm_{1423} is the following sum of products of complementary minors, which (with the shaded regions representing the complementary minors).



To prove the equality, consider the coefficient of the monomial x_σ. Observe that if $\sigma^{-1}([w(n)+1,n])\neq I$, then the term $\mathrm{CM}_{I,[w(n)+1,n]}$ contributes coefficient of zero, since I has the same cardinality as $\sigma^{-1}([w(n)+1,n])$. In particular, notice that the coefficient is equal to 0 if no such I exists. Otherwise, σ permutes the elements of I and $\overline{I}$ separately, meaning that the coefficient of x_σ is equal to $\mathrm{sgn}(\sigma)$ in $\mathrm{CM}_{I,[w(n)+1,n]}$. But notice that I is unique if it exists in the above description, meaning that in

$$\mathrm{Imm}_w^{\%} = \sum_{|I|=n-w(n), I\subset[1,w^{-1}(n)]} \mathrm{CM}_{I,[w(n)+1,n]}$$

the coefficient of x_σ is $\operatorname{sgn}(\sigma)$ if and only if $\sigma^{-1}([w(n)+1,n]) \subset [1, w^{-1}(n)]$.

But notice that the coefficient of x_σ in the %-immanant is $\operatorname{sgn}\sigma$ so long as $\sigma([w(n)+1,n])$ lies in $[1, w^{-1}(n)]$, and zero otherwise. This is precisely what we wanted to prove. $\square$

Proposition 3.6. *Suppose that $w \in \mathfrak{S}_n$ is a 321-, 1324-, 2143- avoiding permutation so that $w(1) \neq 1$ and $w(n) \neq n$. Then*

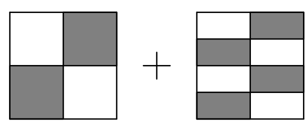
$$\operatorname{Imm}_w^{\%} = \sum_{|I|=w(n),[w^{-1}(n)+1,n]\subset I\subset[w^{-1}(1),n]} \mathrm{CM}_{I,[1,w(n)]}\,.$$

Proof. First, observe that if $w^{-1}(1) < w^{-1}(n)$, then we need, to avoid the 2143 pattern, for $w(1) > w(n)$. But at the same time, to avoid a 321-pattern, we need for $w(1) = w(n)+1$; otherwise, notice that $n, w^{-1}(w(n)+1), 1$ forms an increasing sequence that, when we apply w, becomes descending.

Again, recall that our goal is to show that

$$\operatorname{Imm}_w^{\%} = \sum_{|I|=w(n),[w^{-1}(n)+1,n]\subset I\subset[w^{-1}(1),n]} \mathrm{CM}_{I,[1,w(n)]}\,.$$

Again, as an example we have the following sum when $w = 3142$, with the shaded regions representing which complementary minors we take.



To see this fact, we will compare coefficients of x_σ. Observe that, given a permutation σ, the monomial x_σ has coefficient 0 in the %-immanant unless $\sigma^{-1}([1,w(n)]) \subset [w^{-1}(1), n]$ and $\sigma^{-1}([w(1),n]) \subset [1, w^{-1}(n)]$, which it then has coefficient $\operatorname{sgn}(\sigma)$ (as these are the only monomials which don't take a term from the rectangle of zeros).

As for the complementary minors, notice that this product will only include a x_σ term iff $I = \sigma^{-1}([1,w(n)])$. In particular, notice that the coefficient of our monomial is 0 unless $\sigma^{-1}([1,w(n)]) \subset [w^{-1}(1), n]$. Furthermore, by definition of $\mathrm{CM}_{I,[1,w(n)]}$, the coefficient of x_σ in the above complementary minor, where $I = \sigma^{-1}([1,w(n)])$, is $\operatorname{sgn}(\sigma)$.

Thus, in either case, we have the coefficient of x_σ is $\operatorname{sgn}(\sigma)$ if $\sigma^{-1}([1,w(n)]) \subset [w^{-1}(1), n]$ and 0 otherwise, showing that the two are equal. $\square$

From here, we now parse our complementary minors in terms of Temperley-Lieb immanants. In each case, we will first show that the term Imm_w appears within the sum, before finally showing that the complementary minor sum does equal $\operatorname{sgn}(w)\operatorname{Imm}_w$.

Lemma 3.7. *Suppose that w satisfies the hypotheses of Proposition 3.5. Write the sum of complementary minors given in Proposition 3.5 as a combination of Temperley-Lieb immanants. Then, the coefficient of Imm_w is ± 1.*

Proof. Just like in Proposition 3.7, we can assume that $w(1) = 1$ in this case. We first consider the matching diagram for w. Notice that, by the fact that this is 1324-avoiding, w must be increasing on the range from 1 to $w^{-1}(n)$, and $w^{-1}(i)$ is increasing on 1 to $w(n)$.

In particular, notice that for j between $w(n)+1$ and $n-1$, we notice that $w^{-1}(j)$ has to be less than $w^{-1}(n)$, as else we form a 321 pattern with $w^{-1}(n), w^{-1}(j)$, and n. But then notice that by the increasing pattern, for j in that range, $w^{-1}(j) = w^{-1}(n) - n + j$.

In conclusion, w sends the set $\{1, 2, \ldots, w^{-1}(n)\}$ to $\{1, 2, \ldots, w^{-1}(n)-n+w(n), w(n)+1, \ldots, n\}$ and must be increasing, which gives us that $w(i) = i$ for $1 \leq i \leq w^{-1}(n) - n + w(n)$, and is $i + n - w^{-1}(n)$ for $i > w^{-1}(n) - n + w(n)$. These conditions uniquely determine w. As an aside, observe that they also require that $w^{-1}(n) + w(n) > n$.

Using this, by Lemma 3.4 if we write out the entries 1 to n on the left, and $1'$ to n' on the right, then i is matched with i' if $i \leq w^{-1}(n) - n + w(n)$. Otherwise, notice that $w^{-1}(n)$ is matched with $w^{-1}(n)+1$, $w^{-1}(n)-1$ is matched with $w^{-1}(n)+2$, and so forth, until either n is matched with $2w^{-1}(n)+1-n$, or $w^{-1}(n)-n+w(n)+1$ is matched with $n+w^{-1}(n)-w(n)$, depending on whether $w^{-1}(n)$ or $w(n)$ is larger.

Similarly, $w(n)'$ is matched with $w(n)'+1$, $w(n)'-1$ is matched with $w(n)'+2$, and so forth, until either n' is matched with $2w(n)'+1-n'$, or $(w^{-1}(n)-n+w(n)+1)'$ is matched with $(n-w^{-1}(n)+w(n))'$, again based on whether $w(n)$ or $w^{-1}(n)$ is larger. The remaining elements are then paired so that the kth smallest remaining (unmatched) element in the first column is paired with the kth smallest remaining element in the second column.

We work with the case where $w^{-1}(n) \leq w(n)$; the other direction for the inequality is analogous. Then, $w(n) \geq \frac{n}{2}$ with our additional assumption. We therefore see that $w^{-1}(n) - n + w(n) + 1$ is matched with $n + w^{-1}(n) - w(n)$, and n' is matched with $(2w(n)+1-n)'$, meaning that our unpaired elements are $n+w^{-1}(n)-w(n)+1, \ldots, n$, pairing $n-i$ to $(2w(n)-n-i)'$.

But now notice that, by Proposition 4.3 from [8], we can obtain our immanants by considering the coloring of products of complementary minors, with each product of complementary minors being the sum of Imm_w, where the matching for w is consistent with the coloring given by $I, \overline{J}$ being black, and $\overline{I}, J$ being white. But then we know that in the second (primed) column, $1'$ to $w(n)'$ are black, and $(w(n)+1)'$ to n' are white.

Similarly, since I lies in $[1, w^{-1}(n)]$, we have that $w^{-1}(n)+1$ to n are white. But then notice that this requires that 1 to $w^{-1}(n)-n-w(n)$ are white, $w^{-1}(n)-n+w(n)$ to $w^{-1}(n)$ is black. But then the colorings here are consistent (with the remaining matchings from $n+w^{-1}(n)-w(n)$, to n, all of which are white, are paired with elements from $(2w(n)-n)'$ to $(w^{-1}(n)-n+w(n))'$, all of which are black), meaning that there is exactly one term for Imm_w present here, which is what we wanted here. ☐

Lemma 3.8. *Suppose that w satisfies the hypotheses of Proposition 3.6. Write the sum of complementary minors given in Proposition 3.6 as a combination of Temperley-Lieb immanants. Then, the coefficient of* Imm_w *is* ± 1.

Proof. We will first show that w is uniquely determined given the pattern avoidance and the values $w(1), w^{-1}(1), w(n), w^{-1}(n)$. Notice that since w is 1324-avoiding and 321-avoiding, w^{-1} is increasing from 1 to $w(n)$, and increasing from $w(1)$ to n. Notice in particular that $w(w^{-1}(n)+1), w(w^{-1}(n)+2), \ldots, n$ are a string of increasing consecutive integers, as are $w(1), w(2), \ldots, w(w^{-1}(1)-1)$.

As for the remaining values, notice that w^{-1} needs to be increasing from $w(1)$ to $w(n)$, leaving one unique way to place them (as we have exactly that many open columns left). In particular, we observe that $w(w^{-1}(1)+i) = 1+i$ for $i \leq w(w^{-1}(n)+1)-2$, and $w(w^{-1}(n)-i) = n-i$ for $i \leq n - w(w^{-1}(1)-1) - 1$.

Combining these, we have the following:

$$w(i) = \begin{cases} w(1) + i - 1 \text{ if } i \leq w^{-1}(1) - 1 \\ i + 1 - w^{-1}(1) \text{ if } w^{-1}(1) \leq i \leq w^{-1}(1) + w^{-1}(n) + w(n) - n - 1 \\ i - w^{-1}(n) + n \text{ if } w^{-1}(1) + w^{-1}(n) + w(n) - n \leq i \leq w^{-1}(n) \\ i - n + w(n) \text{ otherwise} \end{cases}.$$

Notice that these cases cover all of the integers from 1 to n since $w(1) = w(n) + 1$. We will refer to the first case as a first block, and so forth, of inputs, and similarly for the outputs; for instance, $w([1, w^{-1}(1) - 1])$ is the first block of outputs, although they are not the primed indices with the smallest values. We will also refer to these blocks as being for different permutations; when we do not specify the permutation, this will default to mean w.

Similarly to the one rectangle case, we consider the coefficient of each immanant in this sum. We again label two columns for our matchings by $1, 2, \ldots, n$ and $1', 2', \ldots, n'$. Notice that we can write our permutation w as being the composition of three permutations. The first is w_1, which does the following:

1. It sends i to $i + 1 - w^{-1}(1)$ if $w^{-1}(1) \leq i \leq w^{-1}(1) + w^{-1}(n) + w(n) - n - 1$,
2. sends $i + w^{-1}(n) + w(n) - n$ for $i \leq w^{-1}(1) - 1$,
3. fixes the rest.

The next is w_2, which sends i to $i - w^{-1}(n) + n$ if $w^{-1}(1) + w^{-1}(n) + w(n) - n \leq i \leq w^{-1}(n)$, to $i + w^{-1}(1) + w(n) - n - 1$ for $i \geq w^{-1}(n) + 1$, and fixes the rest.

Finally, we have w_3, which sends i to $i - w^{-1}(1)$ if $w^{-1}(n) + w^{-1}(1) + w(n) - n \leq i \leq w^{-1}(1) + w(n) - 1$, to $i + w$ if $1 + w^{-1}(n) + w(n) - n \leq i \leq w^{-1}(1) + w^{-1}(n) + w(n) - n - 1$. Notice that we can write $w = w_3 w_2 w_1$. It's not too hard to see that the number of inversions in w is equal to the number in w_1 plus that in w_2 plus that in w_3; if there are a elements in the first block, b in the second, c in the third and d in the fourth, this implies that w_1 has ab inversions, w_2 has cd, w_3 has bd, and w has $a(b + d) + cd$, which matches the sum.

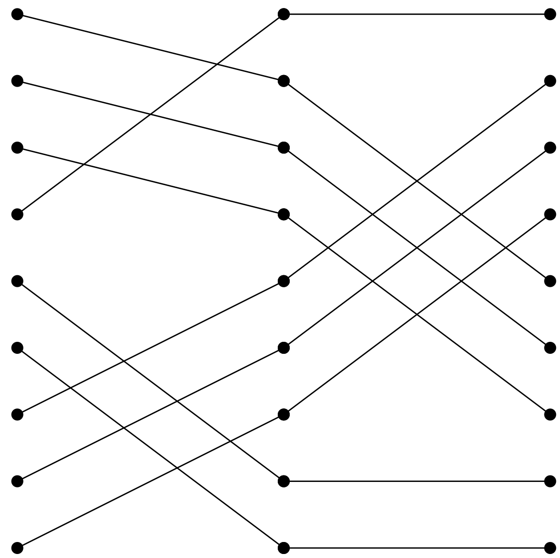
By Lemma 3.4, $w^{-1}(1)$ is paired with $w^{-1}(1) - 1$, and so forth, until either we pair 1 with $2w^{-1}(1) - 2$ or $w^{-1}(1) + w^{-1}(n) + w(n) - n - 1$ with $w^{-1}(1) - w^{-1}(n) - w(n) + n$. A similar argument holds with the third and fourth blocks: either we run out by pairing n with $2w(n) + 1 - n$, or $w^{-1}(1) + w^{-1}(n) + w(n) - n$ with $n + w(n) + 1 - w^{-1}(n) - w^{-1}(1)$. This also happens with the first and fourth blocks of outputs (the second-highest and third-highest along the column), using Lemma 3.4 on $\beta(w_3)$.

For the remaining ones, we use Lemma 3.4. When we consider applying w_3, and consider the effect of multiplying by $\beta(w_3)$, if we are left to pair those in the first and fourth blocks in w, they again will begin to pair up with each other (per the resulting pairing that arises from w_3 due to Lemma 3.4), until again we have paired all of those in one of the blocks.

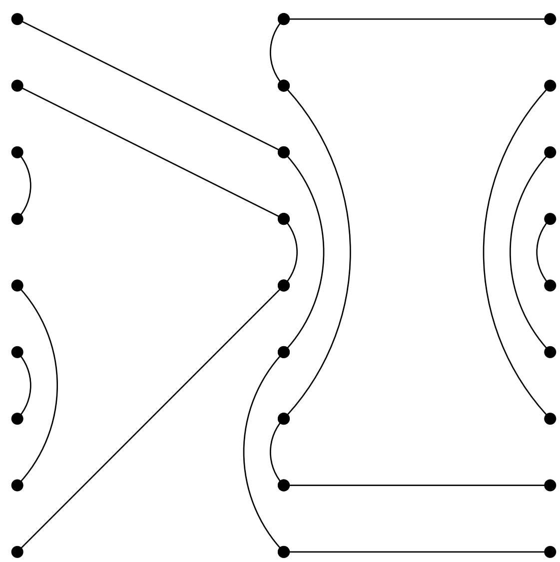
To see that happens, recall that for $\beta(w_2 w_1)$, the other elements in the first block of inputs are paired with those in the first block in the output (but shifted down), and analogously for the fourth block. Then, the effect of $\beta(w_3)$, given by Lemma 3.4, combined with the effect of the pairing $\beta(w_2 w_1)$, translates to the stated effect.

From here, all the remaining unpaired elements that are unprimed will be paired with those that are primed. If they arises from the first or third block, the ith largest element left to be paired among the unprimed elements will be paired with the ith largest element in the third block of outputs (which are the largest-indexed primed labels), and similarly for the second and fourth blocks. Finally, the remaining primed and unpaired elements are just paired amongst themselves, going outwards in.

As an example, we consider what happens when blocks 1 and 4 have size 3, and blocks 2 has size 1, block 3 has size 2. The permutation is the next diagram,



with the corresponding matching diagram below.



From here, notice that the colorings that we have are so that $1', 2', \ldots, w(1)'$ are white, and the rest of the primed ones are black, and furthermore $1, 2, \ldots, w^{-1}(1) - 1$ are white and $w^{-1}(n) +$

$1, \ldots, n$ are black. But we have then uniquely determined the colors of the remaining blocks to yield a compatible coloring (as we only have to specify the second and third blocks, which must be connected either with a first block element, a fourth block element, or something on the primed elements). Thus, we conclude that Imm_w only appears in one term, meaning that its coefficient is $\mathrm{sgn}(w)$. $\square$

With these lemmas, we now argue that our sum of complementary minors contains no other Imm_w terms. Proving this statement is the subject of the next two propositions, which in turn will be able to prove Theorem 3.2.

Proposition 3.9. *Given the conditions of Proposition 3.5 hold, the complementary minors given in that Proposition equals* $\mathrm{sgn}(w)\,\mathrm{Imm}_w$ *.*

Proof. Given Lemma 3.7, it suffices to show that every other Imm_v has coefficient zero when the complementary minors are expanded in terms of the Temperley-Lieb immanants, as then we can compute the global sign factor that we need.

Suppose that $v \neq w$ is a 321- avoiding permutation. We will consider the coefficient of Imm_v in our sum of complementary minors. Notice that given a coloring that corresponds with v, which by our minors has $1'$ to $w(n)'$ are black, and $(w(n)+1)'$ to n' are white, and $w^{-1}(n)+1$ to n are white. But then notice that by construction, if Imm_v appears in the sum corresponding to some product of complementary minors, then it has to be consistent with some coloring. From here, that this matching cannot be the one that corresponds i to i', meaning that at least two elements that are both not primed must be matched together.

Now, observe that $(w(n)+i+1)'$ is paired with $(w(n)-i)'$. To show this fact, if $(w(n)+i+1)'$ is paired with another primed vertex, it has to be $(w(n)-i)'$, since considering the possible values, it has to be j', with $1 \leq j \leq w(n)$; but then we require i white vertices and i black vertices within the matchings to allow for a compatible matching. As for a non-primed vertex, notice that if this number is paired with j, there are at least $w(n)$ black elements strictly above the line pairing $(w(n)+i+1)'$ and j, and at most $w^{-1}(n)-1$ white ones (since we know that there are n white vertices, but $n-w^{-1}(n)$ of these are below j, and one of these is $(w(n)+i+1)'$ itself since j being black means that $j < w^{-1}(n)$). But $w^{-1}(n) \leq w(n)$, contradiction. Thus, it follows that $(w(n)-i)'$ and $(w(n)+i+1)'$ are paired.

We can also apply this argument from $(2w(n)-n)'$ to $(w^{-1}(n)+w(n)-n)'$ to see that these are paired to $n, n-1, \ldots, n+w^{-1}(n)-w(n)$, by seeing that $(2w(n)-n-i)'$ has to be paired with some element j that is unprimed and white. But then notice that if $j \leq w^{-1}(n)$, then there are at least $2w(n)-n-i$ black vertices at or above the line (the number arising from how many are in the primed column), meaning we have at most $2n-2w(n)+i$ below the line. Similarly, we have at least $n-w^{-1}(n)+n-w(n)$ white ones below; but we also know that $i < w(n)-w^{-1}(n)$, meaning that

$$2n-2w(n)+i < 2n-w(n)+w^{-1}(n) \leq 2n-w^{-1}(n)-w(n),$$

and $j > w^{-1}(n)$. But then notice that the number of white vertices below this line is $n-w(n)+n-j$, and the number of black vertices below this line is $n-w(n)+i$. For these to equal, we need $j = n-i$.

After performing the above pairings, the elements we have left to pair are from 1 to $n-w(n)+w^{-1}(n)$, and from $1'$ to $(w^{-1}(n)+w(n)-n)'$. However, there are $2w^{-1}(n)$ of these elements, so either two elements in $\{1, 2, \ldots, w^{-1}(n)\}$ are paired, or each element of these is paired with an element outside. But the latter case is precisely w, since we require the matchings to not cross.

Therefore, there is some pair of elements (i, j), where $1 \leq i < j \leq w^{-1}(n)$, that are connected in the matching for v. In particular, between these two, there must exist some pair of adjacent elements that are connected; let these be i_v and $i_v + 1$. Consider now the following operation on colorings: given a coloring compatible with v, swap the colors of i_v and $i_v + 1$. Notice that this operation is an involution from this subset of compatible colorings to itself.

But the image of the involution goes from $\mathrm{CM}_{I,J}$ to $\mathrm{CM}_{I',J}$, where only one of I, I' contains i_v and the other contains $i_v + 1$. However, this requires that $s(I), s(I')$ have different parities. In particular, this means that if $\mathrm{CM}_{I,J}$ is expanded in terms of Temperley-Lieb immanants with a positive coefficient for Imm_v, then for $\mathrm{CM}_{I',J}$ its expansion has $-\mathrm{Imm}_v$. Indeed, all the $\Delta_{I,J}\Delta_{\bar{I},\bar{J}}$ are sums of Temperley-Lieb immanants, so our argument shows that if $\Delta_{I,J}\Delta_{\bar{I},\bar{J}}$ is a positive sum, $\Delta_{I',J}\Delta_{\bar{I'},\bar{J}}$ is a negative sum (so the coefficients of Imm_v within the expansions for these two are opposite).

But then this involution allows us to argue that for each appearance of an Imm_v, there exists another Imm_v with opposite sign, and this is 1-to-1 correspondence. But then the coefficient of Imm_v must then zero.

Thus, combined with Lemma 3.7, we've shown that our %-immanant is equal to $\mathrm{sgn}(w) imm_w$, by comparing the coefficients of x_w (which is $\mathrm{sgn}(w)$ in $\mathrm{Imm}_w^{\%}$ and 1 in Imm_w), which allows us to finish the proposition. ☐

Proposition 3.10. *Given the conditions of Proposition 3.6 hold, the sum of complementary minors in the lemma equals* $\mathrm{sgn}(w)\,\mathrm{Imm}_w$.

Proof. Once we have Lemma 3.8, it now suffices to show that every other Imm_v has coefficient zero when the complementary minors are expanded in terms of the Temperley-Lieb immanants.

For $v \neq w$, there has to be some matching between two adjacent elements among the elements in blocks two and three among the unprimed elements. To see this fact, suppose that Imm_v appears as some term in any of these products of complementary minors. Thus, there exists some coloring so that v is a compatible non-crossing matching.

Consider the number $(w(n) - i)'$, where $0 \leq i \leq w(n) - 1$, which is colored white. Notice that if $(w(n) - i)'$, is paired with some non-primed element j, then this number is colored black. In addition, notice that the number of white vertices above the matching (namely, above j in unprimed, and $(w(n) - i)'$ in the primed. This is because for non-crossing matchings, these elements all have to be paired with each other) is at least $w^{-1}(1) - 1 + w(n) - i - 1$, since all the white vertices from the first block of inputs (unprimed) and the white vertices from the second block of outputs (primed) must lie above this matching between a white primed and black unprimed-labelled vertex. If j is in the fourth block of inputs (unprimed), notice that all the vertices below j that are unprimed are black, meaning that the number of white vertices can be bounded below further by $n - w(n) + w(n) - i - 1 = n - i - 1$.

But at the same time, the number of black vertices that lie above it is bounded above by $w(n) - 1$ if j lies in the fourth block, and bounded from above by $w(n) - 1 - n + w^{-1}(n)$ if j lies in the second or third block. Indeed, on the primed column, all the black vertices are below $(w(n) - i)'$, and in the unprimed column, there are $w(n)$ black vertices, of which we know that $n - w^{-1}(n)$ are in the fourth block.

However, for the case where j is in the fourth block, we require that $n - i - 1 \leq n - w(n) - 1$, or $i \geq w(n)$, impossible. For the second block, we require $w^{-1}(1) - 1 + w(n) - i - 1 \leq w(n) - 1 - n + w^{-1}(n)$ or that $i \geq n + w^{-1}(1) - w^{-1}(n) - 1$. Hence, for i less than $n + w^{-1}(1) - w^{-1}(n) - 1$, we need to pair up $(w(n)-i)'$ with some primed element; inducting on i allows us to see that $(w(n))'$ pairs with $(w(n)+1)'$, and so forth.

We have paired up twice as many elements, meaning that we have $2n-2(n+w^{-1}(1)-w^{-1}(n)-1) = 2w^{-1}(n)-2w^{-1}(1)+2$ elements left to pair up. But the number of elements in the second and third blocks on the unprimed column is $w^{-1}(n)-w^{-1}(1)+1$, meaning that the only way for none of these elements to be paired with themselves is to be paired with elements not in these blocks. But there is only one way to do this pairing; the largest element in the third block must be paired with the smallest element in the fourth block, and so forth, and similarly with the second and first blocks. The resulting pairing, in fact, is just w, as we saw before.

Thus, it follows that v, like in the case with one rectangle, must have at least one pair of adjacent elements among the second and third block of inputs (over which we can choose any colors) that are matched. Doing the involution of swapping these two elements yields us, just like in the first case, that for each Imm_v there is a $-\mathrm{Imm}_v$ corresponding to the product of complementary minors by swapping adjacent elements, since in the two $\mathrm{CM}_{I,J}$s that we involute between one is a positive sum of Temperley-Lieb immanants and the other is a negative sum.

The case where $w^{-1}(1) > w^{-1}(n)$ yields us with $w^{-1}(1) = w^{-1}(n)+1$, and $w(1) < w(n)$; the argument in that case is analogous.

Finally, notice that the coefficient of w in Imm_w, $f_w(w)$, is necessarily equal to 1, but in the %-immanant the coefficient of w is equal to $\mathrm{sgn}(w)$. Thus, Imm_w is $\mathrm{sgn}(w)\,\mathrm{Imm}^{\%}_w$. This proves the proposition. ☐

Combining Propositions 3.10 and 3.9 is precisely what we need to prove Theorem 3.2.

3.2 $\mathrm{Imm}_w = \mathrm{sgn}(w)\,\mathrm{Imm}^{\%}_w$ Implies 2143-, 1324-Avoiding

This subsection will prove the only if direction of Theorem 3.1.

Theorem 3.11. *If w is a permutation that avoids 321 such that Imm_w is a %-immanant up to sign, then w must avoid both 1324 and 2143.*

Our strategy is to first show w avoids 1324. Then, we will show that if w avoids 1324, then w must also avoid 2143.

Theorem 3.12. *If w is a 321-avoiding permutation where Imm_w is a linear combination of %-immanants, then w must be 1324-avoiding.*

Proof. Assume for contradiction that w contains the pattern 1324. Then we have indices $i < j < k < l$ such that $w(i) < w(k) < w(j) < w(l)$. Since w avoids 321, $w(t) > w(j)$ or $w(t) < w(k)$ for each $j < t < k$. Define the permutation $w' := w \cdot (j,k)$. By Lemma 2.2, $w' \leq w$. By Lemma 2.5, we have $f_w(w') = 0$ and $f_w(w) = 1$.

By our assumption, we may write $\mathrm{Imm}_w = \sum_{p=1}^{d} c_p \,\mathrm{Imm}^{\%}_{\lambda_p/\mu_p}$ is a linear combination of %-immanants. For each generic $\mathrm{Imm}^{\%}_{\lambda_p/\mu_p}$ in the sequence of %-immanants, if x_w appears with nonzero coefficient in the immanant, we have that each $(i, w(i)) \in \lambda_p/\mu_p$ for $t \in [n]$. In particular, $(i, w(i))$ and $(l, w(l))$ lie in the skew shape. Since all zeros in λ_p/μ_p (the pieces not in the skew shape) concentrate in blocks in the top left and bottom right corner, every $(t, w(t))$ such that $i < t < l$ and $w(i) < w(t) < w(l)$ is in the skew shape! Recall that

$$w'(t) = \begin{cases} w(t) & t \notin \{j, k\} \\ w(k) & t = k \\ w(j) & t = j \end{cases}$$

So for each $t \notin \{j, k\}$, $(t, w'(t))$ and $(t, w(t))$ lie in the skew shape. For $t = k$, $(k, w'(k)), (k, w(j)) \in \lambda_p/\mu_p$ because $i < k < l$ and $w(i) < w(j) < w(l)$. Similarly for $t = j$, $(j, w(j)) \in \lambda_p/\mu_p$. And since w and w' has a length difference of 1, they have reverse signs. So thus $x_{w'}$ also appears in the %-immanant, with the opposite sign.

Otherwise, x_w isn't a term in the %-immanant. If $(t, w(t)) \notin \lambda_p/\mu_p$ for some $t \notin \{j, k\}$, then $(t, w'(t)) \notin \lambda_p/\mu_p$ because $(t, w'(t)) = (t, w(t)) \notin \lambda_p/\mu_p$. Otherwise, assume WLOG that $(j, w(j)) \notin \lambda_p/\mu_p$. Note that $i < j < l$, $w(i) < w(j) < w(l)$, and that λ_p/μ_p is a skew shape. So $(t, w(t)) \notin \lambda_p/\mu_p$ implies that $(i, w(i)) \notin \lambda_p/\mu_p$ or $(l, w(l)) \notin \lambda_p/\mu_p$. So we still have that $x_{w'}$ isn't a nonzero term in the %-immanant. In either case, the signs of the monomials $\operatorname{sgn}(w)x_w$ and $\operatorname{sgn}(w')x_{w'}$ have coefficients negative of each other. Denote $a_p(w)$ to be 1 if the monomial associated to w appears as a term in the %-immanant associated to λ_p/μ_p, and zero otherwise.

Combining the two cases, we have that

$$\begin{aligned} 0 = f_w(w') &= \sum_{p=1}^{d} \operatorname{sgn}(w')c_p a_p(w') \\ &= -\left(\sum_{p=1}^{c} \operatorname{sgn}(w)c_p a_p(w)\right) \\ &= -f_w(w) = -1. \end{aligned}$$

But this is a contradiction, as desired.

□

Now, to finish proving Theorem 3.11, we will show that w avoids 2143. We begin by analyzing what happens if w doesn't avoid 2143.

Lemma 3.13. *Suppose that w contains 2143.*

(a) If w avoids 321, then $w(1) < w(n)$ and $w^{-1}(1) < w^{-1}(n)$.

(b) If w avoids 321, then $w^{-1}(1) + w(1) \leq n + 1$ and $(n + 1 - w^{-1}(n)) + (n + 1 - w(n)) \leq n + 1$.

(c) If w avoids 1324, then $w(1) \neq 1$ and $w(n) \neq n$.

Proof. (a) We will first $w(1) < w(n)$. Suppose for the sake of contradiction that $w(1) > w(n)$. Since there exists a 2143, we can find $a < b < c < d$ such that $w(b) < w(a) < w(d) < w(c)$.

First suppose $a = 1$. Then we must have $d \neq n$, and also $c < d < n$ satisfy $w(n) < w(1) < w(d) < w(c)$, contradiction to w being 321-avoiding. Thus, assume $a \neq 1$.

Next, suppose $d = n$. Then we must have $a \neq 1$, and also $1 < a < b$ satisfy $w(b) < w(a) < w(n) < w(1)$, contradiction to w being 321-avoiding.

Now, suppose $w(a) < w(1)$. Then $1 < a < b$ satisfies $w(b) < w(a) < w(1)$, contradiction to w being 321-avoiding.

Finally, suppose $w(a) > w(1)$. Then $c < d < n$ satisfy $w(n) < w(1) < w(d) < w(c)$, contradiction to w being 321-avoiding. This shows in fact $w(1) < w(n)$, as desired. Then $w^{-1}(1) < w^{-1}(n)$ follows by applying Lemma part (a) to w^{-1}, which also avoids 321 and contains 2143.

(b) Note that $w(i) > w(1)$ for $2 \leq i \leq w^{-1}(1) - 1$ and $i = n$ by 321-avoidance and part 1. Therefore, it follows that $w(1)$ is smaller than $w^{-1}(1)-1$ elements, and so thus $w(1)+w^{-1}(1) \leq n+1$. Then $(n+1-w^{-1}(n)) + (n+1-w(n)) \leq n+1$ follows from applying Lemma part (b) to w_0ww_0, which also avoids 321 and contains 2143.

(c) We will first show $w(1) \neq 1$. If $w(1) = 1$, then there exist $1 < a < b < c < d$ such that $w(b) < w(a) < w(d) < w(c)$. Then $1 < a < b < c$ satisfy $w(1) < w(b) < w(a) < w(c)$, contradiction to w being 1324 avoiding. Thus, $w(1) \neq 1$, and $w(n) \neq n$ follows by applying Lemma to w_0ww_0, which also avoids 1324 and contains 2143. □

We are given that Imm_w is a %-immanant up to sign. We would like to know some properties of the %-immanant, beyond the one given in Lemma 2.3

Lemma 3.14. *Suppose w is nice and contains 2143, and Imm_w is a %-immanant $\mathrm{Imm}^{\%}_{\lambda/\mu}$ up to sign. Then:*

(a) $(1,1),(n,n) \notin \lambda/\mu$*;*

(b) There exists i such that $(1,i),(n+1-i,1),(n,i),(n+1-i,n) \in \lambda/\mu$.

Proof. (a) By Lemma 3.13(c), we have $w(1) \neq 1$. If $(1,1) \in \lambda/\mu$, then consider $w' = w \cdot (1, w^{-1}(1))$. Then w' has fewer inversions than w by Corollary 2.2, hence $w' \not\geq w$. Thus, by Lemma 2.5, we have $f_w(w') = 0$ and $f_w(w) = 1$. But this is impossible, since the coefficients of x_w and $x_{w'}$ in $\mathrm{Imm}^{\%}_{\lambda/\mu}$ are negatives of each other. Thus, $(1,1) \notin \lambda/\mu$, and an analogous argument shows $(n,n) \notin \lambda/\mu$.

(b) Choose $i = \max(w(1), n+1-w^{-1}(n))$. It's not hard to see that $(1,i)$ and $(n+1-i,n) \in \lambda/\mu$. This is because we require that $(j, w(j)) \in \lambda/\mu$ for all λ/μ, and Imm_w is not identically zero (so $(1,n),(n,1)$ must lie in the skew shape).

Now, for (n,i), observe that either $i = w(1) < w(n)$, or $i = n+1-w^{-1}(n) \leq w(n)$ by Lemma 3.13(b). In either case we see that $(n,i) \in \lambda/\mu$. Finally, we can observe that in $(n+1-i,1)$, either $n+1-i = w^{-1}(n) > w^{-1}(1)$, or $n+1-i = n+1-w(1) \geq w^{-1}(1)$, so again in either case $(n+1-i,1) \in \lambda/\mu$. This proves the claim. □

We will now finish the proof of Theorem 3.11.

Proof of Theorem 3.11. Suppose that Imm_w is a %-immanant $\mathrm{Imm}^{\%}_{\lambda/\mu}$, up to a sign. By Theorem 3.12, w is 1324-avoiding. Assume for the sake of contradiction that w is not 2143-avoiding. Then, from Lemma 3.14, we can find i' such that $(1,i'),(n+1-i',1),(n,i'),(n+1-i',n) \in \lambda/\mu$ and $(1,1),(n,n) \notin \lambda/\mu$.

Now, we will construct an $n \times n$ matrix X and show that Imm_w and $\mathrm{Imm}^{\%}_{\lambda/\mu}$ give different results when evaluated on X. Let $X = (x_{ij})$ where $x_{i,n+1-i} = 1$ for all i, and also set $x_{1,n+1-i'} = x_{1,1} = x_{i',1} = x_{i',n} = x_{n,n} = x_{n,n+1-i'} = 1$. Set the rest of the entries to be zero.

We now evaluate $\mathrm{Imm}^{\%}_{\lambda/\mu}(X)$. Notice that, by our assumption, this is equal to the determinant of the matrix X' by setting $x_{1,1}$ and $x_{n,n}$ to zero in X by Lemmas 2.3 and 3.14. In other words, we are evaluating the determinant of this when we apply the %-immanant:

$$\begin{pmatrix} 0 & 0 & \dots & 1 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 1 & \dots & 0 & 0 \end{pmatrix}.$$

Notice that we can re-arrange these rows (which only affects the sign) so that the i'th row and the nth row become the 2nd and 3rd row, respectively. Then, re-arrange the columns so that, again, they become the 2nd and 3rd columns. The resulting determinant we are evaluating is then

$$\begin{pmatrix} 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Notice that this is a block-diagonal matrix, with a matrix with all 1s along the anti-diagonal in one block and the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ as the other. This means that the determinant of this matrix in total is equal to simply the product of the determinants of these two blocks, which is just 1. There are some sign considerations, but this is not important. We conclude $\mathrm{Imm}^{\%}_{\lambda/\mu}(X) = \pm 1$.

On the other hand, since X has three equal rows (rows $1, i$ and n), Proposition 3.14 from [8] tells us that $\mathrm{Imm}_w X = 0$. In particular, for our w, we know that $\mathrm{Imm}_w X = 0$. But this contradicts $\mathrm{Imm}^{\%}_{\lambda/\mu}(X) = \pm 1$. This proves the theorem. $\square$

4 Temperley-Lieb Immanants as Linear Combinations of %-Immanants

This section is devoted to proving the following theorem.

Theorem 4.1. *Let w be a 321-avoiding permutation. The following statements are equivalent:*

1. *The Temperley-Lieb immanant Imm_w is a linear combination of %-immanants;*

2. *The signed Temperley-Lieb immanant $\mathrm{sgn}(w)\,\mathrm{Imm}_w$ is a sum of at most two %-immanants;*

3. *The permutation w avoids the patterns* $24153, 31524, 231564$*, and* 312645*, in addition to avoiding* 321.

As a step towards proving the theorem, we first to classify permutations w that avoid 321 and 1324 but contain 2143, which allows us to compute the immanant coefficients $f_w(u)$, for $u \in \mathfrak{S}_n$ and w avoiding 321 and 1324.

4.1 Classifying 321-, 1324-avoiding, 2143-containing permutations

We will show such permutations satisfy one of two prescribed block structures.

Proposition 4.2. *Let $w \in \mathfrak{S}_n$ be a permutation that avoids* 1324 *and* 321*, but not* 2143*.*

Let $a = w^{-1}(1) - 1, b = w(1) - 1, c = n - w(n), d = n - w^{-1}(n)$.

If $a + b + c + d \leq n$, let $e = n - a - b - c - d$. We have that the one line notation of w has at most 5 ascending strings of consecutive integers given as follows.

- *Block 1 consists of a integers $w(i) = b + i$ for $i \in [1, a]$.*
- *Block 2 consists of b integers $w(i) = i - a$ for $i \in [a + 1, b + a]$.*
- *Block 3 consists of e integers $w(i) = i$ for $i \in [b + a + 1, b + a + e]$.*
- *Block 4 consists of c integers $w(i) = i + d$ for $i \in [b + a + e + 1, b + a + e + c]$*
- *Block 5 consists of d integers $w(i) = i - c$ for $i \in [b + a + e + c + 1, n]$*

In this case, we see w has block structure $[2][1][3][5][4]$.

Otherwise, the permutation w consists of at most six ascending strings of consecutive integers, where the first one or two ascending strings have combined length a and the last one or two ascending strings have combined length d and the middle two strings do not contain any of the values between $b+1$ and $n-c$, inclusive. In this case, the permutation w has block structure $[3][5][1][6][2][4]$.

Proof. We divide into two cases.

Case 1. Suppose that $a+b+c+d \leq n$. Observe that $S = w([1, a : n-d+1, n])$ is a set of $a+d$ elements. We claim that it must be a subset of $[b + 1, n - c]$. This is because otherwise, we have some element y that fails this. Notice that this set contains $n-b-c$ elements. But $n-b-c \geq a+d$, meaning that $w^{-1}([b+1, n-c])$ must contain some element x not inside $[1, a] \cup [n-d+1, n]$. From the fact that w is 321-avoiding, $w([1, a]) \subset [b + 1, n]$ (and in fact w must be increasing on this interval), and similarly $w([n-d+1, n]) \subset [1, n-c]$ (and again must be increasing on the interval).

However, observe that either $y \in [1, a]$, so $1 < y < x < n - d$, but $b + 1 < w(x) < w(y) < n$, contradicting the fact it is 1324-avoiding, or $y \in [n-d+1, n]$, so $a+1 < x < y < n$ but $1 < w(y) < w(x) < n-c$, another contradiction. Thus, we require $w([1, a] \cup [n-d+1, n]) \subset [b+1, n-c]$. Notice that 1324-avoiding also implies that $w([1, a]) = [b + 1, a + b]$; if x is the smallest value in $[1, a - 1]$ so $w(x + 1) > w(x) + 1$, observe then that $1, x + 1, w^{-1}(w(x) + 1), n$ would form the 1324-pattern. Similarly, $w([n - d + 1, n]) = [n - d - c + 1, n - c]$.

Notice that we can employ a similar argument as the above with $w^{-1}([1, b], [n-c+1, n])$ to find that $w^{-1}([1, b]) = [a + 1, a + b]$, and $w^{-1}([n - c + 1, n]) = [n - c - d + 1, n - d]$. But then notice that this specifies all of the values in $[1, n]$ except for those in $[a + b + 1, n - c - d]$, which take on values in $[a+b+1, n-c-d]$, if there are any; 1324-avoiding implies that these must be increasing, which gives the first part of the claim.

Case 2. Notice that $a+b+c+d > n$, meaning that $w([1,a] \cup [n-d+1,n])$ cannot lie in $[b+1,n-c]$, so some y in $[1,a] \cup [n-d+1,n]$ is so that $w(y) \notin [b+1,n-c]$. If $y \in [1,a]$, notice that $w(y) \in [n-c+1,n]$. But then observe that for each value x in $[a+1,n-d]$, if it lies in $[b+1,n-c]$, then we'd form a 1324 pattern with $1, y, x, w^{-1}(n)$, and similarly if $y \in [n-d+1,n]$.

Finally, observe that there can only be at most one value of x in $[1,a]$ so that $w(x+1) > w(x)+1$, and it can't be less by 1324-avoiding again. Indeed, if there were two, with the first being x and y, respectively, then observe that either $w(x+1) \leq n-c$ or $w(y) > n-c$. In the first case, the values $1, x+1, w^{-1}(w(x)+1), n$ forms the pattern; in the latter the values $1, y+1, w^{-1}(w(y)+1), w^{-1}(n)$ form the 1324-pattern. A similar argument holds for the last two strings, which finishes the proposition. □

In fact, we can be a bit more specific about which case we belong to with certain patterns.

Lemma 4.3. *Suppose that w is a nice permutation that contains the pattern* 2143, *and define a, b, c, d as above. Then, suppose that $a+b+c+d > n$. Then, w either contains the pattern* 21453 *or* 31524.

Proof. Notice that, per the statement of proposition 4.2, that when $a+b+c+d > n$, that the six ascending consecutive strings of integers obey a 351624-pattern if all of these strings are non-empty.

If the 2nd or 5th string doesn't exist, then we can simply remove it from this order, but this general order will still be satisfied. The only issue we might have is when both the second and fifth strings don't exist; we'll show that this is impossible.

Otherwise, if the second and fifth strings don't exist, it implies that the first block is of length a and the fourth block is of length d. But then observe that the largest element in the first block is $a+b$ and the smallest element in the last block is $n-c-d+1$ But then we need $a+b < n-c-d+1$ or that $a+b+c+d \leq n$, contradiction.

From here, observe that the existence of at least one of the second and fifth blocks requires one of the two patterns to exist, as desired. □

Remark. These two orderings will be heavily and implicitly used throughout the rest of this section.

In order to compute the coefficients $f_w(u)$ for a $321, 1324$-avoiding, 2143-containing permutation w, we will first express $\mathrm{Imm}_w(u)$ as a sum of complementary minors. (This is motivated by the following consideration: we defined Imm_w such that $\mathrm{CM}_{I,J}$ is a sum of immanants, and we are now reversing this operation.)

4.2 Temperley-Lieb Immanants as a Sum of Complementary Minors

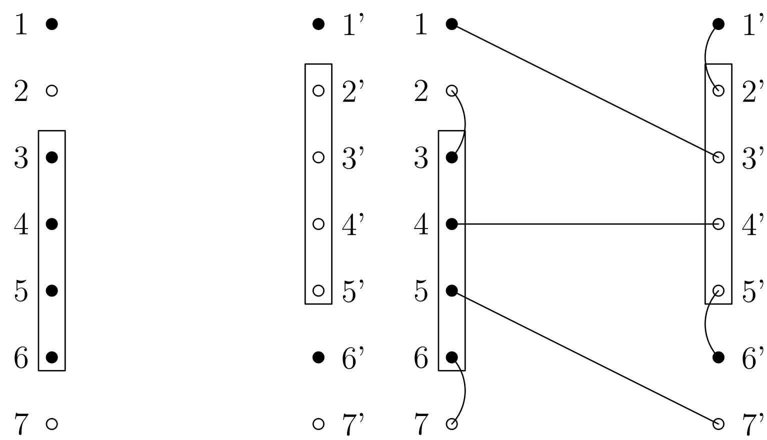
Our main goal in this subsection is to obtain a relatively simple formula for $f_w(v)$ for nice permutations w with the 2143-pattern. In particular, this formula will take the form $\mathrm{sgn}(w)\,\mathrm{sgn}(v)\binom{A+B}{A}$. Our main method for arriving at this formula will be to find a good way to express the immanants Imm_w in terms of products of complementary minors.

To give this explicit expression of complementary minors, there are two cases that we need to consider, along the lines of the cases provided in Proposition 4.2. For each of these cases, we will argue that a unique non-crossing matching exists for a given condition of colorings (Lemmas 4.4 and 4.5), and then show that the unique non-crossing matching corresponds to w (Lemmas 4.6 and 4.7). Finally, we show that a linear combination of products of complementary minors (corresponding to certain colorings) equals our desired Temperley-Lieb immanant (Propositions 4.8 and 4.9).

Lemma 4.4. *Let a, b, c, d, e be nonnegative integers, so $a, b, c, d \geq 1$ and $a+b+c+d+e = n$. Then, there is a unique non-crossing matching and a coloring compatible with it, such that the coloring satisfies*

1. *i is black for $i \in [a+1, n-d]$*
2. *i' is white for $i \in [b+1, n-c]$*
3. *There are exactly a black vertices and b white vertices in $[1, a] \bigcup [1, b]'$*
4. *There are exactly d black vertices and c white vertices in $[n-d+1, n] \bigcup [n-c+1, n]'$*
5. *There are no pairings between two vertices in $[1, a] \bigcup [1, b]'$ (which we will refer to as an "internal pairing")*
6. *There are no internal pairings in $[n-d+1, n] \bigcup [n-c+1, n]'$*

For instance, here is a complementary coloring, with the corresponding unique matching, with $a = c = 2, e = b = d = 1$. The boxed vertices are those that are fixed by conditions 1 and 2.



Proof. Assuming the above conditions are satisfied. Suppose some white vertex $i \in [1, a] \bigcup [1, b]'$ is paired with some black vertex $j \in [a+b+1, a+b+e+c]$. By conditions *1.,3.*, there are at most $b-1$ white vertices to the left of this pair, but at least b black vertices to the left of this pair. $b-1 < b$, so we cannot have a non-crossing matching. Suppose that some white vertex $i \in [1, a] \bigcup [1, b]'$ is paired with some black vertex $j \in [a+b+e+c+1, a+b+e+c+d] \bigcup [b+a+e+d+1, b+a+e+d+c]'$. By conditions *1.,3.,4.*, there are at most $b-1+c$ white vertices to the left of this pair, but at least $b+e+c$ black vertices to the left of this pair. $b-1+c < b+c+e$, so we cannot have a non-crossing matching.

Therefore, any white vertex in $[1, a] \bigcup [1, b]'$ must be paired with a black vertex in $[a+1, a+b]$.

Now, for convenience, we re-index the $2n$ vertices starting from $a+b$ and go clockwise. That is,

- Vertex $i \in [1, a+b]$ has new index $a+b+1-i$
- Vertex $i \in [a+b+1, n]$ has new index $2n+a+b+1-i$
- Vertex $i' \in [1, n]'$ has new index $a+b+i$

Let's consider the first black vertex (with respect to the new indexing). Suppose it has new index less than $2b+1$, then by condition *3.*, there exists a white vertex after i_0. By the above paragraph, j_0 must be paired with a black vertex k in $[a+1, a+b]$ (with new index $[1, b]$). But then, i_0 is a black vertex to the left of the pair j_0k, while the only white vertices to the left of the pair j_0k are in $[1, a] \bigcup [1, b]'$. This contradicts condition *5.* Therefore, vertices with new index $[b+1, 2b]$ are all whites and vertices with new index $[2b+1, 2b+a]$ are all blacks. Similarly, vertices with new index $[a+b+n-c+1, a+b+n-c+d]'$ are all blacks and $[a+b+n-c+d+1, a+b+n+d]'$ are all whites.

Thus, we divide all $2n$ vertices into 7 blocks of the same color.

- Block 1 consists of b black vertices with new index $[1, b]$
- Block 2 consists of b white vertices with new index $[b+1, 2b]$
- Block 3 consists of a black vertices with new index $[2b+1, 2b+a]$
- Block 4 consists of $a+e+d$ white vertices with new index $[2b+a+1, 2b+a+a+e+d]$
- Block 5 consists of d black vertices with new index $[2b+2a+e+d+1, 2b+2a+e+2d]$
- Block 6 consists of c white vertices with new index $[2b+2a+e+2d+1, 2b+2a+e+2d+c]$
- Block 7 consists of $e+c$ white vertices with new index $[2b+2a+e+2d+c+1, 2n]$

Notice that block 2 and 3 cannot pair by condition *5.* Also, block 5 and 6 cannot pair by condition *6.* Therefore, the unique possible non-crossing matching satisfying the given conditions is obtained by

- Pairing block 1 with block 2
- Pairing block 3 with the first a vertices in block 4
- Pairing block 5 with the last d vertices in block 4
- Pairing block 6 with the first c vertices in block 7
- Pairing the remaining e vertices in block 4 and the last e vertices in block 7.

where "first" and "last" are in terms of the new indexing.

For instance, to see that the first two hold, observe that any vertex with new label i in block 2 must be paired with either something in block 5 or block 1. However, notice that there are at most $a+d$ black vertices between these vertices. But there at least $a+e+d$ white vertices, which is a contradiction.

Therefore, we see that anything in block 2 must be paired with block 1. From here, it follows that there is a unique way to form this pairing. $\square$

Lemma 4.5. *Let a, b, c, d, e, f be nonnegative integers where $a, b, c, d, \max(e, f) \geq 1$, and $a+b+c+d+e+f = n$. Then, there is a unique non-crossing matching and a coloring compatible with it, such that the coloring satisfies*

1. i is black for $i \in [1, a+e]$

2. *i is white for $i \in [a+e+b+c+1, n]$*

3. *i' is black for $i \in [1, b+f]$*

4. *i' is white for $i \in [b+f+a+d+1, n]$*

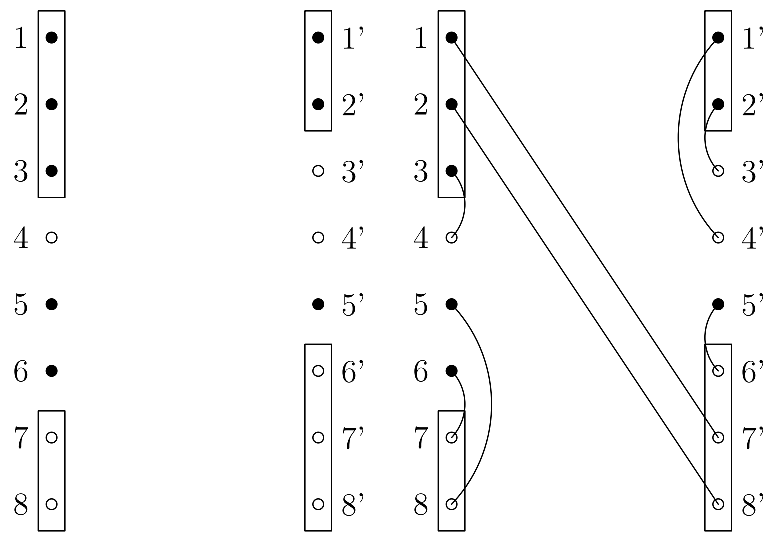
5. *There are exactly c black vertices and b white vertices in $[a+e+1, a+e+b+c]$*

6. *There are exactly d black vertices and a white vertices in $[b+f+1, b+f+a+d]'$*

7. *There are no pairings between two vertices in $[a+e+1, a+e+b+c]$ (which we will refer to as an "internal pairing")*

8. *There are no internal pairings in $[b+f+1, b+f+a+d]'$*

This case has the following diagram, where $a = c = 2, b = d = e = f = 1$, with the boxed vertices again being those fixed (this time for conditions 1 to 4).



Proof. Assuming the above conditions are satisfied. Suppose that some white vertex $i \in [a+e+1, a+e+b+c]$ is paired with some black vertex $j' \in [b+f+1, b+f+a+d]$. By conditions *1.,3.,5.,6.*, there are at most $b+a-1$ white vertices above this pair but at least $a+e+b+f$ black vertices above this pair. $b+a-1 < a+e+b+f$, so we cannot have a non-crossing matching. Therefore, any white vertex in $[a+e+1, a+e+b+c]$ must be paired with a black vertex in $[1, a+e] \bigcup [1, b+f]'$.

Consider the first black vertex (with smallest label) $i_0 \in [a+e+1, a+e+b+c]$. If $i_0 \in [a+e+1, a+e+b]$, then by condition *5.*, there exists white vertices $j_0 \in [i_0+1, a+e+b+c]$ after i_0. By the above paragraph, j_0 must be paired with a black vertex k' in $[1, a+e] \bigcup [1, b+f]'$. But then, i_0 is a black vertex to the left of the pair j_0k', while the only white vertices to the left of the pair j_0k' are in $[a+e+1, a+e+b+c]$. This contradicts condition *7.* Hence, $[a+e+1, a+e+b]$ are all white and $[a+e+b+1, a+e+b+c]$ are all black. Similarly, $[b+f+1, b+f+a]'$ are all white and $[b+f+a+1, b+f+a+d]'$ are all black.

For convenience, we now re-index the $2n$ vertices starting from $a+e+b$ and go clockwise. That is,

- Vertex $i \in [1, a+e+b]$ has new index $a+e+b+1-i$
- Vertex $i \in [a+e+b+1, n]$ has new index $2n+a+e+b+1-i$
- Vertex $i' \in [1, n]'$ has new index $a+e+b+i$

We divide all $2n$ vertices into 6 blocks of the same color.

- Block 1 consists of b white vertices with new index $[1, b]$
- Block 2 consists of $e+a+b+f$ black vertices with new index $[b+1, b+e+a+b+f]$
- Block 3 consists of a white vertices with new index $[b+e+a+b+f+1, b+e+a+b+f+a]$
- Block 4 consists of d black vertices with new index $[b+e+a+b+f+a+1, b+e+a+b+f+a+d]$
- Block 5 consists of $e+c+d+f$ white vertices with new index $[b+e+a+b+f+a+d+1, b+e+a+b+f+a+d+e+c+d+f]$
- Block 6 consists of c black vertices with new index $[b+e+a+b+f+a+d+e+c+d+f+1, 2n]$

Notice that block 1 and 6 cannot pair by condition *7*. Also, block 3 and 4 cannot pair by condition *8*. It follows that the unique possible non-crossing matching satisfying the given conditions is obtained by

- Pairing block 1 with the first b vertices in block 2
- Pairing block 3 with the last a vertices in block 2
- Pairing block 4 with the first d vertices in block 5
- Pairing block 6 with the last c vertices in block 5
- Pairing up the remaining middle $e+f$ vertices in block 2 and the remaining middle $e+f$ vertices in block 5 in reverse order.

where the "first", "last", and "reverse order" are in terms of the new indexing.

To see this, for instance, notice that if the vertex with new index i is paired with j, notice that j is either in blocks 2 or 4, so $j > i$. But then notice that if it is paired with something in block 4, we have at least $a+e+b+f$ black vertices and at most $a+b$ white vertices; but at least one of e, f is 1, which is a contradiction. This gives us the first pairing condition. $\square$

Lemma 4.6. *Let w have 5 ascending strings of consecutive integers as stated in the first case of Proposition 4.2, with a, b, c, d, e defined accordingly. The non-crossing matching of w is exactly the non-crossing matching in Lemma 4.4.*

Proof. Notice that the permutation w can be thought of as the composition of two permutations that commute with each other. Indeed, observe that $w = w_1 w_2$, where we have that

$$w_1(i) = \begin{cases} i+b \text{ if } i \leq a \\ i-a \text{ if } a < i \leq a+b \\ i \text{ otherwise} \end{cases}$$

and

$$w_2(i) = \begin{cases} i - c \text{ if } i \geq n - d \\ i + d \text{ if } n - c - d \leq i < n - d \\ i \text{ otherwise} \end{cases}.$$

But then we may employ Lemma 3.4 on w_1, w_2 to obtain the non-crossing matching for w, since w_1, w_2 commute.

It's not hard to check then that the properties in Lemma 4.4 hold. For instance, notice that $a+i$ is paired with $a-i+1$ for $i \in [1, \min(a,b)]$, and $(b+i)'$ is paired with $(b-i+1)'$ for $i \in [1, \min(a,b)]$. If $a = b$ this determines the entire pairing; otherwise WLOG $a > b$. Then, from Lemma 3.4, we have that $a-b$ is paired with $(a+b)'$, $a-b-1$ with $(a+b-1)'$, and so forth, until 1 is paired with $(2b+1)'$. But then we see that, given conditions 1 and 2, conditions 3 and 5 hold (the white vertices are $a, a-1, \ldots, a-b+1$). The same argument with c, d allow us to conclude that conditions 4, 6 hold.

But by Lemma 4.4, it follows that the non-crossing matching corresponding to w is the unique matching described in the lemma, as desired. ☐

Lemma 4.7. *Let w have 6 ascending strings of consecutive integers as stated in the second case of Proposition 4.2, with a, b, c, d, e, f defined accordingly. The non-crossing matching of w is exactly the non-crossing matching in Lemma 4.5.*

Proof. By uniqueness of the non-crossing matching satisfying the conditions listed in Lemma 4.5, we only need to show that

- The coloring described in the proof of Lemma 4.5 is compatible with the non-crossing matching of w.
- Conditions *7* and *8* in Lemma 4.5 are satisfied.

We wish to apply Lemma 3.3. To do so, we construct the coloring in the Lemma and obtain the following. We divide the left n vertices into 6 blocks given by

- Block 1 consists of a black vertices
- Block 2 consists of e black vertices
- Block 3 consists of b white vertices
- Block 4 consists of c black vertices
- Block 5 consists of f white vertices
- Block 6 consists of d white vertices

Similarly, we divide the right n vertices into 6 blocks given by

- Block $1'$ consists of b black vertices
- Block $2'$ consists of f black vertices
- Block $3'$ consists of a white vertices

- Block $4'$ consists of d black vertices
- Block $5'$ consists of e white vertices
- Block $6'$ consists of c white vertices

By Lemma 3.3, we are done. Note in particular that no vertex in block 3 can pair with a vertex in block 4, and similarly for 3' and 4'.

□

Proposition 4.8. *Let w have 5 ascending strings of consecutive integers as stated in the first case of Proposition 4.2, with a, b, c, d, e defined accordingly.*
Then, we have

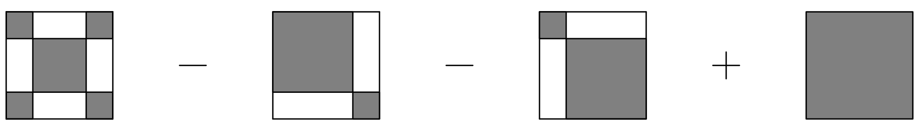
$$\mathrm{Imm}_w = \mathrm{sgn}(w) \sum_{|I_1|=|I_2|,|I_3|=|I_4|} \mathrm{CM}_{I_1\cup I_3, I_2\cup I_4}(-1)^{|I_1|+|I_3|}. \tag{3}$$

where the sum runs over all $I_1 \subset [1,a], I_2 \subset [1,b], I_3 \subset [n-d+1,n], I_4 \subset [n-c+1,n]$.

For instance, with $w = 2143$, we are given that $a = b = c = d = 1$, and so the possible $(I_1 \cup I_3, I_2 \cup I_4)$ are:

$$(\{1,4\},\{1',4'\}),(\{1\},\{1'\}),(\{4\},\{4'\}),(\emptyset,\emptyset).$$

(It is a coincidence in this example that we always have $I_1 = I_2$ and $I_3 = I_4$.) They correspond to the following complementary minors:



Notice that I_1, I_2 could both be taken to be empty.

Proof. Let $\beta(I_1, I_2) := s(I_1) + s(I_2) + s(I_3) + s(I_4) + |I_1| + |I_3|$. First, we claim for some α,

$$\mathrm{Imm}_w = \alpha \sum_{I_1,I_2} (-1)^{\beta(I_1,I_2,I_3,I_4)} \Delta_{\bar{I}_1\cup\bar{I}_3\cup[a+1,n-d],\bar{I}_2\cup\bar{I}_4\cup[b+1,n-c]} \Delta_{I_1\cup I_3, I_2\cup I_4}. \tag{4}$$

Notice that we are taking the sum of product of complementary minors over all possible colorings satisfying conditions *1* through *4* in Lemma 4.4. To show this, we expand each product of complementary minors as a sum of TL-immanants. Take any $u \neq w$ such that Imm_u appears when expanding some product of complementary minors. By Lemmas 4.4 and 4.6, the non-crossing matching of u does not satisfy conditions *7* and *8* in Lemma 4.4. Let the first (in terms of the new indexing introduced in the proof of Lemma 4.4) pair be v_1 and v_2. Then v_1, v_2 must have different parities, so swapping the colors of v_1, v_2 will change the sign of $\beta(I_1, I_2)$. By pairing up complementary minors using this sign-reversing involuion, we see the coefficient of Imm_u is zero in the RHS of (3). Thus, only Imm_w survives in the double sum of Imm_u obtained by expanding every product of complementary minors in (4). This gives us equation (4) up to a global sign. We then get by applying Lemma 2.8:

$$\mathrm{Imm}_w = \alpha \sum_{|I_1|=|I_2|,|I_3|=|I_4|} \mathrm{CM}_{I_1\cup I_3, I_2\cup I_4}(-1)^{|I_1|+|I_3|}. \tag{5}$$

To determine α, we compare coefficients of x_w in (5). There is a unique CM with nonzero x_w coefficient, and that CM has $I_1 = I_2 = I_3 = I_4 = \emptyset$. Then since $f_w(w) = 1$, we get $1 = \alpha \operatorname{sgn}(w)$, so $\alpha = \operatorname{sgn}(w)$. $\square$

Proposition 4.9. *Let w have 6 ascending strings of consecutive integers as stated in the second case of Proposition 4.2, with a, b, c, d, e, f defined accordingly.*
Then, we have

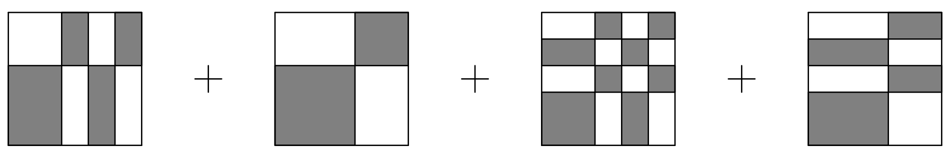
$$\mathrm{Imm}_w = \operatorname{sgn}(w) \sum_{I_1, I_2} \mathrm{CM}_{[1,a+e]\cup I_1, I_2 \cup [b+f+a+d+1,n]'} \tag{6}$$

where the sum runs over all $I_1 \subset [a+e+1, a+e+b+c]$ with $|I_1| = c$ and $I_2 \subset [b+f+1, b+f+a+d]'$ with $|I_2| = a$.

For instance, with $w = 21453$, we are given that $a = b = c = d = f = 1$ and $e = 0$, and so the possible $([1, a+e] \cup I_1, I_2 \cup [b+f+a+d+1, n])$ are:

$$(\{1,2\}, \{3', 5'\}), (\{1,2\}, \{4', 5'\}), (\{1,3\}, \{3', 5'\}), (\{1,3\}, \{4', 5'\}).$$

They correspond to the following complementary minors:



Proof. Let $\beta(I_1, I_2) := s(I_1) + s(I_2)$. First, we claim for some α,

$$\mathrm{Imm}_w = \alpha \sum_{I_1, I_2} (-1)^{\beta(I_1, I_2)} \Delta_{[1,a+e]\cup I_1, I_2 \cup [b+f+a+d+1,n]'} \Delta_{\overline{I}_1 \cup [a+e+b+c+1,n], \overline{I}_2 \cup [1,b+f]'}. \tag{7}$$

Notice that we are taking the sum of product of complementary minors over all possible colorings satisfying conditions *1* through *6* in Lemma 4.5. To show this, we expand each product of complementary minors as a sum of TL-immanants. Take any $u \neq w$ such that Imm_u appears when expanding some product of complementary minors. By Lemmas 4.5 and 4.7, the non-crossing matching of u does not satisfy conditions *7* and *8* in Lemma 4.5. Let the first (in terms of the new indexing introduced in the proof of Lemma 4.5) pair be v_1 and v_2. Then v_1, v_2 must have different parities, so swapping the colors of v_1, v_2 will change the sign of $\beta(I_1, I_2)$. By pairing up complementary minors using this sign-reversing involuion, we see the coefficient of Imm_u is zero in the RHS of (6). Thus, only Imm_w survives in the double sum of Imm_u obtained by expanding every product of complementary minors in (7). This gives us equation (7) up to a global sign. We then get by applying Lemma 2.8:

$$\mathrm{Imm}_w = \alpha \sum_{I_1, I_2} \mathrm{CM}_{[1,a+e]\cup I_1, I_2 \cup [b+f+a+d+1,n]'} \tag{8}$$

To determine α, we compare coefficients of x_w in (8). Then since $f_w(w) = 1$, we get $1 = \alpha \operatorname{sgn}(w)$, so $\alpha = \operatorname{sgn}(w)$.

$\square$

Using the above propositions, we can explicitly express Imm_w as a sum of products of complementary minors. Thereby, it becomes easy to calculate $f_w(u)$ for each $u \in \mathfrak{S}_n$.

Theorem 4.10. *Let w have 5 ascending strings of consecutive integers as stated in the first case of Proposition 4.2, with a, b, c, d, e defined accordingly. Let $u \in \mathfrak{S}_n$. Then*

$$f_w(u) = \begin{cases} 0 & \text{if } \exists i \in [1, a] \text{ s.t. } u(i) \in [1', b'], \\ & \text{or } \exists i \in [(n+1-d), n] \text{ s.t. } u^{-1}(i) \in [(n+1-c)', n'] \\ & \\ \operatorname{sgn}(w)\operatorname{sgn}(u)\binom{A+B}{A} & \text{otherwise, where } A = \#\{i \in [1, a] \text{ s.t. } u(i) \in [(n+1-c)', n']\}, \\ & B = \#\{i \in [1', b'] \text{ s.t. } u^{-1}(i) \in [(n+1-d), n]\}. \end{cases}$$

Proof. Consider the sum of $\mathrm{CM}_{I,J}$ given in Proposition 4.8. Notice that x_u has nonzero coefficient in a $\mathrm{CM}_{I,J}$ if and only if the corresponding coloring satisfies conditions *1* through *4* in Lemma 4.4 and that i and $u(i)$ has different colors for all $i \in [n]$. (In that case, the coefficient is $\operatorname{sgn}(w)\operatorname{sgn}(u)$.) In particular, the coloring of $[n]$ determines the coloring of $[n]'$.

If there exists $i \in [1, a]$ such that $u(i) \in [1', b']$, then swapping the colors of i and $u(i)$ is a sign-reversing involution among complementary minors satisfying $u(I) = J$. In this case, $f_w(u) = 0$. A similar argument holds if there exists $i \in [n+1-d, n]$ such that $u(i) \in [(n+1-c)', n']$. Thus, assume neither condition holds.

Then $a - A$ of the vertices in $[1, a]$ are paired with a vertex in $[b+1, n-c]$, hence they must be black. Similarly, $b - B$ of the vertices in $[1', b']$ are paired with a vertex in $[a+1, n-d]$, hence they must be white. As a result, among the $A + B$ unresolved vertices in $[1, a] \cup [1', b']$, A must be black and B must be white. There are $\binom{A+B}{A}$ ways of choosing colors for the unresolved vertices. Finally, if C of the unresolved vertices in $[1, a]$ are white, then C of them are black in $[1', b']$. Mapping unresolved vertices in $[1', b']$ to unresolved vertices in $[n-d+1, n]$, we observe that C of the unresolved vertices in $[n-d+1, n]$ are white. Thus, $(-1)^{|I_1|+|I_3|} = 1$. $\square$

Theorem 4.11. *Let w have 6 ascending strings of consecutive integers as stated in the second case of Proposition 4.2, with a, b, c, d, e, f defined accordingly. Let $u \in \mathfrak{S}_n$. Then*

$$f_w(u) = \operatorname{sgn}(w)\operatorname{sgn}(u)\binom{A+B}{A},$$

where

$$A = c - \left|\left\{i \in [a+e+1, a+e+b+c] \;\middle|\; u(i) \in [b+f+a+d+1, n]\right\}\right|$$

and

$$B = b - \left|\left\{i \in [a+e+1, a+e+b+c] \;\middle|\; u(i) \in [1, b+f]\right\}\right|$$

Proof. Consider the sum of $\mathrm{CM}_{I,J}$ given in Proposition 4.9. Notice that x_u has nonzero coefficient in a $\mathrm{CM}_{I,J}$ if and only if the corresponding coloring satisfies conditions *1* through *6* in Lemma 4.5 and that i and $u(i)$ has different colors for all $i \in [n]$. (In that case, the coefficient is $\operatorname{sgn}(w)\operatorname{sgn}(u)$.) In particular, the coloring of $[n]$ determines the coloring of $[n]'$. Also, $\left\{i \in [a+e+1, a+e+b+c] \;\middle|\; u(i) \in [b+f+a+d+1, n]\right\}$ must all be blacks and $\left\{i \in [a+e+1, a+e+b+c] \;\middle|\; u(i) \in [1, b+f]\right\}$ must all be whites. Therefore, we have the freedom of choosing exactly A blacks and B whites from $A+B$ vertices. As a result, x_u has coefficient $\operatorname{sgn}(w)\operatorname{sgn}(u)$ in exactly $\binom{A+B}{A}$ of the $\mathrm{CM}_{I,J}$'s, and hence $f_w(u) = \operatorname{sgn}(w)\operatorname{sgn}(u)\dbinom{A+B}{A}$.

□

Corollary 4.12. *Let w be as above, avoiding 1324 and 321 but not 2143. If w_0 is the longest word in $\mathfrak{S}_n$, then $|f_w(w_0)| = \binom{\min(a,c)+\min(b,d)}{\min(b,d)}$.*

Proof. We divide up our work into the two cases, given by Theorems 4.11 and 4.10.

For the first case (in the first theorem), observe that for w_0, A is equal to $c - |[b+f+a+d+1, n] \cap [n-a-e-b-c+1, n-a-e]|$. But $n-a-e-b-c = d+f$, meaning that our set has magnitude $|[b+f+a+d+1, n-a-e]|$. But observe that $n-a-e = b+f+c+d$, meaning that this is equal to $\max(c-a, 0)$. Similarly, for B, this is equal to the size of the set $[d+f, b+f] = \max(b-d, 0)$ subtracted from b.

Therefore, $A = c - \max(c-a, 0) = \min(a,c), B = b - \max(b-d, 0) = \min(b,d)$, whereby Theorem 4.11 gives us the desired result.

For the second case (in the second theorem), it's not hard to see that w_0 lies in the second case in Theorem 4.10, meaning that we need to compute the number of elements in $w_0^{-1}([n+1-c, n]) \cap [1, a]$ and $w_0^{-1}([1, b]) \cap [n+1-d, n]$. However, notice that w_0 sends $[1, b]$ to $[n+1-b, n]$, meaning that the size of the latter set is $\min(b, d)$. Similarly, we see that the size of the former set is $\min(a, c)$.

Combining these together gives us the value of the antidiagonal coefficient, as desired. □

We remark that the 2143-avoiding condition in Theorem 3.11 follows immediately from Theorem 4.10. This gives us a lower bound on the number of %-immanants that we need to use to write our Temperley-Lieb immanants.

4.3 Proof of Theorem 4.1

We restate theorem 4.1.

Theorem. *Let w be a 321-avoiding permutation. The following statements are equivalent:*

1. *The Temperley-Lieb immanant Imm_w is a linear combination of %-immanants;*
2. *The signed Temperley-Lieb immanant $\operatorname{sgn}(w)\mathrm{Imm}_w$ is a sum of at most two %-immanants;*
3. *The permutation w avoids the patterns 24153, 31524, 231564, and 312645, in addition to avoiding 321.*

Proof. We will prove (3) ⇒ (2) ⇒ (1) ⇒ (3). (2) ⇒ (1) is trivial.

(3) ⇒ (2). Since w avoids 321, 24153, 31524, w has 5 ascending strings of consecutive integers as stated in the first case of Proposition 4.2. Since w furthermore avoids 231564 and 312645, we must have $a = 1$ or $c = 1$, and $b = 1$ or $d = 1$. We will first assume $a = 1$.

Case 1 Assume $a = b = 1$. We use Theorem 4.10 to prove that $\operatorname{sgn}(w)\mathrm{Imm}_w$ is the sum of the following two % immanants.

- $\mathrm{Imm}_1^{\%} = \mathrm{Imm}_w^{\%}$.

- The %-immanant $\mathrm{Imm}_2^{\%}$ corresponding to the % shape where we remove the $d\times c$ rectangle in the lower-right corner, and remove the $(n-d)\times 1$ and $1\times(n-c)$ rectangles in the upper-left corner (with the rectangles overlapping at the 1×1 rectangle).

Since in the perfect matching of w, block $[1,a]$ is matched with block $[(b+1)',(b+a)']$, and block $[1',b']$ is matched with block $[a+1,a+b]$, letting $a=b=1$ gives us that $w(1)=2$ and $w(2)=1$. So for any $u\in\mathfrak{S}_n$ with $u(1)=1$, the coefficient of x_u in Imm_w is 0.

Case 2 If $a=d=1$, we take the following two %-immanants:

- $\mathrm{Imm}_1^{\%}=\mathrm{Imm}_w^{\%}$.
- The %-immanant $\mathrm{Imm}_2^{\%}$ corresponding to the shape where we remove the $1\times(n-b)$ rectangle in the lower-right corner, and remove the $(n-c)\times 1$ rectangle in the upper-left corner,

Notice that whenever $u(i)\in[1',b']$ for some $i\in[1,a]$, or $u(i)\in[(n+1-d),n]$ for some $i\in[(n+1-c)',n']$, we have $u\not\geq w$, and thus $f_w(u)=0$. We can also see that in these cases the coefficient of x_u in the sum of the %-immanants is zero.

Now, consider $u\in\mathfrak{S}_n$ where $f_w(u)\neq 0$. Then, $f_w(u)=\binom{A+B}{A}$ as in Theorem 4.10. Since we claimed that for w to avoid the above patterns, w fall into one of the four cases: $a=1,b=1$, or $a=1,d=1$, or $c=1,b=1$, or $c=1,d=1$. Given these constraints, for $\binom{A+B}{A}$ to be nonzero, we can have either $\binom{B}{0}=1$, or $\binom{2}{1}=2$. But this coefficient equals 1 happens when $A=0$ corresponding to the permutations that fit only the first % configuration (for instance, when $a=b=1$, we notice that $A=0$ iff $u(1)\notin[n-c+1,n]$, or that it doesn't fit in the second pattern, and $B=0$ iff $u^{-1}(1)\notin[n-d+1,n]$, or that it doesn't fit in the second pattern), whereas $\binom{2}{1}=2$ are the coefficients of the permutations that have more inversions and fit both configurations. This proves our result for $a=1$.

If $c=1$, then notice that $w'=w_0w^{-1}w_0$ will also consist of 5 ascending blocks and $a'=1$. This is because taking the inverse is the same as reflecting the matching diagram across the perpendicular bisector of 1 and $1'$, while taking conjugation by w_0 is the same as reflecting the matching diagram across the perpendicular bisector of 1 and n. By the discussion of our previous case, $\mathrm{sgn}(w')\,\mathrm{Imm}_{w'}$ is a sum of two %-immanants $\mathrm{Imm}_1^{\%}+\mathrm{Imm}_2^{\%}$. And by Lemma 2.9, $\mathrm{sgn}(w)\,\mathrm{Imm}_w=\mathrm{Imm}_1^{*\%}+\mathrm{Imm}_2^{*\%}$, where $\mathrm{Imm}_i^{*\%}$ is the %-immanant with a zero in (i,j) if and only if $\mathrm{Imm}_i^{\%}$ has a zero in $(n+1-j,n+1-i)$.

(1) $\Rightarrow$ (3). For this, we use the explicit coloring arguments from Proposition 4.8 and 4.9 that gives our Temperley-Lieb immanants as sums and differences of products of complementary minors.

Suppose that we have a permutation that doesn't avoid one of the four patterns. Again, we divide into cases per Proposition 4.2.

First, suppose that we are in the first case, with five ascending strings of consecutive integers. WLOG, suppose that $\min(a,c)\geq 2$. From Proposition 4.8, we know that Imm_w can be written as the sum over complementary minors whose corresponding colorings where the first a unprimed and the first b primed-labelled vertices collectively contain a black vertices and b white vertices, and the last d unprimed and the last c primed-labelled vertices collectively contain d black and c white vertices, and the other vertices with unprimed labels are all black (and the other primed vertices are all white).

Consider the permutation v given by $v(i)=i+(n-a-c)$ for $i\leq a+c$, and $v(i)=n-i$ for all other values of i. Notice that x_v only lies in the product of complementary minors where $i,v(i)'$ have opposite colors, by construction of this coloring (as otherwise row i and column $v(i)$

are associated to different complementary minors, so our product cannot contain this monomial). However, for this to happen, as $n-a-c \geq b$, we need $1, 2, \dots a$ must all be colored black, as $(n-a-c+1)', \dots, (n-c)'$ in our coloring were all forced to be white.

Similarly, $a+1, a+2, \dots, a+c$ are all colored black as $a+c \leq n-d$, and so thus $(n-c+1)', (n-c+2)', \dots, n'$ are all colored white. These conditions uniquely determines our coloring, meaning that the coefficient of our monomial is equal to 1; the complementary minor associated to this is just the determinant (with one of the minors being trivial). Thus, $|f_w(v)| = 1$.

Now, consider the permutation vs_a. Notice that, repeating the above argument, we are forced to have $1, 2, \dots, a-1$ be black, $(n-c+2)', \dots, n'$ be white. This means in particular that, among $1', 2', \dots, b', a$, there is exactly one more black number, which will then uniquely determine our coloring. Similarly to the argument given in Theorem 4.10, notice that the fact that $v(i) = n-i$ for $i > a+c$ determines the colors of all the vertices in the primed column after $\min(b, d)$, meaning that we have $1 + \min(b, d)$ possible colorings in this case.

Now, notice that the sign we associate to the product of complementary minors is $(-1)^{\sum I + \sum J - |J|}$, where I, J are the subsets containing the indices of the rows and columns, respectively, for a complementary minor. But the sign this complementary minor has with respect to the whole determinant is just $(-1)^{\sum I + \sum J}$, meaning that we are summing over all these possible colors the value of $(-1)^{|J|} \operatorname{sgn}(vs_a)$. But in the first coloring, $|J| = n$, and the other colorings, notice that we have $b-1+a+e+d+c-1 = n-2$ white colored vertices, so $|J| = n-2$. Therefore, $|f_w(vs_a)| = \min(b, d) + 1 > 1$.

However, similarly to the argument in Theorem 3.12, notice that in every %-immanant, the sign of the monomials associated to v and vs_a are negative of each other. It's not hard to see that if v is nonzero in a %-immanant, vs_a isn't either. Similarly, if vs_a has nonzero coefficient, this means that entries $(a-1, n-c-1)$ and $(a+2, n-c+2)$ both aren't zero. But then observe that the entries $(a, n-c), (a+1, n-c+1)$ can't be forced to be zeroed out by the %-immanant, as otherwise this would contradict the definition of our %-immanant. Thus, it follows that in every %-immanant, the sign of the monomials associated to v and vs_a must be negative of each other, and so this also holds for every linear combination of them.

We thus see that the magnitudes of these need to be equal if Imm_w was a sum of %-immanants, contradicting what we found above. Thus, w cannot be written as a sum of %-immanants.

We now work in the second case. WLOG assume $e \geq 1$, so we have the 24153 pattern here. Again, from the proof in Theorem 4.10, the colorings we consider are those where $1, 2, \dots, a+e$ are black, $1', 2', \dots, (b+f)'$ are black, $n-f-d, n-f-d+1, \dots, n$ are white, $(n-e-c)', \dots, n'$ are white, and among the remaining uncolored vertices, b of the unprimed and a of the primed ones are white, and the rest are black. We again construct a permutation to yield our contradiction.

In this case, consider the permutation given by $v(i) = v(i) = i + n - a - 2e - c$ for $i = e-1, e, \dots, a+e+c+1$, and $w(i) = n-i$ for the other i. Notice that there are no product of complementary minors that contain this monomial, since this would have to be associated to a coloring where $(n-a-e-c-1)', (n-a-e-c)', \dots, (n-e-c)'$ are all white; but as $n-a-e-c-1 > b+f$, we have $a+1$ primed white vertices when we only should have at most a, which is a contradiction.

Now, consider the permutation vs_{a+e}. This time, notice that we are forced to have $(n-a-e-c-1)', (n-a-e-c)', \dots, (n-e-c-1)'$ white, which uniquely determines the coloring, since then $(b+f+1)', (b+f+2)', \dots, (n-a-e-c-2)', (n-e-c)'$ are all colored black, and we note that $a+e+2, \dots, a+e+c+1$ are all colored black, $a+e+1$ is colored white, and the rest of the yet-uncolored vertices are colored white.

Thus, $f_w(vs_{a+e})$ is equal to 1 or -1. But again, for every %-immanant, the coefficients of the monomials associated to v, vs_{a+e} must be the same, as if one of them has nonzero coefficient, the entries $(a+e-1, n-e-c-1), (a+e+2, n-e-c+2)$ are both not zeroed out, meaning that $(a+e, n-e-c), (a+e+1, n-e-c+1), (a+e+1, n-e-c), (a+e, n-e-c+1)$ can't be zeroed out.

But again, for Imm_w to be a linear combination of %-immanants, we require that the coefficients $f_w(v), f_w(vs_{a+e})$ must be negatives of each other, which as we've constructed is definitively not the case. This proves the desired result. □

In particular, this theorem tells us that Theorems 3.2, 4.1 specify all the possible 321-avoiding permutations for which the corresponding Temperley-Lieb immanant could be expressed as a linear combination of %-immanants.

Example. Let $w = 3416725$. w avoids pattern 1324 but contains pattern 231564. The coefficient of $x_{0(n-1)}x_{1(n-2)}\cdots x_{(n-1)0}$ in Imm_w is -3, but the same coefficient in a %-immanant is ± 1.

5 Kazhdan-Lusztig Immanants as Linear Combinations of %-Immanants

In this section, we expand our scope to Kazhdan-Lusztig immanants Imm_w, where w is allowed to contain a 321-pattern. These immanants were defined first by Rhoades and Skandera in [9].

Definition 16. Let $w \in \mathfrak{S}_n$. The **Kazhdan-Lusztig immanant** $\mathrm{Imm}_w : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$ is given by

$$\mathrm{Imm}_w(M) := \sum_{u\in\mathfrak{S}_n} (-1)^{l(u)-l(w)} P_{w_0u,w_0w}(1) m_{1,u_1} m_{2,u_2} \cdots m_{n,u_n}$$

where $P_{x,y}(q)$ is the Kazhdan-Lusztig polynomial and w_0 is the longest word in $\mathfrak{S}_n$. See [1, Section 5.5] for definitions of Kazhdan-Lusztig polynomials.

One might be concerned that we had previously defined Imm_w as the Temperley-Lieb immanant, but this turns out to not be an issue. From Rhoades and Skandera's paper [9], the following theorem is known:

Proposition 5.1 (Proposition 5 from [9]). *Let w be a 321-avoiding permutation. Then, the Kazhdan-Lusztig immanant of w equals the Temperley-Lieb immanant of w.*

In particular, when we write Imm_w, if w is 321-avoiding, we do not need to worry about whether we are talking about Kazhdan-Lusztig immanants or Temperley-Lieb immanants, as they are equal. As such, in this section when we write Imm_w, we will be referring to the more general Kazhdan-Lusztig immanant, which is defined for all permutations w.

Here, we will list several elementary properties satisfied by the Kazhdan-Lusztig Polynomials, which will be used in our proofs. See for instance [1, Chapter 5] for the first two properties.

Proposition 5.2. *The following properties hold.*

1. *For $w \not\leq u$, $P_{w,u}(1) = 0$.*
2. *For $u \leq w$ such that $l(w) - l(u) \in \{0, 1, 2\}$, $P_{u,w}(q) = 1$.*

3. For $u \le w$, $P_{u,w}(1) \ge 1$.

4. [2, Corollary 3.7] For $u \le v \le w$, $P_{u,w}(q) - P_{v,w}(q) \in \mathbb{N}[q]$.

5. $P_{u,v}(q) = P_{u^{-1},v^{-1}}(q) = P_{w_0uw_0,w_0vw_0}(q)$, where $w_0 \in \mathfrak{S}_n$ is the longest word.

Remark.

- The argument in Theorem 3.12 is a special case of Property (1).
- Property (4) generalizes Lemma 2.9 to KL-immanants.
- It is an easy consequence of the above proposition that $P_{u,w}(1) \ge 1$ for $u \le w$, which will be used extensively throughout the proof of the theorem below.
- To apply the above propositions, notice that $w < u$ if and only if $w_0w > w_0u$.

Now, our goal is to generalize Theorem 4.1 from TL immanants to KL immanants, relaxing the 321-avoiding condition.

Theorem 5.3. *If* Imm_w *is a linear combination of %-immanants, w avoids the patterns* 1324*,* 24153*,* 31524*, and* 426153*.*

Remark. Note that the patterns 1324, 24153, and 31524 already appeared in Theorem 4.1, whereas the new pattern 426153 is not 321-avoiding.

Proof. We prove the contrapositive statement for w containing any one of the four patterns. We first do the case of containing a 1324 pattern, which generalizes our proof of Theorem 3.12 by not assuming w is 321-avoiding. Then, we provide similar arguments for the avoidance of the other three patterns.

Case 1. Assume for sake of contradiction that w contains a pattern of 1324. Then there exists $i < j < k < l$ such that $w(i) < w(k) < w(j) < w(l)$. Consider $v = w \cdot (j, k)$. That is, v is the permutation such that the positions $i < j < k < l$ form a 1234-pattern, while v has the same value as w on all other positions. We have that $v < w$ because (j, k) is an inversion in w. By Proposition 5.2, $P_{w_0v,w_0w}(1) = 0$ and $P_{w_0w,w_0w}(1) = 1$. However, we notice that w and v fit in exactly the same set of %-shapes, because $(i, w(i)) = (i, v(i))$ and $(l, w(l)) = (l, v(l))$ fitting into some %-shape guarantees that $(j, w(j)), (k, w(k)), (j, v(j)), (k, v(k))$ all fit into the same %-shape. So the coefficients of x_v and x_w in any %-immanant are negatives of each other, and thus the coefficients of x_v and x_w in any linear combination of %-immanants are negatives of each other. This contradicts with $P_{w_0w,w_0w}(1) = 1$ and $P_{w_0v,w_0w}(1) = 0$. Hence, Imm_w is not a linear combination of %-immanants.

Case 2. Assume for sake of contradiction that w contains a pattern of 24153. Then there exists $i < j < k < l < m$ such that $w(k) < w(i) < w(m) < w(j) < w(l)$. Let $u = w \cdot (k, m)$. That is, u is the permutation such that the positions $i < j < k < l < m$ form a 24351-pattern, while u has the same value as w on all other positions. We have that $w < u$ because (k, m) is not an inversion in w. By Proposition 5.2, $P_{w_0u,w_0w}(1) \ge 1$.

We now consider $v = u \cdot (j, k)$. That is, v is the permutation such that the positions $i < j < k < l < m$ form a 23451-pattern, while v has the same value as w and u on all other positions. Since 23451 and 24153 are not comparable, v and w are not comparable by Lemma 2.1. By Proposition 5.2, $P_{w_0v,w_0w}(1) = 0$. However, notice that v and u fit in the same set of %-immanants, because we switched a pattern of 1324 in u into 1234 to form v. Similarly as before, the coefficients of x_u and x_v in any linear combination of %-immanants are negatives of each other. But earlier, we had that $P_{w_0u,w_0w(1)\geq 1}$. Therefore, Imm_w is not a linear combination of %-immanants.

Case 3. Assume for sake of contradiction that w contains a pattern of 31524. Apply the same argument for the previous case to u having the pattern 51324 and v having the pattern 51234 on the same 5 positions of 31524 in w, while u and v have the same value as w on all other positions. The whole argument is symmetric up to taking inverses, so we do not repeat it here. (Alternatively, apply case 2 to w^{-1} and use the fourth property of Proposition 5.2.)

Case 4. Assume for sake of contradiction that w contains a pattern of 426153. Let $i < j < k < l < m < n$ be the positions of the pattern, and $w(l) < w(j) < w(n) < w(i) < w(m) < w(k)$. Let $u = w \cdot (i,k) \cdot (l,n)$. That is, u is the permutation such that the positions $i < j < k < l < m < n$ form a 624351-pattern, while u has the same value as w on all other positions. By Lemma 2.2, we obtain $w < u$ because (i,k) and (l,n) are both non-inversions in w. By the remark under Proposition 5.2, $P_{w_0u,w_0w}(1) \geq 1$.

But consider $v = u \cdot (k,l)$. That is, v is the permutation such that the positions $i < j < k < l < m < n$ form a 623451-pattern, while v has the same value as w and u on all other positions. Notice that v and w are not comparable by Lemma 2.1, because 426153 and 624351 are not comparable. By Proposition 5.2, $P_{w_0v,w_0w}(1) = 0$. However, notice that v and u fit in the same set of %-immanants, because we switched a pattern of 1324 in u into 1234 to form v. Similarly to before, the coefficients of x_u and x_v in any linear combination of %-immanants, are negatives to each other, but $P_{w_0u,w_0w} \geq 1$. Therefore, Imm_w is not a linear combination of %-immanants.

Thereby, we finished the proof of all four cases. $\square$

Remark. For each $i \in [n]$, place a rook at position $(i, w(i))$ on an $n \times n$ board. Take the maximal % shape λ/μ such that w fits into λ/μ. Then, the set of all permutations that fit into λ/μ is an upper ideal in the Bruhat poset. Indeed, using the second equivalent definition for Bruhat order, it suffices to show that if $v \in \lambda/\mu$ and $v \cdot (i,j)$ covers v, then $v \cdot (i,j) \in \lambda/\mu$. But this fact is true for any non-inversion (i,j) of v, and follows from geometric considerations. For more details, see Proposition 12 of [10] (note the different convention for λ/μ).

The condition to avoid the patterns 1324, 24153, 31524, and 426153 is equivalent to the condition that this upper ideal is the principal ideal generated by w itself. This is shown in [10, Theorem 4]. This is not a coincidence. If w contains any of the four patterns, the upper ideal is non principal, which means that there exists some other minimal element v that lies in the upper ideal. And notice that in our proof of Theorem 5.3, our choices of v are exactly these minimal elements that are not w's.

This result, combined with our result in the previous section, Theorem 4.1, suggests the following conjecture.

Conjecture. *The Kazhdan-Lusztig immanant for a permutation $w \in \mathfrak{S}_n$, Imm_w, can be expressed as a linear combination of %-immanants if and only if w avoids the patterns*

$$1324, 24153, 31524, 231564, 312645, 426153.$$

In addition to considering Kazhdan-Lusztig immanants as %-immanants, it is also an interesting question to consider ways to express %-immanants as Kazhdan-Lusztig immanants. This leads to the following positive result.

Theorem 5.4. *Suppose that w is a 321-avoiding permutation that also avoids the patterns $1324, 24153, 31524,$ and 34127856. Then, there exists some w' such that*

$$\mathrm{Imm}_w^{\%} = \pm(\mathrm{Imm}_w + \mathrm{Imm}_{w'}),$$

where $w' = \tau w$. Here, τ is the 4-cycle $(w(1)-1, w^{-1}(1)+w(1)-2, w(n)+1, w^{-1}(n)+w(n)+1-n)$.

Proof. To prove this, we employ the rooted tree formula for Kazhdan-Lusztig polynomials given by Lascoux [7]. Our proof will largely follow Brenti's exposition[3].

Since w is a permutation that avoids those patterns, from Lemma 4.3, we require our permutation to be in the case with at most five blocks, with block sizes a, b, e, c, d in order. Furthermore, since w is assumed here to avoid 34127856, one of a, b, c, d must be one. We can assume WLOG that $a = 1$, so then $w^{-1}(1) = 2$. Also, let $w' = \tau w$ as given in the statement.

For the purpose of computing $\mathrm{Imm}_{w'}$ is, we would need to first compute $P_{w_0 v, w_0 w'}$ for all $v \geq w'$. Notice that w has the one-line notation

$$(b+1)[1,b][b+2,n-c-d][n-c+1,n][n-c-d+1,n-c].$$

Therefore, since $b = w(1) - 1, a = w^{-1}(1) - 1, c = n - w(n), d = n - w^{-1}(n)$, we have $\tau = (b, b+1, n-c+1, n-c-d+1)$, so our permutation w' has one-line notation

$$(n{-}c{+}1)12\dots(b{-}1)(b{+}1)(b{+}2)(b{+}3)\dots(n{-}c{-}d)(n{-}c{-}d{+}1)(n{-}c{+}2)\dots nb(n{-}c{-}d{+}2)(n{-}c{-}d{+}3)\dots(n{-}c).$$

Now, we wish to compute $P_{w_0 v, w_0 w'}$ for a permutation v. Notice that if $v \geq w'$ doesn't hold, then $w_0 v \leq w_0 w'$ doesn't hold either, as we noted above, below Proposition 5.2. Acting on the left, observe that $w_0 w'$ is equal to

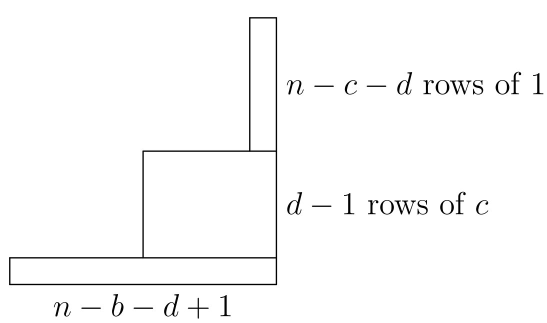
$$cn(n{-}1)\dots(n{-}b{+}2)(n{-}b)(n{-}b{-}1)(n{-}b{-}2)\dots(c{+}d{+}1)(c{+}d)(c{-}1)\dots1(n{-}b{+}1)(c{+}d{-}1)(c{+}d{-}2)\dots(c{+}1).$$

We now employ the construction provided in Section 2.3 of [3]. Indeed, we construct a tree associated to this permutation. First, we consider the permutation $w_0 w' w_0$, which is equal to

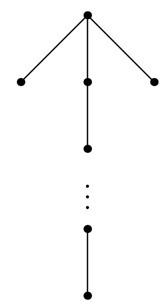
$$(c{+}1)(c{+}2)\dots(c{+}d{-}1)(n{-}b{+}1)12\dots(c{-}1)(c{+}d)(c{+}d{+}1)\dots(n{-}b)(n{-}b{+}2)(n{-}b{+}3)\dots nc.$$

Next, we need the inversion table of this permutation. This turns out to be $(c, c, c, \dots, n-b-d+1, 0, 0, \dots, 0, 1, 1, \dots, 1, 0)$, as we know in this case that $a+b+c+d+e \leq n$. Notice that we have $d-1$ cs, and $n-c-d$ 1s.

Forming the sequence of parentheses associated to the nondecreasing partition formed out of these integers, we get the sequence of parentheses $())\dots)(((\dots())\dots)(((\dots()$. From [3], this is formed by moving from the SW to the NE corner along the partition above. Here, this is the following partition.



From here, we form a tree by matching parentheses, with nested parentheses representing a descendant. For our particular case, notice that we have a lot of unmatched parentheses, but we have two single nested parentheses surrounding one that is nested with $\min(d-1, c-1)$. This is the following tree:



From here, to each leaf we associate a permutation $u \leq w_0 w' w_0$ that is maximal under Bruhat order and is bigrassmannian, which as noted in [3] is a permutation of the form

$$12 \ldots a(b+1) \ldots c(a+1) \ldots b(c+1) \ldots n;$$

notice that this is like a 1324 permutation but in block form. From the exposition under Example 17 from [3], this association is done by associating this permutation to the value $u(i) - i$, where i is the unique descent of u (where $u(i) > u(i+1)$). This value then associates to a leaf (as noted in [3] as well, each leaf corresponds to a corner, or in other words a distinct value in the partition).

The values that we have here are $1, c, n-b-d+1$. We thus seek to find the maximal bigrassmanian permutations underneath $w_0 w' w_0$ with these descents. We claim that the permutations are as follows:

- The value 1 is associated to the permutation

$$123 \ldots (c-1)(c+1)(c+2) \ldots nc.$$

- The value c is associated to the permutation

$$(c+1)(c+2) \ldots (c+d)12 \ldots c(c+d+1) \ldots n.$$

- The value $n-b-d+1$ is associated to the permutation

$$12 \ldots (d-1)(n-b+1)d(d+1) \ldots (n-b)(n-b+2)(n-b+3) \ldots n.$$

It's not hard to see that the map described forms this association. For maximality, we show that there isn't a larger element.

For instance, notice that the second is maximal. Indeed, applying transpositions, the values of $u(1)$ and $u^{-1}(1)$ are nondecreasing. But then this implies that, since these values match for the bigrassmanian element and $w_0 w' w_0$ that we must have $c+1$ first, and 1 at the end. But this uniquely specifies the bigrassmanian. A similar logic holds for the first element.

For the third one, recall from Definition 4 that if $u' \leq u$, then $|u'([1,d+1]) \cap [n-b,n]| \leq |u([1,d+1]) \cap [n-b,n]|$. We can also show that the sum of the elements in this set is also monovariant. To do this, observe that

$$s(u'([1,d+1]) \cap [n-b,n]) = (n-b)|u'([1,d+1]) \cap [n-b,n]| + \sum_{j=n-b+1}^{n} |u'([1,d+1]) \cap [j,n]|,$$

and using third part of Definition 4, we can see that this is bounded above by

$$(n-b)|v([1,d+1]) \cap [n-b,n]| + \sum_{j=n-b+1}^{n} |v([1,d+1]) \cap [j,n]| = s(v([1,d+1]) \cap [n-b,n]).$$

Furthermore, if $u' \leq u$ and $|u'([1,d+1]) \cap [n-b,n]| = |u([1,d+1]) \cap [n-b,n]|$, then observe by the same logic that

$$s([1,d+1] \cap u'^{-1}([n-b,n])) = (d+1)|u'([1,d+1]) \cap [n-b,n]| - \sum_{j=1}^{d} |u'([1,j]) \cap [j,n]|$$

$$(d+1)|u([1,d+1]) \cap [n-b,n]| - \sum_{j=1}^{d} |u([1,j]) \cap [j,n]| = s([1,d+1] \cap u^{-1}([n-b,n])).$$

Then, observe that for both the bigrassmanian and $w_0 w' w_0$, there is only one such element, and the sum of the elements is $n-b+1$, and furthermore the sum of the indices is d. Therefore, in the maximal bigrassmanian u', $|u'([1,d+1]) \cap [n-b,n]| = 1$, and it has to have $n-b+1$ in position d, with $u'(d-1), u'(d+1)$ both smaller than $n-b$. Thus, in the maximal bigrassmanian, we have 1 to $d-1$ as the first ones (by bigrassmanian), and then our descent is at $n-b+1$, which will be d. But this uniquely specifies the bigrassmanian, proving maximality.

From here, we consider $w_0 v$. For each leaf associated to bigrassmanian permutation

$$12 \ldots a(b+1) \ldots c(a+1) \ldots b(c+1) \ldots n$$

we label it according to the maximal value i so that

$$12 \ldots (a-i)(b+1) \ldots (c+i)(a-i+1) \ldots b(c+i+1) \ldots n \leq w_0 v w_0.$$

Notice that for the first two in our list that this value $i = 0$, since we cannot perform this operation of shrinking.

Finally, to obtain the Kazhdan-Lusztig polynomial, we consider functions f evaluated on the edges of our graph, with values in $\mathbb{N}$, that are non-decreasing along paths from the root, and the value at the final edge is at most that of the label of the leaf vertex. Then, Theorem 18 from [3] tells us that $P_{v,w'}(q) = \sum_f q^{|f|}$, where $|f|$ is the sum of the values on edges f takes. For us, as we only really care about the coefficient, $P_{v,w'}(1)$, we just need the number of possible f.

However, notice that the only thing that is left to be specified is the label that we associate to the leaf with permutation

$$12 \ldots (d-1)(n-b+1)d(d+1) \ldots (n-b)(n-b+2)(n-b+3) \ldots n$$

associated to it.

We claim that the label is equal to $|v([n-d+1,n]) \cap [1,b]|-1$. To show this, we show that this value of i works, but no larger one does. Let this value be B.

First, notice that with this value, we're considering the permutation u with one-line notation

$$12\dots(d-B)(n-b+1)\dots(n-b+1+B)(d-B+1)\dots(n-b)(n-b+B+1)(n-b+B+2)\dots n.$$

However, w_0vw_0 has one line notation $(n+1-v(1))(n+1-v(2))\dots(n+1-v(1))$. We will show that w_0vw_0 is larger than u. Notice that our condition now translates to saying that $|w_0vw_0([1,d]) \cap [n-b+1,n]|-1 = B$. Suppose furthermore that $w_0vw_0([1,d]) \cap [n-b+1,n] = \{v'_0, v'_1, \dots, v'_B\}$, in increasing order.

To show this, notice that we increase in Bruhat order by applying transpositions that increase the inversion count (in other words, swapping a, b so that, initially, $a < b$ and a was earlier in the one-line notation). We can first apply some transpositions which increase the number of inversions to yield

$$12\dots(d-B-1)v'_0v'_1\dots v'_B(d-B)\dots(n-b)(n-b+1)\dots B,$$

with the last block missing $v'_0, v'_1, \dots, v'_B$ (for instance, swapping $n-b+B$ with v'_B, and so forth).

From here, the first d elements are in increasing order, so we can permute them going in increasing Bruhat order so $v'_0, v'_1, \dots, v'_B$ are in the correct places, and $1, 2, \dots, d-B-1$ appear in that order, by using Lemma 2.1 where the last $n-d$ elements are all fixed.

Finally, again, the remaining numbers within the one-line notation are in increasing order, so by Lemma 2.1, fixing the first d elements, we see that w_0vw_0 is at least as large as u. Thus, the label is at least B. We now show it can't be more than B.

To show it can't be larger than B, given a permutation u, consider the number of elements in $u([1,d]) \cap [n-b+1,n]$, similar to our maximality argument. But for our bigrassmanian it has value $i+1 > B+1$, but v has $B+1$. This is a contradiction. Therefore, the label is B, meaning that $P_{w_0v,w_0w'}(q) = B+1 = |v([n-d+1,n]) \cap [1,b]|$. This requires the coefficient of x_v to be $\operatorname{sgn}(w')\operatorname{sgn}(v)|v([n-d+1,n]) \cap [1,b]|$ if $v \geq w'$, and 0 otherwise.

At the same time, however, from Theorem 4.10, we know that the coefficient of x_v in Imm_w is equal to

$$\operatorname{sgn}(w)\operatorname{sgn}(v)\binom{|v([n-d+1,n]) \cap [1,b]| + |v([1,1]) \cap [n-c+1,n]|}{|v([1,1]) \cap [n-c+1,n]|},$$

as $a = 1$. But τ is the product of three transpositions: in order, we are swapping $w(1)-1$ and $w^{-1}(1)+w(1)-2$, then swapping $w(1)-1$ with $w(n)+1$, and finally $w(1)-1$ with $w^{-1}(n)+w(n)+1-n$. We then have that $\operatorname{sgn}(w) = -\operatorname{sgn}(w')$.

The coefficient of x_v in $\mathrm{Imm}_w + \mathrm{Imm}_{w'}$ is hence $\operatorname{sgn}(w)\operatorname{sgn}(v)$ if $v \geq w'$, and

$$\operatorname{sgn}(w)\operatorname{sgn}(v)\binom{|v([n-d+1,n]) \cap [1,b]| + |v([1,1]) \cap [n-c+1,n]|}{|v([1,1]) \cap [n-c+1,n]|},$$

if $v \not\geq w'$, but $v \geq w'$, and 0 otherwise. We know that $|v([n-d+1,n]) \cap [1,b]| + |v([1,1]) \cap [n-c+1,n]| \geq 1$, which follows from the fact that $P_{w_0v,w_0w}(1) \geq 1$ here. To show that this isn't at least 2, notice that this can only happen either when $|v([1,1]) \cap [n-c+1,n]| = 1$ and $|v([n-d+1,n]) \cap [1,b]| \geq 1$. We will show that this cannot happen when $v \not\geq w'$.

To see this, we know that by [10][Theorem 4] that $v \geq w$ means that it fits within the %-immanant. In other words, we require $v(1) \geq b+1$ and $v([n-d+1,n]) \subset [1,n-c]$.

For the second case, notice that, with w' having one-line notation

$$(n{-}c{+}1)12\dots(b{-}1)(b{+}1)(b{+}2)(b{+}3)\dots(n{-}c{-}d)(n{-}c{-}d{+}1)(n{-}c{+}2)\dots nb(n{-}c{-}d{+}2)(n{-}c{-}d{+}3)\dots(n{-}c),$$

can apply the following transpositions: first, we know there is some $v_1 < b$ that appears in the last d positions, and a $v_2 > n - c + 1$ that appears in the first position.

We can then apply transpositions to arrive at

$$v_2 12\dots(b+3)\dots(n-c-d+1)(n-c+1)\dots nv_1(n-c-d+2)(n-c-d+3)\dots(n-c),$$

then swap v_1 until it is in the correct place. From there, the remaining $n-2$ elements that need to be arranged are in increasing order, so (again, with the identity permutation being in increasing order and the lowest in Bruhat order), there is a sequence of transpositions that will reach v from w'.

This finishes the proof of the theorem. $\square$

Notice that we can specify the sign by comparing coefficients; for x_v this requires us to have that the sign be $\mathrm{sgn}(v)$, and so our argument above allows us to conclude that this choice of sign makes all signs for each monomial agree.

6 Positivity Conditions

In this section, we consider a type of positivity which we call polynomial positivity. We will first review what this means, and then go over a result that fall out of Theorem 5.4.

First, we consider the notion of a network from n source nodes to n sink nodes, using the notation of [8]. We then proceed to define what it means to be polynomial positive.

Definition 17 (See §2 of [8])**.** A **planar network of order** n is an acyclic directed planar multigraph $X = (V, E)$ with $2n$ vertices on the boundary, which we can again label as $1, 2, 3, \dots, n, n', (n-1)', \dots, 1'$, going clockwise, where the first n are source nodes and the last n are sink nodes.

Given a network X, with a variable associated to each edge giving its weight, and a subgraph γ, we define the **weight** of the subgraph as

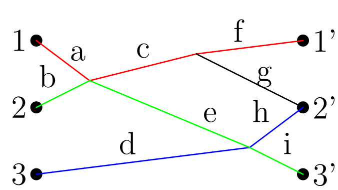
$$\mathrm{wt}(\gamma) = \prod_{e\in\gamma} \mathrm{wt}(e).$$

Of particular interest will be the weights of paths.

From here, to this network and set of weights we associate a matrix, where the (i, j)th entry is the sum of the weights over all paths from i to j'.

We will also refer to **twisted paths of type** $\sigma \in \mathfrak{S}_n$ which is a set of n paths, where the ith path goes from i to $\sigma(i)'$.

For instance, we can consider the following network, with the following edge variables:



The path colored red is of weight acf, and the paths colored red green and blue form twisted paths of type s_2. This yields us with the following matrix:

$$\begin{pmatrix} acf & aeh+acg & aei \\ bcf & bcg+beh & bei \\ 0 & dh & di \end{pmatrix},$$

Given these notions of networks, we can now define what it means to be polynomial positive on networks.

Definition 18. An immanant Imm_f is said to be **polynomial positive** if, when evaluated on the matrix of any network (so substituting in the polynomial in the (i,j)th entry for the variable $x_{i,j}$ in the immanant), the resulting polynomial in the edge weight variables has only nonnegative coefficients.

For instance, the immanant given by $\mathrm{Imm}_f = x_{1,1}x_{2,2}\cdots x_{n,n}$ is polynomial positive, since each of the $x_{i,i}$ is a polynomial with nonnegative coefficients in terms of the weights, and so their product will be as well.

Remark. Notice that this is not just positivity, since there are polynomials that don't have just positive coefficients but that are always positive when evaluated at any real value. For instance, x^2-2x+1 is not a polynomial with positive coefficients, but it equals $(x-1)^2$ and so cannot ever be negative when any choice of real numbers are substituted for x.

In fact, Lemma 6.1 and [9, Proposition 2], tells us that all Kazhdan-Lustzig immanants are polynomial positive.

Lemma 6.1. *Let $f : \mathfrak{S}_n \to \mathbb{R}$ be a function, and extend f to $\mathbb{R}[\mathfrak{S}_n]$ by linearity. If $f(z) \geq 0$ for all $z = z_{[i_1,j_1]}z_{[i_2,j_2]}\cdots z_{[i_k,j_k]}$, then* Imm_f *is polynomial positive on path networks.*

Proof. Let N be the network, and let X_σ be the sum of the weights of twisted paths of type σ. From definition, we have

$$\mathrm{Imm}_f(A) = \sum_{\sigma\in\mathfrak{S}_n} f(\sigma)X_\sigma,$$

where A is the path matrix described above. Choose a monomial m. This m will tell us how many times each edge is used. Let $X_{\sigma,m}$ be the number of twisted paths of type σ and weight m; then the coefficient of m in Imm_f is $\sum_{\sigma\in\mathfrak{S}_n} f(\sigma)X_{\sigma,m}$.

Now we construct a new network N', such that if e appears with multiplicity $k_e \geq 0$ in m, turn e into k_e edges $e_1, e_2, \cdots, e_{k_e}$. Let $Y_{\sigma,m}$ be the number of twisted paths of type σ in N' such that each edge is used exactly once. By symmetry, we have

$$Y_{\sigma,m} = \left(\prod_{e\in N'}(k_e)!\right) X_{\sigma,m}.$$

We will encode the information in N' using a product of $z_{[i,j]}$'s. Let $I_1, I_2, \cdots, I_k$ be the intersections of N' from right to left; each I_a will involve edges i_a through j_a inclusive, ordering edges from top to bottom. (This ordering will depend on which intersection you are on.) Then

$$z_{[i_1,j_1]}z_{[i_2,j_2]}\cdots z_{[i_k,j_k]} = \sum_{\sigma\in\mathfrak{S}_n}\sigma Y_{\sigma,m} = \sum_{\sigma\in\mathfrak{S}_n} C\sigma X_{\sigma,m},$$

where $C = \prod_{e\in N'}(k_e)!$. Applying f to both sides, we obtain $\sum_{\sigma\in\mathfrak{S}_n} f(\sigma)X_{\sigma,m} \geq 0$, as desired. $\square$

Using this result, we can show that the %-immanants given in Theorem 5.4 are either polynomial positive or polynomial negative.

Corollary 6.2. *Given that w is a nice permutation that avoids the patterns* $24153, 31524$, *and* 34127856. *Then,* $\operatorname{sgn}(w)\operatorname{Imm}_w^{\%}$ *is polynomial positive on networks.*

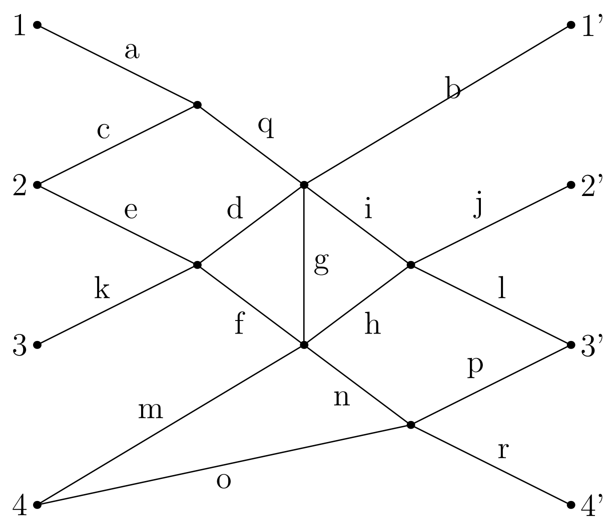
Proof. From Theorem 5.4, these signed %-immanants are equal to a sum of a Temperley-Lieb and a Kazhdan-Lusztig immanant. But both of these are polynomial positive, meaning that the entire %-immamant is polynomial positive, as desired. □

Notice that not all %-immanants are polynomial positive. For instance, we have the following counterexample to the polynomial positivity of %-immanants.

In particular, we can consider the following immanant:

$$\begin{pmatrix} 0 & 0 & * & * \\ * & * & * & * \\ * & * & * & 0 \\ * & * & * & 0 \end{pmatrix}$$

and the following network:



The resulting path matrix is the following matrix:

$$\begin{pmatrix} aqb & aqghj + aqij & aqghl + aqgnp + aqil & aqgnr \\ cqb + edb & cqghj + cqij + edghj + edij + efhj & cqghl + cqgnp + cqil + edghl + edgnp + edil + efhl + efnp & cqgnr + edgnr + efnr \\ kdb & kdghj + kdij + kfhj & kdghl + kdgnp + kdil + kfhl + kfnp & kdgnr + kfnr \\ 0 & mhj & mhl + mnp + op & mnr + or \end{pmatrix}.$$

Evaluating the immanant on this matrix yields us with the following polynomial, which has both positive and negative coefficients:

$$abd^2eg^2h^2jklmnqr + abd^2eg^2hjkmn^2pqr + abcdg^2h^2jklmnq^2r + abcdg^2hjkmn^2pq^2r$$

$$+abdefgh^2jklmnqr + abd^2eghijklmnqr + abdefghjkmn^2pqr + abcdghijklmnq^2r + abdefhijklmnqr - abcfghjknopq^2r.$$

As such, we can see that not all %-immanants are polynomial positive up to sign. A natural next step would be to generalize the result from Corollary 6.2. In particular, we have the following conjecture concerning when $\mathrm{Imm}_w^{\%}$ is polynomial positive:

Conjecture. *A %-immanant of the form* $\mathrm{Imm}_w^{\%}$ *is polynomial positive (or polynomial negative) if it avoids the patterns* $1324, 24153, 31524, 426153$.

Notice that our example doesn't come from a permutation w.

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