SIMPLICIAL COMPLEXES AND JEU DE TAQUIN THEORY

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Contents

1. Introduction

Young's lattice is a well-studied poset that consists of partitions ordered by inclusion, usually represented as Young diagrams. One intriguing object of study is the order complex associated with intervals in Young's lattice, as these simplicial complexes have nice geometric properties, such as thinness and shellability. Björner and Brenti [BB05] state that shellable and thin simplicial complexes are homeomorphic to balls or spheres, which allows one to deduce that the order complexes of intervals in Young's lattice are also balls or spheres. A natural question is then to think about other substructures within Young's lattice whose induced simplicial complex structure may also have nice geometric properties. Such substructures may be motivated by existing theory and operations that may be done to Young diagrams and tableaux.

The chains in the simplicial complex $\Delta([\mu, \lambda])$ are naturally described using some tableaux of shape λ/μ, with the maximal faces bijecting with the standard skew tableaux of shape λ/μ. A classical operation to perform on standard skew tableau are jeu de taquin slides, which provide a way to transform one standard skew tableau into another by iteratively sliding boxes within the tableau. This process not only gives rise to visually appealing algorithms but also establishes connections to other combinatorial structures. However, when applying a jeu de taquin slide to standard skew tableaux of the same shape, the shapes of the resulting standard skew tableaux may not be the same. To better understand and analyze these transformations, we restrict our focus to the dual equivalent tableaux, explored by Haiman [Hai92]: two tableaux are dual equivalent if every sequence of slides applied to S and T output tableaux of the same shape.

In this report, we aim to look at the geometric implications of jeu de taquin slides on simplicial complexes whose maximal faces are dual-equivalent standard skew tableaux.

1.1. Main Results and Organization.

In Section 2, we give preliminary background on Jeu de Taquin theory and simplicial complexes, as well as introducing our main technical lemmas. In Section 3, we introduce the main object of our study – dual equivalence complexes, which are simplicial complexes arising from dual equivalence classes of tableaux. We explain our motivating conjecture for these complexes, and show that the motivating conjecture is false. In Section 4, we study operations on indexing tableaux that yield isomorphic dual equivalence complexes, namely transposition and evacuation. In Section 5, we show that certain classes of standard tableaux do satisfy our motivating conjecture, namely superstandard tableaux, rectangular tableaux, dual reading tableaux, and certain hook-shaped tableaux. In Section 6, we study how K-jdt acts on the interior faces of dual equivalence complexes. Finally, in Section 7, we discuss possible future work and unsolved problems.

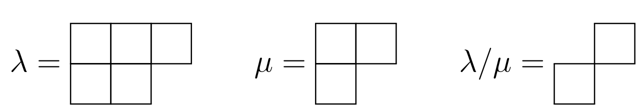
2. Preliminaries

2.1. Young Tableaux and Jeu de Taquin Theory.

In this subsection, we recall some standard definitions in jeu de taquin theory. First, recall that the shape of a ***Young diagram*** (which we sometimes refer to as a ***straight Young diagram***) is given by a sequence $\lambda = (\lambda_1, \lambda_2 \dots \lambda_k)$ of positive integers in weakly decreasing order. Young diagrams may be represented pictorially as left-justified arrays of boxes, with λ_i boxes in row i. We refer to the number of boxes in a Young diagram λ as $|\lambda|$.

A ***skew Young diagram*** is determined by a pair of straight diagrams (λ, μ) such that the Young diagram of λ contains the Young diagram of μ. In terms of sequences, this means that $\lambda_i \geq \mu_i$ for all i. Then, the skew diagram is given by the set theoretic difference of boxes in λ and μ.

Example 2.1. Let $\lambda = (3, 2)$ and $\mu = (2, 1)$. Then, (λ, μ) defines a skew diagram:



Definition 2.2. Let λ be a straight Young diagram. A ***tableau***[1] of shape λ is a labeling of the boxes of the diagram with integers from $[k] := \{1, \ldots, k\}$ such that the entries weakly increase along any row and along any column, and every integer in $[k]$ is the entry of at least one box. An ***increasing tableau*** is a tableau such that the entries strictly increase along any row and along any column. A ***standard tableau*** is a tableau with distinct entries.

These definitions are analogous for skew shapes $\lambda \setminus \mu$.

1	1	3
1	3	

(A) Example of a non-tableau.

1	1	3
1	2	

(B) Example of a tableau.

1	2	3
2	3	

(C) Example of an increasing tableau.

1	2	4
3	5	

(D) Example of a standard tableau.

FIGURE 1. Examples of tableaux and non-tableaux of shape $\lambda = (3, 2)$.

The following notation is often useful in describing the entries of a tableau:

Definition 2.3. The ***row reading word*** of a tableau T, abbreviated as $w_{\text{row}}(T)$, is the word obtained by concatenating the rows (from left to right), starting from the bottom and going up. The ***column reading word*** of T, $w_{\text{col}}(T)$, is the word obtained by concatenating the columns (read bottom up) from left to right.

For example, for the tableau in Figure 1D, its row reading word is 35124, and its column reading word is 31524.

Jeu de taquin, which we will often abbreviate as ***jdt***, is an algorithm that takes in a standard skew tableau and transforms it into a standard straight tableau via a series of moves, sometimes called ***slides***. We briefly sketch steps of jeu de taquin here, but we refer the reader to [Ful97, Chapter 1] for more details.

		1
		2
3	4	5

(A) A standard tableau of skew shape.

		1
	2	5
3	4	

(B) Applying one jeu de taquin move to the inner corner.

	1	5
	2	
3	4	

(C) A second jeu de taquin move.

	1	5
2	4	
3		

(D) A third jeu de taquin move.

1	4	5
2		
3		

(E) The tableau has been fully rectified.

FIGURE 2. Iterated jeu de taquin slides on a standard skew tableau of shape $(3, 3, 3)/(2, 2)$.

[1]What we call tableaux may be known as weakly-increasing tableaux.

As shown in Figure 2, at each step of jeu de taquin on a skew shape, one chooses an inner corner of the empty cells and slides it out of the diagram. In the first step, we slide out the empty corner adjacent to 2 and 4 by first swapping it with 2, since $2 < 4$, and then swapping it with 5. At each step of the process, the number of empty boxes in the diagram decreases by one; when there are no empty boxes left, we have rectified the tableau.

It is a nontrivial fact that if we start with a standard tableau, then at each step of jeu de taquin, we still have a standard tableau. It is an even more remarkable and non-obvious fact that the rectification is well-defined: we can pick inner corners to slide out in any order, and the resulting straight tableau will always be the same [Ful97, Chapter 3]

The ***Robinson-Schensted correspondence*** (abbreviated as RS) will also be a key tool for our report. Again, we will briefly describe the main idea of RS and refer the reader to [Ful97, Chapter 4] for details (and to [Ful97, Chapter 1] for details on insertion on tableaux, a preliminary for RS). There are many equivalent ways to formulate RS, but for us, RS is a bijection between permutations of $[n]$ and pairs (P, Q) of standard tableaux of a given shape λ with $|\lambda| = n$. We call $P(w)$ the ***insertion tableau*** and $Q(w)$ the ***recording tableau***. Dual equivalence classes, described in the next subsection, were originally motivated by their nice relationship with RS.

$$P(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad Q(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

FIGURE 3. The pair (P, Q) corresponding to the permutation $w = 12534$.

2.2. **Dual Equivalence.** In this subsection, we introduce ***dual equivalence classes*** of tableaux, which were defined by Haiman [Hai92].

First, we recall the definition of ***Knuth moves*** [Ful97, Chapter 2]. Given a word w, let xyz be a substring of w. A Knuth move on w is given by the following equivalences:

$$\begin{array}{lcll} \cdots xzy \cdots & \leftrightarrow & \cdots zxy \cdots & \text{if } x < z < y, \text{ or} \\ \cdots yzx \cdots & \leftrightarrow & \cdots yxz \cdots & \text{if } x < y < z. \end{array}$$

The motivation for studying dual equivalence is that it is precisely dual to Knuth equivalence, in a way made more precise by Proposition 2.4.

Proposition 2.4 ([Ful97, p. 191])**.** *Let w and z be permutations of $[n]$. The following statements are equivalent:*

- (i) *w^{-1} and z^{-1} are Knuth equivalent.*
- (ii) *w and z have the same recording tableau, i.e. $Q(w) = Q(z)$.*
- (iii) *w and z can be transformed into each other by a finite sequence of elementary dual Knuth transformations, where an elementary dual Knuth transformation on a permutation $x_1 \dots x_n$ is the interchange of two letters $x_i = k$ and $x_j = k + 1$, provided that one of the letters $k - 1$ or $k + 2$ occurs between them in the word.*

We say permutations w and z of $[n]$ are ***dual equivalent***, denoted $w \sim z$, if they satisfy the equivalent properties in Proposition 2.4. Dual equivalence is an equivalence relation on the set of permutations of $[n]$.

Given a standard straight tableau Q with n boxes, the set of permutations w of $[n]$ having recording tableau Q forms a dual equivalence class, by Proposition 2.4(ii). Hence, we define this set to be the ***dual equivalence class indexed by*** Q.

Example 2.5. The permutations 2341 and 1342 are dual equivalent, since they differ by an elementary dual Knuth transformation that interchanges 1 and 2, since 3 appears between them. Moreover, the recording tableau of both 2341 and 1342 is $\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$, and hence 2341 and 1342 are elements of the dual equivalence class indexed by $\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$.

Proposition 2.6 ([Hai92, Theorem 2.6, Lemma 2.11])**.** *Let S and T be standard skew tableaux of the same shape. The following statements are equivalent:*

- (i) *The permutations $w_{\text{row}}(S)$ and $w_{\text{row}}(T)$ are dual equivalent.*
- (ii) *Every sequence of slides, when applied to S and T, yields two tableaux of the same shape.*
- (iii) *There is a rectifying sequence of slides which, when applied to S and T, yields pairs tableaux of the same shape at each intermediate step.*
- (iv) *S and T can be transformed into each other by a finite sequence of elementary dual Knuth transformations on their reading words.*

We say standard skew tableaux S and T of the same shape are ***dual equivalent***, denoted $S \sim T$, if they satisfy the equivalent properties in Proposition 2.6.

We lay out some useful known facts about the dual equivalence relation.

Corollary 2.7 ([Hai92, Corollary 2.5])**.** *Any pair of standard straight tableaux of the same shape are dual equivalent.*

Corollary 2.7 follows from applying Proposition 2.6(ii) to the empty sequence of slides.

Corollary 2.8 ([Hai92])**.** *Let S and T be standard skew tablaeux, and let S' and T' be obtained from S and T by the same sequence of slides. Then, S and T are dual equivalent if, and only if, S' and T' are dual equivalent.*

The forwards direction of Corollary 2.8 follows from applying Proposition 2.6(ii) to every sequence of slides starting with the given sequence of slides, while the backwards direction follows from appending the given sequence of slides to the sequence obtained from Proposition 2.6(iii).

Definition 2.9. Let T be a standard tableau of shape λ/μ. The ***dual equivalence class of tableau*** containing T is denoted $[T]_D$.

2.3. **Simplicial Complexes.** A ***simplicial complex*** Δ with vertex set V is a nonempty family of subsets of V, called ***faces***, such that if $F \subset G \in \Delta$, then $F \in \Delta$. Moreover, we assume that $\{v\} \in \Delta$ for all $v \in V$. A ***facet*** of Δ is a maximal face of the complex. The ***dimension of a face*** F is $\dim F \coloneqq |F| - 1$, and the ***codimension of a face*** F is $\operatorname{codim} F \coloneqq \dim \Delta - \dim F$. The ***dimension of a simplicial complex*** Δ is given by $\dim \Delta \coloneqq \max_{F \in \Delta} \dim F$. A simplicial complex is called ***pure*** if all of its facets have the same dimension.

Given a finite poset P, we can define its associated ***order complex*** $\Delta(P)$ as the simplicial complex of all chains of P. The dimension of a face of $\Delta(P)$ is one less than the length of its corresponding chain; so, vertices of $\Delta(P)$ correspond to elements of P, and facets of $\Delta(P)$ correspond to maximal chains of P.

In particular, we obtain an order complex from ***Young's Lattice***, the poset of Young diagrams with the ordering given by inclusion. However, since this simplicial complex is

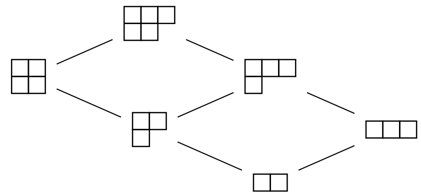


FIGURE 4. Hasse diagram of [$\ydiagram{2}$, $\ydiagram{3,2}$] in Young's lattice.

infinite, we instead consider the order complex of intervals $[\mu, \lambda]$ in Young's lattice, which are finite.

Recall that the faces of the order complex of $\Delta([\mu, \lambda])$ containing μ and λ are chains of partitions of the form

$$\mu = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k = \lambda.$$

We associate this chain to the tableau of shape λ/μ that assigns the integer ℓ to the boxes in $\lambda_\ell/\lambda_{\ell-1}$. This correspondence also gives a bijection between facets of $\Delta([\mu, \lambda])$ and standard tableaux of shape λ/μ.

Definition 2.10. The ***boundary*** of a pure simplicial complex Δ consists of all faces $F \in \Delta$ such that there exists a codimension 1 face G with the property that $F \subseteq G$ and G lies in exactly one facet of Δ. The ***interior*** of Δ consists of all faces of Δ not in the boundary.

Proposition 2.11. *The correspondence between faces of $\Delta([\mu, \lambda])$ and weakly-increasing tableaux of shape λ/μ determines a bijection between interior faces of $\Delta([\mu, \lambda])$ and increasing tableaux of shape λ/μ.*

Proof. First, we prove that all interior faces in $\Delta([\mu, \lambda])$ contain both μ and λ. Equivalently, we prove that all faces not containing either μ or λ are a boundary face. Let G be a face not containing μ (resp. λ), and let F be a facet containing G. Since $\Delta([\mu, \lambda])$ is pure, F has codimension 0. Let F' be F without the vertex μ (resp. λ). Then, the codimension of F' is 1, the only facet containing F' is F, and G is contained in F'. Hence, G is a boundary face.

We now prove that interior faces are mapped to increasing tableaux via this association. Let G be an interior face. Consider two boxes b_1, b_2 of weakly increasing tableau G with the same filling ℓ. Consider the Young diagram ρ_1 (resp. ρ_3) containing all boxes with numbers strictly less than ℓ in the tableau G, and all boxes on the upper-left to b_1, b_2, exclusively (resp. inclusively). Note that $G' = G \cup \{\rho_1, \rho_3\}$ is a face, since the new vertices contain $G^{(\ell-1)}$ and are contained in $G^{(\ell)}$. Let F' be a facet containing G', say

$$F' = \{\mu \subset \cdots \subset \rho_1 \subset \rho_2 \subset \rho_3 \subset \cdots \subset \lambda\}.$$

Without loss of generality, $\rho_2 = \rho_1 \cup \{b_1\}$. Let $F = F' \setminus \{\rho_2\}$. Note that $F \supset G$ has codimension 1. Since G is an interior face, there is a facet F'' which is distinct from F' and also contains F. By the structure of the Young's lattice, this facet must be

$$F'' = \{\mu \subset \cdots \subset \rho_1 \subset \rho_1 \cup \{b_2\} \subset \rho_3 \subset \cdots \subset \lambda\}.$$

Since both $\rho_1 \cup \{b_1\}$ and $\rho_1 \cup \{b_2\}$ are Young diagrams, it follows that b_1 and b_2 are not in the same column or row. Therefore, the tableau G is an increasing tableaux.

Next, we prove that boundary faces containing μ and λ are mapped to weakly increasing tableaux that are not increasing tableaux. Let G be a boundary face. Hence, there exists a face F' with codimension 1, say

$$F = \{\mu \subset \cdots \subset G^{(\ell-1)} \subset \cdots \subset \rho_1 \subset \rho_3 \subset \cdots \subset G^{(\ell)} \subset \cdots \subset \lambda\},$$

where $\rho_3 \setminus \rho_1 = \{b_1, b_2\}$, and such that there's only one facet F' that contains it, say

$$F' = \{\mu \subset \cdots \subset G^{(\ell-1)} \subset \cdots \subset \rho_1 \subset \rho_1 \cup \{b_1\} \subset \rho_3 \subset \cdots \subset G^{(\ell)} \subset \cdots \subset \lambda\}.$$

Therefore,

$$F'' = \{\mu \subset \cdots \subset G^{(\ell-1)} \subset \cdots \subset \rho_1 \subset \rho_1 \cup \{b_2\} \subset \rho_3 \subset \cdots \subset G^{(\ell)} \subset \cdots \subset \lambda\}$$

is not a face, meaning that $\rho_1 \cup \{b_2\}$ is not a Young diagram, while ρ_1 and $\rho_3 = \rho_1 \cup \{b_1, b_2\}$ are Young diagrams. Consequently, boxes b_1 and b_2, which both are assigned ℓ in G, since they both are in $G^{(\ell)} \setminus G^{(\ell-1)}$, must either be in the same column or the same row. Hence, G is not an increasing tableaux. $\square$

Theorem 2.12 is a classical result about such order complexes.

Theorem 2.12 ([BB05, Theorem 2.7.7])**.** *The order complex of the closed interval $[\mu, \lambda]$ of Young's lattice is piecewise-linear homeomorphic to a ball. Furthermore, the order complex of the open interval (μ, λ) is piecewise-linear hoomeomorphic to either a ball or a sphere.*

2.4. **Simplicial Isomorphisms.**

Definition 2.13. Let Δ_1, Δ_2 be simplicial complexes with vertex sets V_1, V_2. An injection $f\colon V_1 \to V_2$ is a ***simplicial map of*** Δ_1 ***on*** Δ_2 if, for all $S \in \Delta_1$, $\{f(v) : v \in S\} \in \Delta_2$. A bijection $f\colon V_1 \to V_2$ is a ***simplicial isomorphism of*** Δ_1 ***on*** Δ_2 if f and f^{-1} are simplicial maps.

Let B_1, B_2 be finite sets. Let $f\colon B_1 \to B_2$ be a function. As an abuse of notation, define $f\colon 2^{B_1} \to 2^{B_2}$ to be the map given by $f(v) = \{f(b) : b \in v\}$, and define $f\colon 2^{2^{B_1}} \to 2^{2^{B_2}}$ to be the map given by $f(S) = \{f(v) : v \in S\}$. Note that, if $f\colon B_1 \to B_2$ is an injection (resp. bijection), then $f\colon 2^{B_1} \to 2^{B_2}$ and $f\colon 2^{2^{B_1}} \to 2^{2^{B_2}}$ are also injections (resp. bijections).

Propositions 2.14 and 2.15 are key technical tools of this report.

Proposition 2.14 (Element-to-element injection induces simplicial map)**.** *Let B_1, B_2 be finite sets. Let Δ_1, Δ_2 be simplicial complexes with vertex sets $V_1 \subset 2^{B_1}, V_2 \subset 2^{B_2}$. Let $f\colon B_1 \to B_2$ be an injection. Assume that the injection $f\colon 2^{2^{B_1}} \to 2^{2^{B_2}}$ maps facets of Δ_1 to faces of Δ_2. Then, the image of V_1 is a subset of V_2 under the injection $f\colon 2^{B_1} \to 2^{B_2}$. Moreover, the injection $f\colon V_1 \to V_2$ is a simplicial map of Δ_1 on Δ_2.*

We say that the simplicial map $f\colon V_1 \to V_2$ of Δ_1 on Δ_2 is induced by the injection $f\colon B_1 \to B_2$. The main application of Proposition 2.14 is when B_1, B_2 are sets of boxes in $\mathbb{N} \times \mathbb{N}$ (in particular, these sets of boxes are skew diagrams), V_1, V_2 are sets of skew diagrams, and Δ_1, Δ_2 are (isomorphic to) simplicial subcomplexes of Young's lattice.

Proof. First, we prove that the injection $f\colon 2^{2^{B_1}} \to 2^{2^{B_2}}$ maps faces of Δ_1 to faces of Δ_2. Let $S \in \Delta_1$ be a face, contained by a facet $F \in \Delta_1$. Hence, $f(S) = \{f(v) : v \in S\} \subset \{f(v) : v \in F\} = f(F)$. Since $f(F)$ is a face of Δ_2, it follows that $f(S)$ is a face of Δ_2.

Second, we prove that the injection $f\colon 2^{B_1} \to 2^{B_2}$ maps vertices of V_1 to vertices of V_2. Let $v \in V_1$ be arbitrary. Hence, $\{v\} \in \Delta_1$, consequently $f(\{v\}) = \{f(v)\} \in \Delta_2$, and finally $f(v) \in V_2$. We can now consider the restriction of $f\colon 2^{B_1} \to 2^{B_2}$ to the injection $f\colon V_1 \to V_2$. This restriction is a simplicial map of Δ_1 on Δ_2. $\square$

Proposition 2.15 (Element-to-element bijection induces simplicial isomorphism)**.** *Let B_1, B_2 be finite sets. Let Δ_1, Δ_2 be simplicial complexes with vertex sets $V_1 \subset 2^{B_1}, V_2 \subset 2^{B_2}$. Let $f\colon B_1 \to B_2$ be a bijection. Assume that the bijection $f\colon 2^{2^{B_1}} \to 2^{2^{B_2}}$ bijects facets of Δ_1 to facets of Δ_2. Then, the image of V_1 is V_2 under the bijection $f\colon 2^{B_1} \to 2^{B_2}$. Moreover, the bijection $f\colon V_1 \to V_2$ is a simplicial isomorphism of Δ_1 on Δ_2.*

Proposition 2.15 follows from Proposition 2.14 by applying it on f and on f^{-1}.

3. Simplicial Complexes from Dual Equivalence Classes and the Motivating Conjecture

3.1. **Simplicial Complexes from Dual Equivalence Classes.** In this paper, the main objects of study are the order complexes generated by shape sequences of standard skew tableaux belonging to the same dual equivalence class.

Definition 3.1. Let T be a standard skew tableau of shape λ/μ. Let $\Delta_{\text{tab}}(T)$ be the simplicial complex with facets given by the standard skew tableaux in the dual equivalence class containing T. We call this the ***dual equivalence complex containing*** T.

Note that $\Delta_{\text{tab}}(T)$ is a simplicial subcomplex of $\Delta([\mu, \lambda])$.

The other notion of dual equivalence, defined on permutations rather than standard skew tableaux, gives rise to an similar class of complexes. We first adapt a corresponding notion of chains of tableaux to permutations.

Definition 3.2. Given a permutation w of $[n]$, the ***shape sequence of*** w is the sequence of nested sets

$$\varnothing \subseteq w^{-1}([1]) \subseteq \cdots \subseteq w^{-1}([n-1]) \subseteq w^{-1}([n]).$$

Definition 3.3. Let Q be a standard straight tableau with n boxes. Let $\Delta_{\text{word}}(Q)$ be the simplicial complex generated by the shape sequences of permutations of $[n]$ in the dual equivalence class indexed by Q. We call this the ***dual equivalence complex indexed by*** Q.

Lemma 3.4 explicitly transfers the similarities of dual equivalence on tableaux and on permutations to this new setup of simplicial complexes.

Lemma 3.4. *Let T be a standard skew tableau. Let Q be the recording tableau of $w_{\text{row}}(T)$. Then $\Delta_{\text{tab}}(T) \cong \Delta_{\text{word}}(Q)$.*

Proof. Let λ/μ be the shape of T, and let $n = |\lambda/\mu|$, the number of boxes in T. We identify the vertices of $\Delta_{\text{tab}}(T)$ with skew shapes skewed by μ. Let f be the bijection from $[n]$ to the skew diagram of λ/μ that sends i to the i^{th} box of the skew diagram of λ/μ in row reading order. Let f be the bijection from the skew diagram of λ/μ to $[n]$ that sends the i^{th} box of the skew diagram of λ/μ in row reading order to i. Note that f sends the shape sequence of a standard skew tableau S to the shape sequence of the reading word of S. Therefore, it maps the facets of $\Delta_{\text{tab}}(T)$ to the facets of $\Delta_{\text{word}}(Q)$. From Proposition 2.15, f induces a simplicial isomorphism of $\Delta_{\text{tab}}(T)$ on $\Delta_{\text{word}}(Q)$. □

Lemma 3.4 presents two notations for the simplicial complexes arising from dual equivalence classes. Consequently, in each context, we opt for the notation that best suits our purposes. In-text, we may abuse language and refer to both as a ***dual equivalence complex*** if the method of referring to such a complex is clear, as they refer to isomorphic notions.

3.2. The Motivating Conjecture.

Conjecture 3.5 (False)**.** *Let Q be a standard straight tableaux. Then, $\Delta_{\text{word}}(Q)$ is homeomorphic to a ball.*

This conjecture is motivated by the fact that all tableaux of a straight shape λ form a dual equivalence class [Hai92, Proposition 2.4] and that the order complex associated to λ is homeomorphic to a ball [Bjö80, Theorem 2.7.7]. Furthermore, JDT preserves dual equivalence classes [Hai92, Lemma 2.11]. Therefore, one might guess that JDT induces isomorphisms of the simplicial complexes associated to dual equivalence classes, and thus that after rectifying a skew shape via JDT, the resulting isomorphic simplicial complex is homeomorphic to a ball.

By exhaustive computation, $\Delta_{\text{word}}(Q)$ is shellable, thus homeomorphic to a ball, for all standard straight tableaux Q with at most 6 boxes. However, there are standard straight tableaux Q for which $\Delta_{\text{word}}(Q)$ is not shellable; computed exhaustively, all counterexamples with 7 boxes are displayed in Figure 5, and all counterexamples with 8 boxes are displayed in Figure 6.

$$\begin{ytableau} 1 & 2 & 3 & 7 \\ 4 & 6 \\ 5 \end{ytableau} \qquad \begin{ytableau} 1 & 4 & 5 \\ 2 & 6 \\ 3 \\ 7 \end{ytableau}$$

FIGURE 5. Tableaux Q with 7 boxes that are counterexamples of Conjecture 3.5

$$\begin{ytableau} 1 & 2 & 3 & 7 & 8 \\ 4 & 6 \\ 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 4 & 8 \\ 5 & 7 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 5 & 7 \\ 2 & 4 & 8 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 & 7 \\ 3 & 4 & 8 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 4 & 8 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 4 & 8 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 5 & 7 \\ 2 & 6 \\ 4 & 8 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 & 7 \\ 2 & 6 \\ 5 & 8 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 & 8 \\ 2 & 7 \\ 5 \\ 6 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 & 4 & 8 \\ 3 & 7 \\ 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 8 \\ 4 & 7 \\ 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 4 & 5 & 8 \\ 2 & 6 \\ 3 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 8 \\ 4 & 6 \\ 5 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 4 & 5 & 7 \\ 2 & 6 \\ 3 \\ 8 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 7 \\ 4 & 6 \\ 5 \\ 8 \end{ytableau} \quad \begin{ytableau} 1 & 4 & 5 & 6 \\ 2 & 7 \\ 3 \\ 8 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 5 & 6 \\ 2 & 7 \\ 4 \\ 8 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 7 \\ 4 \\ 8 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 & 5 \\ 3 & 6 & 8 \\ 4 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 \\ 3 & 6 & 8 \\ 5 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 7 \\ 2 & 4 \\ 5 & 8 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 7 \\ 3 & 4 \\ 5 & 8 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 6 \\ 2 & 4 \\ 5 & 8 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 6 \\ 3 & 4 \\ 5 & 8 \\ 7 \end{ytableau} \quad \begin{ytableau} 1 & 5 & 6 \\ 2 & 7 \\ 3 \\ 4 \\ 8 \end{ytableau} \quad \begin{ytableau} 1 & 4 & 5 \\ 2 & 6 \\ 3 \\ 7 \\ 8 \end{ytableau}$$

FIGURE 6. Tableaux Q with 8 boxes that are counterexamples of Conjecture 3.5

Letting Q being the second tableau in Figure 5, we explicitly show that the simplicial complex $\Delta_{\text{word}}(Q)$ is not Cohen-Macaulay, and hence not homeomorphic to a ball, which disproves the motivating conjecture.

To look at $\Delta_{\text{word}}(Q)$ for

$$Q = \begin{ytableau} 1 & 4 & 5 \\ 2 & 6 \\ 3 \\ 7 \end{ytableau},$$

consider a word w such that $Q(w) = Q$ — for our purposes, consider $w = 4325761$. We may compute the dual equivalence class of w and $[w]_D$, and by taking the inverses of these words (as permutations), we may compute the shape sequences that form the facets of $\Delta_{\text{word}}(Q)$. Here is a list of all these inverted words:

3247651	3274651	3276415	3276451	3427651	3472651	3476215
3476251	3724651	3726415	3726451	3742165	3742615	3742651
3746215	3746251	3762145	3762415	3762451	7321465	7321645
7324165	7324615	7324651	7326145	7326415	7326451	7342165
7342615	7342651	7346215	7346251	7362145	7362415	7362451

One easy computational check we can perform to determine whether or not a simplicial complex is shellable is to test if the simplicial complex is also Cohen-Macaulay, defined in Definition 3.7.

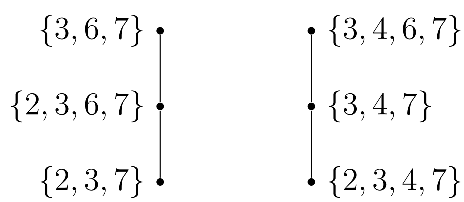
Definition 3.6. The ***link*** of a face $F \in \Delta$ is the simplicial complex $\operatorname{link}_\Delta(F)$ consisting of all faces $G \in \Delta$ such that $G \cap F = \varnothing$ and $G \cup F \in \Delta$.

Definition 3.7. A simplicial complex Δ is ***Cohen-Macaulay*** if for every $F \in \Delta$ and $i < \dim \operatorname{link}_\Delta(F)$, the relative homology group $\tilde{H}_i(\operatorname{link}_\Delta(F)) = 0$.

For $Q = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$, consider the following face F' in $\Delta_{\text{word}}(Q)$:

$$F' = \{\{3\}, \{3,7\}, \{2,3,4,6,7\}, \{1,2,3,4,6,7\}\}.$$

The link of this face is a collection of edges:



The zeroth reduced homology of this link is nontrivial since it has two connected components. Taking the suspension on this by the vertices $\{3\}$ and $\{7\}$ lifts to nontrivial first homology, and taking the suspension again by the vertices $\{1,2,3,4,6,7\}$ and $\{2,3,4,5,6,7\}$ gives nontrivial second homology. In any case, this simplicial complex is not Cohen-Macaulay.

From [Wac04], triangulations of a ball or a sphere must be Cohen-Macaulay, so this simplicial complex is not homeomorphic to a ball or a sphere.

4. Operations on Indexing Tableaux

Note that the two indexing tableaux with 7 boxes in Figure 5, which contradict Conjecture 3.5, are transposes of each other. Moreover, they are ***self-evacuating*** tableaux (evacuation is defined in Definition 4.2)[2]. This observation motivates Question 4.1.

Question 4.1. Which operations on indexing tableaux give isomorphic dual equivalence complexes?

In this section, we show that transposition and evacuation are operations which produce these isomorphisms.

[2]This observation was due to Vic Reiner.

4.1. **Evacuation.** We first describe the process of evacuation on straight tableaux. Evacuation can be defined in several equivalent ways; we follow the definition from [But94].

Definition 4.2. Let T be a straight standard tableau with n boxes. Replace each entry j of T with $j^* = n+1-j$, rotate the tableau 180°, and rectify the resulting standard skew tableau to obtain a standard straight tableau, called the ***evacuation*** of T, denoted by $\epsilon(T)$.

Example 4.3. This example shows the steps of evacuation; the leftmost tableau is T and the rightmost is $\epsilon(T)$.

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{\text{replace entries}} \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{\text{rotate } 180^\circ} \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{2} \\ \hline \multicolumn{1}{|c|}{1} & 3 \\ \hline\end{array} \xrightarrow{\text{rectify}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \epsilon(T)$$

Theorem 4.4. *Let Q be a straight standard tableaux. Then, $\Delta_{\text{word}}(Q) \cong \Delta_{\text{word}}(\epsilon(Q))$.*

Proof. Recall that vertices v of $\Delta_{\text{word}}(Q)$ are subsets of $[n]$, since they are elements of shape sequences of permutations. Define the reverse complement map $f\colon 2^{[n]} \to 2^{[n]}$ by

$$f(v) = \{a^* : a \notin v\}.$$

Let S be a facet in $\Delta_{\text{word}}(Q)$. There exists a permutation w with recording tableau Q such that S is the shape sequence of w. Explicitly, S is the set consisting of

$$\varnothing, \quad \{w^{-1}(1)\}, \quad \dots, \quad \{w^{-1}(1), \dots, w^{-1}(n-1)\}, \quad \{w^{-1}(1), \dots, w^{-1}(n-1), w^{-1}(n)\}.$$

Hence, $f(S)$ is the set consisting of

$$\{w^{-1}(1)^*, w^{-1}(2)^*, \dots, w^{-1}(n)^*\}, \quad \{w^{-1}(2)^*, \dots, w^{-1}(n)^*\}, \quad \dots, \quad \{w^{-1}(n)^*\}, \quad \varnothing,$$

that is,

$$\{w^{*-1}(1^*), w^{*-1}(2^*), \dots, w^{*-1}(n^*)\}, \quad \{w^{*-1}(2^*), \dots, w^{*-1}(n^*)\}, \quad \dots, \quad \{w^{*-1}(n^*)\}, \quad \varnothing.$$

Note that this corresponds to the elements of the shape sequence of $w^* = w_n^* \cdots w_2^* w_1^*$. Now, the Duality Theorem [Ful97, p. 184] says that $\mathrm{RS}(w^*) = (\epsilon(P), \epsilon(Q))$. Hence, w^* is a permutation in the dual equivalence class indexed by $\epsilon(Q)$, and thus $f(S)$ is a face in $\Delta_{\text{word}}(\epsilon(Q))$.

Therefore, f is a simplicial map. By the same argument applied to $\epsilon(Q)$, f^{-1} is also a simplicial map, which finally implies that f is an simplicial isomorphism between $\Delta_{\text{word}}(Q)$ and $\Delta_{\text{word}}(\epsilon(Q))$. ☐

Example 4.5. Take Q to be $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$; then, we previously computed $\epsilon(Q) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ in Example 4.3.

The permutations with recording tableau Q are 312 and 213, whose shape sequences are, respectively, $\varnothing, 2, 23, 123$ and $\varnothing, 2, 12, 123$. The permutations with recording tableau $\epsilon(Q)$ are 132 and 231, whose shape sequences are, respectively, $\varnothing, 1, 13, 123$ and $\varnothing, 3, 13, 123$. The simplicial isomorphism $f\colon 2^{[n]} \to 2^{[n]}$, defined in the proof of Theorem 4.4, sends

$$\varnothing \mapsto 1^*2^*3^* = 123, \quad 1 \mapsto 2^*3^* = 12, \quad 2 \mapsto 1^*3^* = 13, \quad 3 \mapsto 1^*2^* = 23,$$
$$12 \mapsto 3^* = 1, \quad 13 \mapsto 2^* = 2, \quad 23 \mapsto 1^* = 3, \quad 123 \mapsto \varnothing.$$

Hence, the simplicial isomorphism f sends the facet in $\Delta_{\text{word}}(Q)$ corresponding to 312 to the facet in $\Delta_{\text{word}}(\epsilon(Q))$ corresponding to 132, and sends the facet in $\Delta_{\text{word}}(Q)$ corresponding to 213 to the facet in $\Delta_{\text{word}}(\epsilon(Q))$ corresponding to 231.

4.2. **Transposition.** We may use the above result for evacuation to show that transposition on Q-tableaux is also a simplicial isomorphism.

Theorem 4.6. *Let Q be a standard straight tableaux. Then $\Delta_{\mathrm{word}}(Q) \cong \Delta_{\mathrm{word}}(Q^\top)$.*

Proof. Recall that vertices v of $\Delta_{\mathrm{word}}(Q)$ are subsets of $[n]$, since they are elements of shape sequences of permutations. Define the complement map $g\colon 2^{[n]} \to 2^{[n]}$ by

$$g(v) = [n] \setminus v.$$

Let S be a facet in $\Delta_{\mathrm{word}}(Q)$. There exists a permutation w with recording tableau Q such that S is the shape sequence of w. Explicitly, S is the set consisting of

$$\varnothing, \left\{w^{-1}(1)\right\}, \ldots, \left\{w^{-1}(1), \ldots, w^{-1}(n-1)\right\}, \left\{w^{-1}(1), \ldots, w^{-1}(n-1), w^{-1}(n)\right\}.$$

Hence, $f(S)$ is the set consisting of

$$\left\{w^{-1}(1), \ldots, w^{-1}(n-1), w^{-1}(n)\right\}, \left\{w^{-1}(2), \ldots, w^{-1}(n-1), w^{-1}(n)\right\}, \ldots, \left\{w^{-1}(n)\right\}, \varnothing,$$

that is

$$\varnothing, \left\{w^{-1}(n)\right\}, \ldots, \left\{w^{-1}(2), \ldots, w^{-1}(n-1), w^{-1}(n)\right\}, \left\{w^{-1}(1), \ldots, w^{-1}(n-1), w^{-1}(n)\right\}.$$

As a shape sequence, this is generated from taking prefixes of the permutation $w^{-1}w_0$, where w_0 is the reversal permutation $n, \ldots, 2, 1$, which we see corresponds to the shape sequence of the permutation $w_0 w$.

It is a fact that if $\mathrm{RS}(w) = (P, Q)$, then for the reversal of w, $w^{rev} = ww_0$ where w_0 is the reversal permutation, we have that $\mathrm{RS}(ww_0) = (P^\top, \epsilon(Q^\top))$ ([Ful97, p.208]). Applying the Duality Theorem and replacing w with w^*, we see that $\mathrm{RS}(w^* w_0) = \mathrm{RS}(w_0 w) = (\epsilon(P^\top), Q^\top)$, so indeed $Q(w_0 w) = Q^\top$. As such, $g(S)$ is a facet in $\Delta_{\mathrm{word}}(Q^\top)$, and therefore g is a simplicial map.

Using the same argument as above, we see that g^{-1} is similarly a simplicial map, so g is a simplicial isomorphism between $\Delta_{\mathrm{word}}(Q)$ and $\Delta_{\mathrm{word}}(Q^\top)$.

□

Example 4.7. For a slightly larger example, consider $Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, so $Q^\top = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$. The words such that $\mathrm{RS}(w) = Q$ are $2341, 1342$, and 1243. Inverting and taking prefixes gives the vertex set $\{\varnothing, \{1\}, \{4\}, \{1,2\}, \{1,4\}, \{1,2,4\}, \{1,2,3,4\}\}$. On the other hand, the words such that $\mathrm{RS}(w) = Q^\top$ are $3214, 4213$, and 4312. Doing the same procedure gives the vertex set $\{\varnothing, \{3\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$, which consists of the complements of the vertices in $\Delta_{\mathrm{word}}(Q)$ by Theorem 4.6. One can check that taking the complements also sends facets to facets – for instance, the facet $\{\varnothing, \{4\}, \{1,4\}, \{1,2,4\}, \{1,2,3,4\}\}$ is sent to the facet $\{\varnothing, \{3\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$. Doing this for the other two facets illustrates that this is indeed an isomorphism.

5. Special Tableaux Families

We now pivot to investigating which straight tableaux Q index dual equivalence complexes that are homeomorphic to balls. We study four classes of such special tableaux.

5.1. **Dual Reading Tableaux.** In this subsection, we show that a special class of tableaux, ***dual reading tableaux***, index (via RSK) simplicial complexes which are homeomorphic to balls. Dual reading tableaux were first defined by Edelman and Greene [EG87]:

Definition 5.1 ([EG87])**.** Let λ be a straight Young diagram. Then, the (unique) dual reading tableau of shape λ, $Q(\lambda)$, is constructed by the following process:

(i) Label the boxes of λ with $1, 2, \ldots, n$ in the order of the row reading word.
(ii) Sort the columns in increasing order.

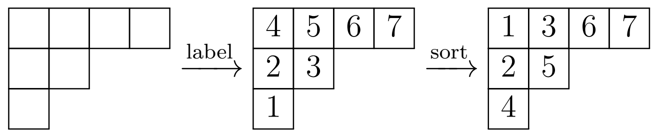


FIGURE 7. Algorithm for obtaining the dual reading tableau of shape $(4, 2, 1)$, the rightmost tableau in the figure.

In [EG87], the motivation for looking at such tableaux is a nice RSK property they exhibit. The same property will be useful to us:

Lemma 5.2 ([EG87, Lemma 6.16])**.** *For any tableau P of shape λ, we have*

$$\mathrm{RS}(w_{row}(P)) = (P, Q(\lambda)).$$

Now, we can prove the main result of this subsection:

Theorem 5.3. *If Q is the dual reading tableau of shape λ, then $\Delta_{word}(Q)$ is isomorphic to $\Delta([\emptyset, \lambda])$ and thus homeomorphic to a ball.*

Proof. We will apply Proposition 2.15 to give an isomorphism $\Delta_{\text{word}}(Q) \to \Delta([\emptyset, \lambda])$. As in the previous application of Proposition 2.15, let $\mu_1 \setminus \mu_2$ be some skew shape, and consider the dual equivalence class of tableaux of shape $\mu_1 \setminus \mu_2$ indexed by Q; let B_1 be the set of boxes of $\mu_1 \setminus \mu_2$, and let B_2 be the set of boxes of λ. We define $f : B_1 \to B_2$ to send the ith box of B_1, where the order is given by the row reading order, to the ith box of λ. Since f is a bijection, by Proposition 2.15 it suffices to show that f maps facets to facets.

A facet of the dual equivalence complex is given by a standard tableau T of shape $\mu_1 \setminus \mu_2$. Applying JDT to T, by Lemma 5.2, we obtain the tableau P whose row reading word is $w_{\text{row}}(T)$. Thus, on facets, f restricts to JDT, which maps standard Young tableaux to standard Young tableaux, and thus facets to facets, as desired. □

5.2. **Rectangular Tableaux.** The main result of this section is:

Theorem 5.4. *Let Q be a standard tableau of rectangular shape λ. Then $\Delta_{\text{word}}(Q)$ is isomorphic, as a simplicial complex, to the order complex $\Delta[\emptyset, \lambda]$. Moreover, for any skew tableau T that rectifies to a rectangular tableau, any JDT move on T yielding a tableau T' is the restriction of a simplicial isomorphism $\Delta_{\text{tab}}(T) \to \Delta_{\text{tab}}(T')$.*

By [Bjö80], this implies that $\Delta(Q)$ for Q standard of rectangular shape is homeomorphic to a ball.

We start by proving a lemma about RSK and rectangular tableaux.

Lemma 5.5. *Let P, Q be standard tableaux of the same rectangular shape. Let Q^{flip} be obtained by reflecting Q through a horizontal axis, and let w be the word obtained by reading the entries of P in the order of the boxes of Q^{flip}. Then, $\mathrm{RS}(w) = (P, Q)$.*

Example 5.6. Let

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 7 & 8 & 9 \\ \hline 4 & 10 & 11 & 12 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 9 \\ \hline 4 & 6 & 8 & 11 \\ \hline 5 & 7 & 10 & 12 \\ \hline \end{array}$$

Then,

$$Q^{\text{flip}} = \begin{array}{|c|c|c|c|}\hline 5 & 7 & 10 & 12 \\ \hline 4 & 6 & 8 & 11 \\ \hline 1 & 2 & 3 & 9 \\ \hline \end{array}$$

and

$$\mathrm{RS}^{-1}(P, Q) = [4, 10, 11, 3, 1, 7, 2, 8, 12, 5, 9, 6].$$

Proof. First, assume that Q is row superstandard. Then, w is just the standard row reading word of P. Furthermore, when λ is a rectangle we see that $Q(\lambda)$ is row superstandard. Therefore, by [EG87, Lemma 6.16], we have $\mathrm{RS}(w) = (P, Q)$, as desired.

Now inductively that for some w whose insertion tableau is P, $\mathrm{RS}(w)$ gives us (P, Q), where w is obtained by reading the entries of P in an order $a_1, a_2 \dots a_k$ given by the order of the boxes of Q^{flip}. Applying some Knuth move to w transposes two of its consecutive entries, giving a new order $a_1, a_2 \dots a_{i+1}, a_i \dots a_k$ on the entries of P. However, by lemma 2.11 of [Hai92], applying a Knuth move to w alters the recording table Q by a dual equivalence move, swapping the corresponding boxes $i, i+1$. Thus, the new order $a_1, a_2 \dots a_{i+1}, a+i \dots a_k$ of the entries of P is still given by the order of the boxes of the recording tableau.

Knuth moves preserve the insertion tableau P, and furthermore every word w with insertion tableau P can be obtained by applying Knuth moves to the row reading word of P [Ful97, Chapter 4]. Thus, this argument inductively shows the claim for all w such that $\mathrm{RS}(w)$ gives tableaux with rectangular shapes, as desired.

□

Now, we can prove Theorem 5.4:

Proof. Let Q be a rectangular shape λ with m rows, n columns, and entries indexed by $q_{i,j}$. Let be any skew shape $\mu_1 \setminus \mu_2$ with n boxes, and consider the dual equivalence class of tableaux of shape $\mu_1 \setminus \mu_2$ indexed by Q. We will use Proposition 2.15 to give a simplicial isomorphism from the simplicial complex defined by this dual equivalence class to the order complex $\Delta([\emptyset, \lambda])$.

Let B_1 be the set of boxes of $\mu_1 \setminus \mu_2$ and let B_2 be the set of boxes of λ. Let $f : B_1 \to B_2$ send the ith box of $\mu_1 \setminus \mu_2$ (in the row reading order) to the box in B_2 that contains i in Q. Since f is a bijection, by Proposition 2.15, it suffices to show that f maps facets to facets.

A facet of the first simplicial complex is just a standard tableau T of shape $\mu_1 \setminus \mu_2$. Suppose that T rectifies to P, a standard tableau of shape λ. Then, we know that

$$\mathrm{RS}(w_{\text{row}}(T)) = (P, Q).$$

Furthermore, by Lemma 5.5, we know that the ith box of T is sent to the ith box of P, in the reading order defined by the Q. Thus, $f(T)$ is just the rectification of T under any sequence of JDT moves. Since this is a standard tableau of shape λ, $f(T)$ is a facet of $\Delta([\emptyset, \lambda])$.

Furthermore, suppose T rectifies to a a rectangular tableau of shape λ. Then, by the above, there is a simplicial isomorphism $f : \Delta_{\mathrm{tab}}(T) \to \Delta([\emptyset, \lambda])$ which restricts to JDT on the facets of $\Delta_{\mathrm{tab}}(T)$. After applying any JDT move to obtain T', we still have that T' rectifies to a rectangular tableau of shape λ. Therefore, there is similarly a simplicial isomorphism g which restricts to JDT on the facets of $\Delta_{\mathrm{tab}}(T')$. Then, the map

$$g^{-1} \circ f : \Delta_{\mathrm{tab}}(T) \to \Delta_{\mathrm{tab}}(T')$$

is a simplicial isomorphism, and on facets of T, it restricts to the JDT move we applied to T to obtain T'. Therefore, every JDT move on T is the restriction of a simplicial isomorphism $\Delta_{\mathrm{word}}(T) \to \Delta_{\mathrm{word}}(T')$, as desired. ☐

5.3. Linear Slides and Superstandard Tableaux.

5.3.1. *Linear Slides.* A ***row (resp. column) slide*** is a slide such that the inserted box and the removed box are in the same row (resp. column). A ***linear slide*** is either a row slide or a column slide. Note that the row-ness, column-ness, and linear-ness of a slide from T to T' are determined by shape T and shape T'. Refer to Figure 8 for an example of a linear slide.

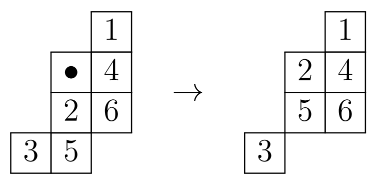


FIGURE 8. Example of a column slide.

Lemma 5.7. *Let T be a standard skew tableau. Let T' be obtained from T via a slide on a corner b. Suppose such a slide is a row (resp. column). Then, for all standard skew tableau U dual equivalent to T, the slide from U on b is also row (resp. column).*

Proof. Let U be a standard skew tableau dual equivalent to T. Let U' be obtained from U via a slide on b. Hence, shape U = shape T and, by Proposition 2.6(ii), shape U' = shape T'. Therefore, since row-ness (resp. column-ness) only depends on the shape of the tableaux, the slide from U to U' is also row (resp. column). ☐

Theorem 5.8. *Let T be a standard skew tableau. Let T' be obtained from T via a linear slide. Then, $\Delta_{\mathrm{tab}}(T) \cong \Delta_{\mathrm{tab}}(T')$.*

Proof. Assume T' is obtained from T by performing the sliding algorithm on an inner corner b_0. Assume that the algorithm slides the entry of box b_i to box b_{i-1}, for all $i \in [k]$, where k is the number of slid boxes. Define the bijection f that maps b_i to b_{i-1}, for all $i \in [k]$, and otherwise fixes a box.

Lemma 5.7 guarantees that $f(U)$ is obtained from U by performing the same sliding algorithm on the inner corner b_0, since a row (resp. column) slide must slide all boxes in the row (resp. column) of b_0 leftward (resp. upward). In other words, f induces a bijection between the facets of $\Delta_{\mathrm{tab}}(T)$ and the facets of $\Delta_{\mathrm{tab}}(T')$, which, by Proposition 2.15, implies that f induces a simplicial isomorphism of $\Delta_{\mathrm{tab}}(T)$ on $\Delta_{\mathrm{tab}}(T')$. ☐

Corollary 5.9. *Let T be a standard skew tableau. If T rectifies to the standard straight tableau P using only linear slides, then $\Delta_{\text{tab}}(T) \cong \Delta_{\text{tab}}(P)$. Moreover, $\Delta_{\text{tab}}(T)$ is homeomorphic to a ball.*

Corollary 5.9 follows by repeatedly applying Theorem 5.8 and using that $\Delta_{\text{tab}}(P)$ is homeomorphic to a ball since P is a standard straight tableau.

5.3.2. *Superstandard Recording Tableaux.* The ***row superstandard tableau of shape*** λ is the standard tableau of shape λ with numbers in increasing order when reading row by row, from the top row to the bottom row, from left to right. The ***column superstandard tableau of shape*** λ is the standard tableau of shape λ with numbers in increasing order when reading column by column, from the leftmost column to the rightmost column, from bottom to top. Refer to Figure 8 for an example of a row and column superstandard tableau.

1	2	3
4	5	
6	7	
8		

1	5	8
2	6	
3	7	
4		

.

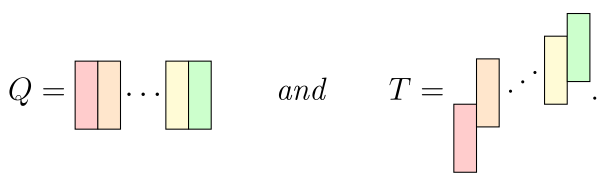
FIGURE 9. The row and column superstandard tableaux of shape $(3, 2, 2, 1)$.

The main result of this subsection is:

Theorem 5.10. *Let Q be a column superstandard tableau of shape λ. Then $\Delta(Q)$ is isomorphic as a simplicial complex to $\Delta[\emptyset, \lambda]$. By [Bjö80, Theorem 2.7.7], this implies that $\Delta(Q)$ is homeomorphic to a ball. Furthermore, there is an isomorphism induced by a sequence of JDT moves.*

To prove this theorem, we use the following lemma:

Lemma 5.11. *Let Q be a column superstandard tableau, and let T be a skew tableau where the i^{th} column of T is a shifted copy of the i^{th} the corresponding column at Q. In other words,*



Any rectification process starting at T ends at Q and only uses column slides.

Proof. Since Q is a column superstandard tableau, all entries in the i^{th} column of Q are smaller than the entries in the $(i+1)^{\text{th}}$ column of Q, and the number of boxes in the i^{th} column of Q is greater than or equal to the number of boxes in the $(i+1)^{\text{th}}$ column of Q. By construction of T, the analogous results for T also hold.

The proof is by induction on the number of inner boxes of T.

If T has 0 inner boxes, that is, T is straight, then $T = Q$ and the only rectification is the empty one, which vacuously only uses column slides.

Otherwise, assume that the result is true for any tableau T' with fewer inner boxes than T. Consider the first slide of a given rectification of T, and an intermediate step of such sliding algorithm. If there is a box below the hole, either its entry will be smaller than the entry to the right of the hole, or there is no entry to the right of the hole; in either case, algorithm slides the box below the hole upwards. If there is not a box below the hole, then there's also not a box to the right of the hole (otherwise the column of the right of the hole's column would have more boxes than the hole's column), hence the sliding algorithm stops. Therefore, the first slide of the given rectification is a column slide.

After the first (column) slide $T \to T'$, the resulting tableau T' has one less inner box than T and satisfies the conditions of the induction hypothesis. Therefore, by the induction hypothesis, the remaining slides of the given rectification are also column slides, as desired.

By induction, the result follows. $\square$

Now we can prove the theorem:

Proof. Let Q be a column superstandard tableau of shape λ. We prove this theorem by giving a chain of simplicial isomorphisms to $\Delta([\emptyset, \lambda])$ induced by a certain sequence of jdt moves.

It is a folklore fact that $\mathrm{RS}(w_{\mathrm{col}}(Q)) = (Q, Q)$. First, let T be the skew tableau of n disjoint diagonal boxes with reading word $w_{\mathrm{col}}(Q)$. We can perform jdt moves to get T into the form of Lemma 5.11 without changing the reading word of T, and so by Lemma 3.4, this sequence of jdt moves induces a simplicial isomorphism.

Now, by Lemma 5.11, we can finish rectifying T by performing only column slides, and by Corollary 5.9, we are done.

$\square$

Corollary 5.12. *Row superstandard tableaux index simplicial complexes which are homeomorphic to balls.*

Proof. The transpose of a row superstandard tableau is a column superstandard tableau; therefore by Theorem 5.10 and Lemma 4.6, we are done. $\square$

Example 5.13. Consider the column superstandard Q of shape $(2, 2, 1)$, which has column reading word 32154.

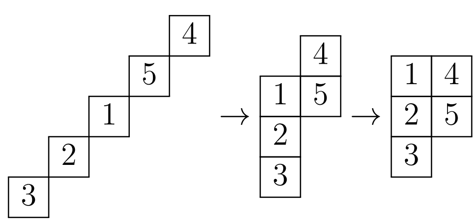


FIGURE 10. Starting with the diagonal tableau T with reading word 32154, we first apply a sequence of jdt moves to get T' into the form of Lemma 5.11. Then, we perform one column slide to the first column to finish rectifying to Q. Each of the jdt moves induces a simplicial isomorphism.

5.4. **Friendly Tableaux.** For this last class of tableaux, we study a specific subset of hook tableaux that index dual equivalence complexes that are homeomorphic to balls. This requires us to describe some specific kinds of skew tableaux that have these nice indexing tableaux in detail and study their properties.

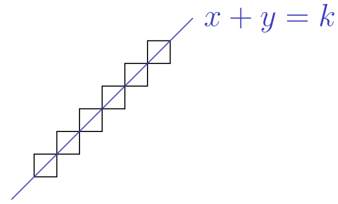


FIGURE 11. Example of a diagonal diagram of height k.

For this section, diagrams live in $\mathbb{Z} \times \mathbb{Z}$. A ***diagonal diagram of height*** k consists of a subset of $\{(x, y) : x + y = k\}$ connected via adding $\pm(1, -1)$, using matrix-like coordinates.

A ***double diagonal diagram*** consists of the union of two diagonals, one of height k, called the ***lower diagonal***, and one of height $k + 1$, called the ***upper diagonal***.

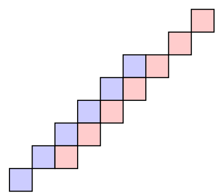


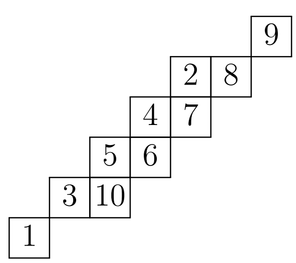
FIGURE 12. Example of a double diagonal diagram. Its lower diagonal is in blue and its upper diagonal is in red.

Given a word $w = w_1 w_2 \cdots w_n$, a triplet (w_{i-1}, w_i, w_{i+1}) is a ***valley*** if $w_{i-1} > w_i < w_{i+1}$, and is a ***peak*** if $w_{i-1} < w_i > w_{i+1}$. A ***no-valley (resp. no-peak)*** word is a word with no valleys (resp. peaks).

Definition 5.14. A ***friendly tableau*** is a standard tableau on a double diagonal diagram such that:

(i) All entries in the lower diagonal are smaller than all entries in the upper diagonal.
(ii) The reading word of the restriction of the tableau to the lower diagonal does not have a valley.
(iii) The reading word of the restriction of the tableau to the upper diagonal does not have a peak.
(iv) The maximum entry of the lower diagonal is adjacent to the minimum entry of the upper diagonal. These entries are called the ***lower friend*** and ***upper friend***, respectively.

For example,



is a friendly tableau, since $[1, 3, 5, 2, 4]$ has no valleys, $[10, 6, 7, 8, 9]$ has no peaks, and the friends 5 and 6 are adjacent.

Proposition 5.15. *Any no-valley word can be obtained as the reading word of a friendly tableau of a shape with upper diagonal of size* 1*; and conversely, the reading word of any friendly tableau of a shape with upper diagonal of size* 1 *is a no-valley word.*

For example, the no-valley word 135642 is the reading word of the friendly tableau

$$\begin{array}{ccccc} &&&&\boxed{2}\\ &&&\boxed{4}&\\ &&\boxed{5}&\boxed{6}&\\ &\boxed{3}&&&\\ \boxed{1}&&&& \end{array}\quad.$$

Proposition 5.16. *Any no-peak word can be obtained as the reading word of a friendly tableau of a shape with lower diagonal of size* 1*; and conversely, the reading word of any friendly tableau of a shape with lower diagonal of size* 1 *is a no-peak word.*

For example, the no-peak word 641235 is the reading word of the friendly tableau

$$\begin{array}{ccccc} &&&&\boxed{5}\\ &&&\boxed{3}&\\ &\boxed{1}&\boxed{2}&&\\ &\boxed{4}&&&\\ \boxed{6}&&&& \end{array}\quad.$$

Proposition 5.17. *Let D be a double diagonal diagram. The set of friendly tableaux on D forms a dual equivalence class.*

Proof. Let n be the number of boxes in the double diagonal diagram, with k boxes in the lower diagonal.

First, we prove that elementary dual equivalence transformations preserve friendliness, that is, if T is a friendly tableau, and T' is obtained from T via an elementary dual equivalence transformation, then T' is also a friendly tableau.

Note that an elementary dual equivalence transformation cannot swap the friends k and $k+1$, since they are consecutive in $w_{\text{row}}(T)$. If such elementary dual equivalence transformation swaps elements of $\{1, 2, \dots, k-1\}$, then the only condition to check is *(ii)*, which is still satisfied. If such elementary dual equivalence transformation swaps elements of $\{k+1, k+2, \dots, n\}$, then the only condition to check is *(iii)*, which is still satisfied. If such elementary dual equivalence transformation swaps $k-1$ and k, then the conditions to check are *(ii)*, which is still satisfied, and *(iv)*, which is still satisfied because the upper friend $k+1$ must be adjacent to both $k-1$ and the lower friend k in order for the elementary dual equivalence transformation to be allowed, hence the friends are still adjacent. If such elementary dual equivalence transformation swaps $k+1$ and $k+2$, then the conditions to check are *(iii)*, which is still satisfied, and *(iv)*, which is still satisfied because the lower friend k must be adjacent to both $k+2$ and the upper friend $k+1$ in order for the elementary dual equivalence transformation to be allowed, hence the friends are still adjacent.

Second, we prove that any two friendly tableau of equal shapes are dual equivalent.

Note that, by only performing elementary dual equivalence transformations that swap elements of $\{1, 2, \dots, k-1\}$, one can obtain any no-valley word in the lower diagonal with the lower friend in its position. Similarly, by only performing elementary dual equivalence transformations that swap elements of $\{k+1, k+2, \dots, n\}$, one can obtain any no-peak word in the upper diagonal with the upper friend in its position. Finally, by performing elementary dual equivalence transformations that swap $k-1$ and k, and $k+1$ and $k+2$, one can obtain friendly tableaux where the friends occupy any possible pair of adjacent boxes. $\square$

Definition 5.18. An ***zig-zag tableau*** is a hook-shaped standard straight tableau with n boxes such that there exists $k \leq \ell \leq n$ such that:

- all entries in $[k]$ are in the first row or in the first column;
- consecutive entries in $[k, \ell]$ alternate between being in the first row and being in the first column; and
- all entries in $[\ell, n]$ are in the first row or in the first column.

For example,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 6 & 8 & 10 \\ \hline 5 \\ \cline{1-1} 7 \\ \cline{1-1} 9 \\ \cline{1-1} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} 14 \\ \cline{1-1} \end{array}$$

is a zig-zag tableau, where $k = 4$ and $\ell = 11$.

Proposition 5.19. *The recording tableau of the reading word of a friendly tableau is a zig-zag tableau; and conversely, any zig-zag tableau is the recording tableau of the reading word of some friendly tableau.*

Sketch. We may subdivide any friendly tableau T into three (possibly empty) sections – one disconnected diagonal of boxes in the upper diagonal, one disconnected diagonal of boxes in the lower diagonal, and a connected component of boxes consisting of boxes from the upper and lower diagonals. WLOG suppose the entries of the upper diagonal are in the bottom-left of T, and suppose the entries of the lower diagonal are in the upper-right of T. Then since $w_{\text{row}}(T)$ reads off these bottom entries in order, the corresponding recording tableau for $w_{\text{row}}(T)$ corresponds to a line of consecutive entries across the top row of the tableau, since these entries are in ascending order. Next, the block of connected entries must start with an entry from the upper diagonal, and alternates between small entries that will be inserted into the top row and large entries that will bump each other down in the first column under RS. We can see this as a result of the relative ordering of the entries in this connected block because the upper diagonal is a word with exactly 1 peak and the lower diagonal is a word with 1 valley, as the extrema of each diagonal must be contained within this connected section. Finally, depending on the last-inserted entry in the top row, the remaining large entries in the disconnected upper-right diagonal will either be inserted in that row (if that entry came from the upper diagonal) or will bump entries down in the first column (if the last inserted entry came from the lower diagonal). Based on the placement of each new inserted entry under Schensted insertion, this matches exactly the description of the zig-zag hook tableaux.

$\square$

Example 5.20. Consider the friendly tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & 2 & 9 \\ \hline & & 4 & 5 & \\ \hline & 3 & 6 & & \\ \hline 1 & 7 & & & \\ \hline 8 & & & & \\ \hline \end{array} .$$

Note that $w_{\text{row}}(T) = 817364529$ and

$$Q(w_{\text{row}}(T)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 7 & 9 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 6 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

which is a zig-zag tableau where all of the entries of [1] are in the first row/column, and the entries $[1, 9]$ alternate between being in the first row and column.

The ***funneling*** of a diagonal diagram D of height k is the diagonal of height $k-1$ consisting of all boxes directly above to some box in D *and* directly to the left of some box in D. The funneling of a non-empty diagonal diagram decreases the number of boxes by one.

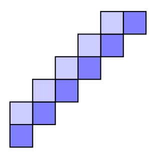


FIGURE 13. Example of funneling. The light blue diagonal diagram is the funneling of the dark blue diagonal diagram.

The ***spreading*** of a diagonal diagram D of height k is the diagonal of height $k-1$ consisting of all boxes directly above to some box in D *or* directly to the left of some box in D. The spreading of a non-empty diagonal diagram increases the number of boxes by one.

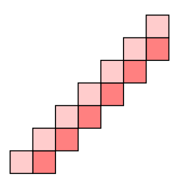


FIGURE 14. Example of spreading. The light red diagonal diagram is the spreading of the dark red diagonal diagram.

The ***lowering*** of a double diagonal diagram is the union of the funneling of its lower diagonal and the spreading of its upper diagonal. Note that the lowering of a double diagonal diagram preserves the number of boxes.

Theorem 5.21. *If D is a double diagonal diagram and D' is the lowering of D, then the simplicial complex generated by the shape sequences of the friendly tableaux of shape D is isomorphic to the simplicial complex of the shape sequences of the friendly tableaux of shape D'.*

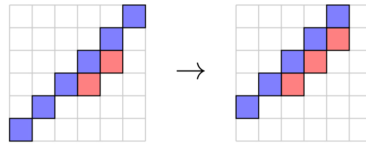


FIGURE 15. Example of lowering. The light red diagonal diagram is the spreading of the dark red diagonal diagram.

Sketch. Let Δ, Δ' be simplicial complex generated by the shape sequences of the friendly tableaux of shape D, D'. Hence, a vertex of Δ, Δ' is a shape in the shape sequence of some friendly tableau T of shape D, D'.

Let L, L' and U, U' be the lower and upper diagonals of D, D'. Let n be the number of boxes in D, and let i be the number of boxes in L. We explicitly define the simplicial isomorphism f.

Consider an arbitrary shape $S \in V(\Delta)$ with $k < i-1$ boxes. Hence, S is the set of boxes whose entries are $\{1, \ldots, k\}$ in some friendly tableau T of shape D, which must be a subset of the lower diagonal L of D. Consider the complement $L - S$, which is the set of boxes whose entries are $\{k+1, \ldots, i\}$ in T. By the no-valley property, $L - S$ is connected via diagonals. Hence, it splits S into two (not necessarily non-empty sides): one to the northeast of $L - S$ and another to the southwest of $L - S$. We declare $f(S)$ to consist of the boxes directly to the right of the boxes of S to the northeast of $L - S$ and of the boxes directly above to the boxes of S to the southwest of $L - S$.

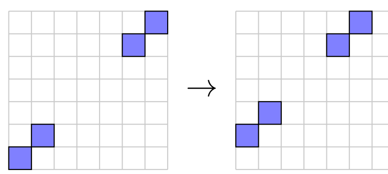


FIGURE 16. Isomorphism when S has less than $i - 1$ boxes.

Consider an arbitrary shape $S \in V(\Delta)$ with $i-1$ boxes. Hence, S is the set of boxes whose entries are $\{1, \ldots, i-1\}$ in some friendly tableau T of shape D, which must be a subset of the lower diagonal L of D. Consider the complement $L - S$, which is the singleton with the box whose entry is i in T. Hence, $L - S$ splits S into two (not necessarily non-empty sides): one to the northeast of $L - S$ and another to the southwest of $L - S$. We declare $f(S)$ to consist of the boxes directly to the right of the boxes of S to the northeast of $L - S$ and of the boxes directly above to the boxes of S to the southwest of $L - S$, *as well as* $L - S$. In other words, $f(S) = L' \cup (L - S)$.

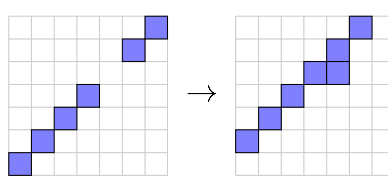


FIGURE 17. Isomorphism when S has $i - 1$ boxes.

Note that the only shape in $V(\Delta)$ with i boxes is L. We define $f(L) = L'$, the only shape in $V(\Delta')$ with $i - 1$ boxes.

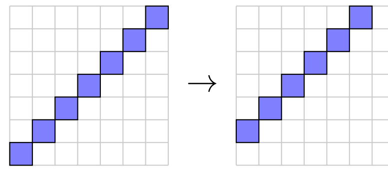


FIGURE 18. Isomorphism when S has i boxes.

Consider an arbitrary shape $S \in V(\Delta)$ with $k > i$ boxes. Hence, S is the set of boxes whose entries are $\{1, \ldots, k\}$ in some friendly tableau T of shape D. Therefore, $S \supset L$ Then, $S - L \subset U$ consists the boxes whose entries are $\{i+1, \ldots, k\}$ in T. By the no-peak property, S is connected via diagonals. We declare $f(S)$ to consist of the boxes in L' and the boxes in the spreading of $S - L$.

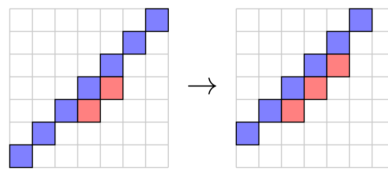


FIGURE 19. Isomorphism when S more than i boxes.

Now, f is a simplicial isomorphism of Δ on Δ', because it bijects the friendly tableaux T of diagram D and the friendly tableaux T' of diagram D'. $\square$

6. Relationships with K-JDT

Given our simplicial isomorphisms established thus far for jeu de taquin slides acting on $\Delta_{\text{word}}(Q)$ for certain special Q-tableaux, one might ask if these isomorphisms give us interesting substructure on the subfaces of $\Delta_{\text{word}}(Q)$. An analogue of jeu de taquin called K-jeu de taquin was introduced in [TY09] by Thomas and Yong, which acts on increasing tableaux.

Definition 6.1. A ***alternating ribbon*** is a connected skew tableau with no 2×2 subbox filled with exactly two symbols such that any two adjacent boxes are filled with different symbols.

Definition 6.2. Let T be an increasing tableau and C be a set of its inner corners, each of which is filled with a $\bullet$. Then the ***K-jeu de taquin slide*** on this set of inner corners is given by the following algorithm – if the entries in I range from 1 up to m for $m \leq |I|$, keep track of a collection of sets of boxes $\{B_i\}_{i=1}^m$ with $B_1 = C$, and for every $1 \leq i \leq m$ (in increasing) order:

- (i) Let T_i be the boxes in T with entry i. Form the union of alternating ribbons consisting of a subset of the boxes of I containing the boxes B_i (which should currently contain $\bullet$s) and T_i.
- (ii) For all sets of edge-connected boxes in $B_i \cup T_i$, swap the positions of the symbols in each connected component. Let B_{i+1} be the new positions of the $\bullet$s.

Example 6.3. As an example, consider the increasing tableau $I = \begin{array}{cccc} & \bullet & 1 & 2 \\ \bullet & 2 & & \\ 2 & & & \end{array}$. Consider the following K-jeu de taquin slide with the shown bullets:

$$\begin{array}{cccc} & \bullet & 1 & 2 \\ \bullet & 2 & & \\ 2 & & & \end{array} \xrightarrow{\text{swap with 1s}} \begin{array}{cccc} & 1 & \bullet & 2 \\ \bullet & 2 & & \\ 2 & & & \end{array} \xrightarrow{\text{swap with 2s}} \begin{array}{ccc} & 1 & 2 \\ 2 & & \end{array}$$

As such, recalling Proposition 2.11, one might ask whether or not K-jdt appears acting on the interior faces of $\Delta_{\text{word}}(Q)$:

Question 6.4. Given a standard skew tableau T with reading word w, where $\mathrm{RS}(w) = (P, Q)$. Suppose some sequence of jdt moves on T on a set of inner corners C yields a tableau T', with a simplicial isomorphism $\Delta_{\text{tab}}(T) \cong \Delta_{\text{tab}}(T')$. Does K-jdt with bullets at C on the increasing tableaux corresponding to the interior faces of $\Delta_{\text{tab}}(T)$ correspond to the interior faces of $\Delta_{\text{tab}}(T')$?

For certain Q for which we have explicit simplicial isomorphisms, we can answer this question.

Proposition 6.5. *Let Q be column superstandard and T be a skew tableau where the ith column of T is a shifted copy of the ith corresponding column of Q. Then each column slide induces K-jdt moves on the interior faces of $\Delta_{\text{tab}}(T)$.*

Proof. Take an increasing tableau I of shape T corresponding to an interior face of $\Delta_{\text{tab}}(T)$. Consider WLOG a column-slide in the leftmost column of I. Since T was a superstandard tableau with its columns shifted vertically to make it skew, we can see that for any given box b in this column of I, no other box above its current row in the column to its right has the same entry as it. Suppose not – since I is increasing, all of the boxes from b to the bottom of this column must be the same number and so in particular b must be the last row in its column and the column to the right must have the same height as the first column to achieve this. However, the condition that I be increasing causes a contradiction, either by the column to right not having strictly increasing entries or that the rows become non-increasing.

As such, the K-jeu de taquin move in this column is then forced to do the same box slides as the jeu de taquin move, since as we progress through the numbers in order, we never have a bullet with the same numbers below and to the right of it. Therefore, the K-jeu de taquin moves continue to give increasing tableaux which correspond to interior faces of the resulting simplicial complexes. ∎

Proposition 6.6. *Let T be a standard skew tableau, and let T' be obtained from T' via one slide. Assume that the function f of boxes satisfying $f(T) = T'$ induces a simplicial isomorphism from $\Delta_{\text{tab}}(T)$ on $\Delta_{\text{tab}}(T')$. Then, for any interior tableau $I \in \Delta_{\text{tab}}(T)$, the tableau $f(I) \in \Delta_{\text{tab}}(T')$ can be obtained from I by performing the K-sliding algorithm on b_0.*

Proof. Assume T' is obtained from T by performing the sliding algorithm on an inner corner b_0. Assume that the algorithm slides the entry of box b_i to box b_{i-1}, for all $i \in [k]$, where k is the number of sliden boxes. Hence, the bijection f maps b_i to b_{i-1}, for all $i \in [k]$, and otherwise fixes a box.

Let c_i be the box distinct from b_i below or to the right of b_{i-1}. Hence, the entry of T in the box c_i (if it exists) is smaller than the entry of T in the box b_i. Moreover, since f induces a simplicial isomorphism, for all standard skew tableau S dual equivalent to T, the tableau $f(S)$ is obtained from S via a slide on b_0, and consequently, the entry of S in the box c_i (if it exists) is smaller than the entry of S in the box b_i.

Let $I \in \Delta_{\text{tab}}(T)$ be an interior face. Hence, $f(I) \in \Delta_{\text{tab}}(T')$ is an interior face. In particular, this implies that $f(I)$ is an increasing tableau. Since c_i (if it exists) is in below or to the right of b_{i-1}, then the entry of $F(I)$ in the box c_i (if it exists) is smaller than the entry of $f(I)$ in the box b_{i-1}, and, consequently, the entry of I in the box c_i (if it exists) is smaller than the entry of I in the box b_i.

Therefore, when we perform the K-sliding algorithm from I on b_0, the bullet first swaps with the entry in b_1 but not with the entry in c_1 (if it exists), and so on, until it swaps with the entry in b_k but not with the entry in c_k (if it exists), which ultimately outputs the tableau $f(I)$, as desired. □

6.1. **Unique Rectification Targets.** A salient feature of K-jeu de taquin is that, unlike jeu de taquin, the order in which subsets of the inner corners are chosen does matter. In particular, under two different orders of K-jeu de taquin slides, the same skew tableau may rectify to straight tableaux of different shapes. However, a remarkable fact is that certain classes of tableaux form ***unique rectification targets***:

Definition 6.7. A standard straight tableau T is a unique rectification target if for any increasing tableau T', if T' rectifies to T under some sequence of K-jdt slides, then T' rectifies to T under *any* sequence of K-jdt slides.

In [TY09, Theorem 1.2], it is shown that all superstandard tableaux are unique rectification targets. Furthermore, in [Gae+16, Corollary 66] it is shown that all rectangular tableaux are unique rectification targets. Since our work also shows that superstandard and rectangular tableaux index simplicial complexes homeomorphic to balls, the following conjecture may be a natural guess:

Conjecture 6.8 (False). *Let T be a standard skew tableau such that $Q(w_{\text{row}}(T))$ is a unique rectification target. Let T' be obtained from T by a slide on a inner corner b. Then, there is an isomorphism $f\colon V(\Delta(T)) \to V(\Delta(T'))$ such that the induced map $\tilde{f}\colon \Delta(T) \to \Delta(T)$ sends an interior face of $\Delta(T)$ (necessarily an increasing tableau) to an interior face of $\Delta(T')$ via the K-slide on the inner corner b.*

The conjecture is false, and here's a counterexample. Let

$$T = \begin{array}{cccc} & & \boxed{1} & \boxed{4} \\ & & \boxed{5} & \\ \boxed{2} & \boxed{3} & \boxed{6} & \end{array}.$$

Note that

$$Q(w_{\text{row}}(T)) = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{4} & \boxed{6} & \\ \boxed{5} & & \end{array}$$

is a unique rectification target. The only possible slide from T is the following

$$\begin{ytableau} \none & \none & 1 & 4 \\ \none & \none[\bullet] & 5 \\ 2 & 3 & 6 \end{ytableau} \to \begin{ytableau} \none & \none & 1 & 4 \\ \none & 3 & 5 \\ 2 & 6 \end{ytableau} = T'.$$

However, note that

$$I = \begin{ytableau} \none & \none & 1 & 2 \\ \none & \none[\bullet] & 2 \\ 1 & 2 & 3 \end{ytableau} \to \begin{ytableau} \none & \none & 1 & 2 \\ \none & 2 & 3 \\ 1 & 3 \end{ytableau} = I',$$

but I is in the boundary of $\Delta_{\text{tab}}(T)$, while I' is an interior face of $\Delta_{\text{tab}}(T')$, hence the K-jeu de taquin process cannot be induced by a simplicial isomorphism.

7. Further Work

There are many directions in which this line of research could be continued. Below are some problems that remain unsolved at the time of writing this report:

Problem 7.1. Our work on friendly tableaux shows that some hook-shaped standard tableaux index complexes homeomorphic to balls. Do all hook-shaped standard tableaux do so?

Problem 7.2. In the specific case of skew tableaux that rectify to a rectangular shape, we showed that *all* JDT moves induce simplicial isomorphisms of the corresponding dual equivalence classes. In the case of dual equivalence classes indexed by column superstandard tableaux, our proof shows that many sequences of JDT moves induce simplicial isomorphisms, but not necessarily all. However, for all superstandard tableaux with ≤ 10 boxes, by exhaustive search we showed that all JDT moves induce simplicial isomorphisms. Is this statement true in general?

Problem 7.3. We can generate exhaustive lists of Q-tableaux whose $\Delta_{\text{word}}(Q)$ do not give rise to shellable or Cohen-Macaulay simplicial complexes. We currently have partial results as to why certain subclasses of Q do give these nice desirable complexes, but are there ways to determine which dual equivalence complexes do not without computing their topological structure explicitly? Can we characterize these Q-tableaux by way of pattern-avoidance?

Problem 7.4. Our Conjecture 6.8 seems to indicate that the Q-tableaux that give rise to simplicial isomorphisms upon jeu de taquin slides are a smaller subset of those tableaux that give rise to unique rectification targets. Is the property of a Q-tableau giving rise to simplicial isomorphisms under jeu de taquin slides stronger than being a URT?

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