ON $\mathrm{GL}_n(k)$-STABLE IDEALS IN POSITIVE CHARACTERISTIC

BJØRN CATTELL-RAVDAL, ERIN DELARGY, AKASH GANGULY, SEAN GUAN, AND SAISUDHARSHAN SIVAKUMAR

ABSTRACT. Stephen Doty [1] determined the $\mathrm{GL}_n(k)$ submodule structure of the degree-d homogeneous component of $k[x_1, \ldots, x_n]$ when k is a field of positive characteristic. We build on this work and prove a decomposition theorem (via Algorithm 1) for $\mathrm{GL}_n(k)$-stable ideals, and we show that the depth of $\mathrm{GL}_n(k)$-stable ideals is 0. Furthermore, we provide the minimal free resolution for the inclusion-minimal $\mathrm{GL}_2(k)$-stable ideal generated in a single degree in any positive characteristic (Theorem 1.4).

CONTENTS

1. INTRODUCTION

The general linear group $\mathrm{GL}_n(k)$ acts on the ring of polynomials $S = k[x_1, \ldots, x_n]$ by linear substitution of variables. Explicitly, for $A \in \mathrm{GL}_n(k)$ and $f \in S$ we have $A \cdot f(x_1, \ldots, x_n) = f(Ax_1, Ax_2, \ldots, Ax_n)$ where we view x_i as the ith standard basis vector. We study the ideals of S that are closed, or *stable*, under this action when k is an infinite field of positive characteristic. Linear substitution preserves polynomial degree, and thus we may regard S as a graded $\mathrm{GL}_n(k)$ module $S = \bigoplus_d S_d$.

When k is a field of characteristic 0, the modules S_d are well understood. In particular, the simple $\mathrm{GL}_n(k)$-submodules of S_d are the famous Schur modules, indexed by integer partitons of d (see for instance [4] for further details). When k is an algebraically closed field of characteristic $p > 0$, the story is less well known.

Stephen Doty [1] initiated the study of the $\mathrm{GL}_n(k)$ submodules of S_d. In particular, S_d is no longer simple in general. Doty shows that there is a lattice isomorphism between the $\mathrm{GL}_n(k)$-submodule lattice of S_d and the lattice of order ideals $J(P)$ of a poset P whose elements are called *carry patterns*. These carry patterns depend both on the characteristic of the field and the number of variables, and are associated to each monomial. (See Section 2 for the definition of carry pattern.) More recently, a line of inquiry has studied the submodule structure of the degree-d component of $k[x, y]$ where k is a finite field [2].

We continue the study this action by considering the action on ideals, rather than only on homogeneous components of the modules. For the remainder of the paper k will be an infinite field of characteristic $p > 0$.

Definition 1.1. Let B be a set of carry patterns. Define $T_{B,d}$ (or T_B when d is clear from context) as the $\mathrm{GL}_n(k)$-submodule of S_d generated by the monomials with carry patterns in B.

Our first main theorem provides a finite decomposition of stable ideals into a sum of stable ideals.

Theorem 1.2 (Structure of stable ideals)**.** *For I a stable ideal, I may be written as a sum $\sum_{d \in F} I_{B_d,d}$ for F a finite set of natural numbers, and $I_{B_d,d} = \langle T_{B_d,d} \rangle$.*

Moreover, we provide an explicit algorithm (Algorithm 1) to determine the set F indexing the sum.

One important ingredient of Doty's characterization is the base-p expansion of a natural number. Recall that the base-p expansion of m is the sum

$$\sum_{i\geq 0} d_i p^i, \text{ where } 0 \leq d_i \leq p-1.$$

Let M be the largest integer for which $d_i > 0$. Then we also write $(d_0, \cdots, d_M)$ as the base p expansion of d, keeping in mind that $d_i = 0$ for $i > M$ and $i < 0$. In order to describe the structure of $\mathrm{GL}_2(k)$ stable ideals, break the base-p expansion of d into pieces.

Definition 1.3. Let d be a natural number and $(d_0, \ldots, d_M)$ be the base p expansion of d. A block of d is a subsequence $(d_r, d_{r+1}, \ldots, d_s)$ of the base p expansion of d where:

(1) $d_{r-1} \neq p-1$
(2) $d_r = d_{r+1} = \cdots = d_{s-1} = p-1$
(3) $0 \leq d_s \leq p-2$

We aim to understand the minimal free resolutions of $\mathrm{GL}_2(k)$-stable ideals generated in a single degree d using the base-p expansion of d and its associated blocks. We show in Corollary 4.9, the free resolution of a $\mathrm{GL}_2(k)$-stable ideal has length two. Thus, understanding first syzygies is enough to understand the minimal resolutions. Our second main theorem makes this understanding explicit.

Theorem 1.4. *Let I denote the inclusion-minimal $GL_2(k)$-stable ideal of $S = k[x, y]$. Let $\{b_{i_1}, \ldots, b_{i_\ell}\}$ be the set of all nonzero blocks of d, where the block b_{i_j} starts at i_j. Let $\mathcal{I} = \{i_1, \ldots, i_\ell\}$.*

Then

(1) *The number of generators of I is $\prod_j (d_j + 1)$.*
(2) *The number of distinct degrees of syzygies of the minimal generators of I is the number of nonzero blocks of d.*
(3) *The distinct degrees of syzygies of the minimal generators of I are $p^i - |(d_0, \ldots, d_{i-1})|$ for each $i \in \mathcal{I}$. Equivalently, this is the consecutive differences in the set of subexpansion numbers.*
(4) *The number of syzygies of degree $p^i - |(d_0, \ldots, d_{i-1})|$ for each $i \in \mathcal{I}$ is $|b_i| \prod_{j>i}(d_j + 1)$.*

The structure of the paper is as follows. In Section 2 we define carry patterns following [1] and show that the poset of carry patterns is a lattice. We also determine how the carry pattern of a monomial changes upon multiplication by an indeterminate. In Section 3 we detail the algorithm to decompose stable ideals, proving Theorem 1.2. In Section 4 we prove Theorem 1.4. In Section 5 we pose questions raised by our work and suggest future work.

2. Carry patterns

Given a monomial $x^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n} \in S$, the degree of of the monomial is $d = b_1 + \ldots + b_n$. The base p expansion of d, and of each b_i is

$$d = \sum_{j\geq 0} d_j p^j \quad \text{and} \quad b_i = \sum_{j \geq 0} b_{ij} p^j.$$

If M is the largest j such that $d_j \neq 0$, then for $j > M$, every $b_{ij} = 0$. Define the integers $c_\ell(\mathbf{b})$ for $1 \leq \ell \leq M$ by the equation

$$\sum_i \sum_{j<\ell} b_{ij} p^j = c_\ell(\mathbf{b}) p^\ell + \sum_{j<\ell} d_j p^j. \tag{1}$$

Definition 2.1. The *carry pattern* $c(\mathbf{b})$ (or $c(x^{\mathbf{b}})$) of a monomial $x^{\mathbf{b}}$ is the tuple of integers $(c_1(\mathbf{b}), c_2(\mathbf{b}), \ldots, c_M(\mathbf{b}))$. Let $c_i(\mathbf{b})$ for $i > M$ or $i < 1$ be 0.

Example 2.2. Let k have characteristic 3, and let $S = k[x, y]$. Consider the monomial $x^4 y^6$, of degree $10 = 1 \cdot 3^0 + 0 \cdot 3^1 + 1 \cdot 3^2$, which gives $M = 2$. The base p expansions of the exponents of

x^4y^6 are $4 = 1 \cdot 3^0 + 1 \cdot 3^1$ and $6 = 0 \cdot 3^0 + 2 \cdot 3^1$. Using Equation (1), the carry patterns $c_1(4,6)$ and $c_2(4,6)$ are given by

$$1 = c_1(4,6) \cdot 3 + 1 \quad \text{and} \quad 1 + 1 \cdot 3 + 2 \cdot 3 = c_2(4,6) \cdot 3^2 + 1.$$

Thus $c_1(4,6) = 0$, and $c_2(4,6) = 1$, so $c(4,6) = (0,1)$.

Another way to understand a carry pattern is to view $c_j(\mathbf{b})$, as the amount carried to the p^j column when performing base p addition of the entries of $\mathbf{b}$. Suppose the sum $\sum_i b_{i,j-1}p^{j-1} + c_{j-1}(\mathbf{b})$ is equal to $qp+r$, where q is a positive integer (possibly greater than p) and $0 \le r \le p-1$. Then q is carried to the p^j column; that is, $c_j(\mathbf{b}) = q$.

Consider again the monomial x^4y^6 from above. Add the base three expansion of each of the exponents, keeping track of how much is carried to the next column:

$$\begin{array}{cc} b_{11} & b_{10} \\ b_{21} & b_{20} \end{array} \quad \longrightarrow \quad \begin{array}{cccc} & \mathbf{1} & \mathbf{0} & \\ & & 1 & 1 \\ + & & 2 & 0 \\ \hline & 1 & 0 & 1 \end{array}$$

So $c(4,6) = (c_1(4,6), c_2(4,6)) = (0,1)$.

Example 2.3. Let $S = k[x,y,z]$ where $\mathrm{char}(k) = 2$, and consider the monomial $x^3y^3z^3$. The degree of the monomial is $9 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^3$, so $M = 3$. Add the base two expansion of each of the exponents of $x^3y^3z^3$, keeping track of how much is carried to the next column.

$$\begin{array}{ccccc} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \\ & & 0 & 1 & 1 \\ & & 0 & 1 & 1 \\ + & & 0 & 1 & 1 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

Note that the amount carried to the next column can be greater than p.

Definition 2.4. The set $C(d)$ is the set of all carry patterns of degree d monomials.

The set $C(d)$ is partially ordered under the digitwise order; we have $c < c'$ if $c_i < c'_i$ for all i. It turns out that not all tuples of length M correspond to a carry pattern for a given d. The following lemma determines which tuples are valid carry patterns.

Lemma 2.5 ([1] Lemma 3)**.** *Let $c = (c_1, c_2, ..., c_M)$ be an M-tuple of integers, and let $c_i = 0$ for $i > M$. The tuple c is a carry pattern if and only if*

$$0 \le c_i \le \sum_{j \le i} d_i p^{j-i}$$

and

$$0 \le d_i + p\ c_{i+1} - c_i \le n(p-1),$$

for all integers i.

Corollary 2.6. *[1] There always exists a minimal carry pattern, namely the carry pattern (0,...,0). When $d \le n(p-1)$, $C(d)$ has a maximum element.*

We extend this to show that $C(d)$ always has a maximum element.

Definition 2.7. For two carry patterns c, c' define

$$\mathrm{lcm}(c,c') := (\max(c_1,c'_1), \max(c_2,c'_2), ..., \max(c_m,c'_m)).$$

Proposition 2.8. *If $c, c' \in C(d)$, then $\mathrm{lcm}(c,c') \in C(d)$.*

Proof. We consider two carry patterns with $c = (c_1, ..., c_M)$ and $c' = (c'_1, ..., c'_M)$. We will show that the $\mathrm{lcm}(c,c')$ satisfies the two conditions given by Lemma 2.5. First, as c_i is a carry in c, we must have that $0 \le c_i \le \sum_{j \le i} d_i p^{j-i}$. The same is true for c'_i. As $\max(c_i, c'_i)$ is either c_i or c'_i, $0 \le \max(c_i, c'_i) \le \sum_{j \le i} d_i p^{j-i}$.

We now consider the second condition. Since c, c' are valid carry patterns for each i we have

$$0 \le d_i + p\ c_{i+1} - c_i \le n(p-1)$$

and

$$0 \leq d_i + p\ c'_{i+1} - c'_i \leq n(p-1)$$

If $\max(c_i, c'_i) = c_i$ and $\max(c_{i+1}, c'_{i+1}) = c_{i+1}$ then we are done. Similarly if $\max(c_i, c'_i) = c'_i$ and $\max(c_{i+1}, c'_{i+1}) = c'_{i+1}$ then we are done. So without loss of generality, it remains to consider the case where $\max(c_i, c'_i) = c_i$ and $\max(c_{i+1}, c'_{i+1}) = c'_{i+1}$. But since $c_i \geq c'_i$ and $c_{i+1} \leq c'_{i+1}$, we have

$$d_i + p\ c_{i+1} - c_i \leq d_i + p\ c'_{i+1} - c_i \leq d_i + p\ c'_{i+1} - c'_i$$

thus

$$0 \leq d_i + p\ c'_{i+1} - c_i \leq n(p-1)$$

as desired.

□

Corollary 2.9. *For a set of carry patterns $C(d)$, there exists a unique maximal carry pattern.*

Proof. Assume by way of contradiction that we have two maximal carry patterns c, c'. Note that $\mathrm{lcm}(c, c')$ is in $C(d)$ and is greater than both c, c'. □

This implies that the structure of the carry pattern poset for any $C(d)$ is in fact a lattice.

Example 2.10. Let $S = k[x, y]$, with $\mathrm{char}(k) = 2$. Consider the set of all carry patterns of degree ten monomials:

$$C(10) = \{(0,0,0), (1,0,0), (0,0,1), (1,0,1), (1,1,1)\}$$

The monomials generating S_{10} are paired with their carry patterns:

$c(x^{\mathbf{b}})$	$x^{\mathbf{b}}$
$(1,1,1)$	x^7y^3, x^3y^7
$(1,0,1)$	x^5, y^5
$(1,0,0)$	x^9y, xy^9
$(0,0,1)$	x^6y^4, x^4y^6
$(0,0,0)$	$x^{10}, x^8y^2, x^2y^8, y^{10}$

Note that the set of carry patterns of S_{10} does not include all possible 3-tuples. For example, $(0,1,1)$ is not in $C(10)$.

We have the following lattice of carry patterns for $C(10)$:

$$\begin{tikzcd}[arrows=dash]
 & (1,1,1) \arrow[d] & \\
 & (1,0,1) \arrow[dl] \arrow[dr] & \\
(1,0,0) \arrow[dr] & & (0,0,1) \arrow[dl] \\
 & (0,0,0) &
\end{tikzcd}$$

Notice that the carry pattern $(0,1,1)$ does not appear in $C(10)$. However, it does appear in $C(9)$, where $c(x^7y^2) = (0,1,1)$.

When working with a polynomial ring in two variables, the degree d of a monomial and the power on x determines the monomial uniquely. Namely, it is $x^a y^{d-a}$. This simplifies the possible carry patterns and allows us to understand when there is a single carry pattern possible.

Proposition 2.11. *For $S = k[x, y]$ the only possible carry pattern for S_d is (0,...,0), when $d = ap^m - 1$ for some positive integer m and $1 \leq a \leq p-1$.*

Proof. Consider $d = ap^m - 1$. The base p expansion of d is

$$(a-1)p^m + \sum_{i=0}^{m-1}(p-1)p^i.$$

By way of contradiction consider some carry pattern with a nonzero entry, say e in position j. Thus our carry pattern is of the form $(0, ..., e, ...0)$. As our expansion of d contains only $p-1$ in these positions we must have that $b_{1j} + b_{2j} = ep + (p-1)$. However, as $b_{1j}, b_{2j} \leq p-1$, we have that $b_{1j} + b_{2j} \leq 2p - 2 = p + p - 2$ which contradicts $b_{1j} + b_{2j} = ep + (p-1)$. □

We now explain the significance of carry patterns in the main result of [1]. Consider a set B of carry patterns from $C(d)$. Say B is *order-closed* if for every carry pattern c in B, and every c' in $C(d)$, if $c' < c$, then c' is in B. That is, all descending chains in $C(d)$ of every carry pattern in B are also in B. Note that B need not have a unique maximal carry pattern to be order closed.

Example 2.12. In Example 2.10, let $B = \{(0,0,0), (1,0,0), (0,0,1)\}$. The set B is an order-closed subset of $C(10)$, where $(1,0,0)$ and $(0,0,1)$ are both maximal elements.

Lemma 2.13. *If a monomial has only two variables, its carry pattern will contain only* 1*'s and* 0*'s.*

Proof. If we add two digits together in the p^0 place, then both must be less than $p-1$, so the sum is less than $2p-2$. In order for a 2 to appear in the carry pattern, the sum would have had to be at least $2p$. In higher places, we either have the same argument, or we add a 1 carried from a lower place. If we add a 1, we have the result being $2p-1$, which is still less than $2p$. □

Recall that T_B is the $\mathrm{GL}_n(k)$-submodule of S_d generated by the monomials with carry pattern in $B \subseteq C(d)$. We are now ready to state the main theorem of [1].

Theorem 2.14 ([1])**.** *The correspondence $B \to T_B$ defines a lattice isomorphism between the lattice of order-closed subsets of $C(d)$ and the lattice of G-submodules of S_d, where G is $\mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$.*

That is, the $\mathrm{GL}_n(k)$-submodules of S_d are indexed by order-closed sets of carry patterns. By reading off the carry patterns of monomials generating a submodule M, one may deduce the set B for which $M = T_B$.

Example 2.15. Consider $\mathrm{GL}_n(k)$-submodules of S_{10} corresponding to order-closed subsets of $C(10)$. We index each order-closed subset of $C(10)$ by its maximal carry patterns:

maximal carry patterns	$\mathrm{GL}_n(k)-$submodule of S_10
(1,1,1)	S_{10}
(1,0,1)	$k\langle x^{10}, x^9y, x^8y^2, x^6y^4, x^5y^5, x^4y^6, x^2y^8, xy^9, y^{10}\rangle$
(1,0,0),(0,0,1)	$k\langle x^{10}, x^9y, x^8y^2, x^6y^4, x^4y^6, x^2y^8xy^9x^{10}\rangle$
(1,0,0)	$k\langle x^{10}, x^9y, x^8y^2, x^2y^8, xy^9, x^{10}\rangle$
(0,0,1)	$k\langle x^{10}, x^8y^2, x^6y^4, x^4y^6, x^2y^8, x^{10}\rangle$
(0,0,0)	$k\langle x^{10}, x^8y^2, x^2y^8, x^{10}\rangle$

Note that in the third row, there are two maximal carry patterns and the linear basis is the union of the linear basis for each of the maximal carry patterns appearing.

Theorem 2.16. *Any $\mathrm{GL}_n(k)$-submodule of S_d is indecomposable.*

Proof. By Theorem 2.14, we have that every $\mathrm{GL}_n(k)$-submodule of S_d must contain the submodule associated to the carry pattern consisting of all zeroes. In particular, the action of $\mathrm{GL}_n(k)$ guarantees that the monomials x^d and y^d are in the $\mathrm{GL}_n(k)$-orbit of any monomial in S_d, so a direct sum of $\mathrm{GL}_n(k)$-submodules in the same degree is not possible. □

[1] explored the $GL_n(k)$-module structure on S_d, but did not discuss the ring/ideal structure. To this end, we wish to understand how the carry patterns (and thus $\mathrm{GL}_n(k)$-modules) appearing in degree d affect the $\mathrm{GL}_n(k)$-modules appearing in degree $d+1$. We can move from degree d to degree $d+1$ by muliplying by a single variable. Thus, we now explore how carry patterns change when multiplying an arbitrary monomial in degree d by a single variable.

Lemma 2.17. *Given a field of characteristic p, fix a monomial $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$ of degree d with carry pattern $(c_1, \cdots, c_M)$. Let $(c'_1, \cdots, c'_M, c'_{M+1})$ be the carry pattern of $x_i x^{\mathbf{b}}$ (we need c'_{M+1} when $d = p^{M+1} - 1$). Let $0 \leq l \leq M+1$ be the smallest integer where $b_{il} \neq p-1$, then:*

(1) *For $1 \leq k \leq l$:*

$$c'_k = \begin{cases} c_k & c'_{k-1} = c_{k-1} \text{ and } d_{k-1} = p-1 \\ c_k - 1 & c'_{k-1} = c_{k-1} \text{ and } d_{k-1} \neq p-1 \\ c_k - 1 & c'_{k-1} = c_{k-1} - 1 \end{cases}$$

(2) *For $k = l+1$:*

$$c'_{l+1} = \begin{cases} c_{l+1} + 1 & c'_l = c_l \text{ and } d_l = p-1 \\ c_{l+1} & c'_l = c_l \text{ and } d_l \neq p-1 \\ c_{l+1} & c'_l = c_l - 1 \end{cases}$$

(3) *For $l+1 < k \leq M+1$:*

$$c'_k = \begin{cases} c_k + 1 & c'_{k-1} = c_{k-1} + 1 \text{ and } d_{k-1} = p-1 \\ c_k & c'_{k-1} = c_{k-1} + 1 \text{ and } d_{k-1} \neq p-1 \\ c_k & c'_{k-1} = c_{k-1} \end{cases}$$

Proof. First note that by hypothesis, $b_i + 1$ has base-p expansion:

$$b_i + 1 = (p-1) + \cdots + (p-1)p^{l-1} + (b_{il} + 1)p^l + b_{i,l+1}p^{l+1} + \cdots + b_{iM}p^M$$

Next recall that for any monomial $x_1^{b_1} \cdots x_n^{b_n}$ with carry pattern $(c_1, \cdots, c_M)$, we have:

$$c_k = \left\lfloor \frac{c_{k-1} + \sum_{s=1}^n b_{s,k-1}}{p} \right\rfloor \tag{2}$$

and

$$c_{k-1} + \sum_{s=1}^n b_{s,k-1} = d_{k-1} + pc_k \equiv d_k \mod p \tag{3}$$

We will prove (1), (2), and (3) in this order by induction on k. Note that for different values of l, some cases can be empty: When $l = 0$, case (1) is empty. When $l = M$, case (3) is empty. When $l = M+1$, case (2) and (3) are both empty. Otherwise, all three cases are nonempty.

In case (1) when $k = 1$, since $l \geq 1$, we know the 0th coefficient of b_i is 0, so by applying Equation (2) to $x_i x^{\mathbf{b}}$ we have:

$$c'_1 = \left\lfloor \frac{c_0 - (p-1) + \sum_{s=1}^n b_{s0}}{p} \right\rfloor$$

So by Equation (3) with $k = 1$, we get:

$$c'_1 = \begin{cases} c_1 & d_0 = p-1 \\ c_1 - 1 & d_0 \neq p-1 \end{cases}$$

Now suppose $1 < k \leq l$ and statement (1) is true for every carry before c'_k. So by induction c'_{k-1} is either c_{k-1} or $c_{k-1} - 1$. Since $k - 1 < l$, we know the $(k-1)$th coefficient of $b_i + 1$ is 0, applying 2 to $x_i x^{\mathbf{b}}$ we have:

$$c'_k = \left\lfloor \frac{c'_{k-1} - (p-1) + \sum_{s=1}^n b_{s,k-1}}{p} \right\rfloor$$

So by 3:

$$c'_k = \begin{cases} c_k & c'_{k-1} = c_{k-1} \text{ and } d_{k-1} = p-1 \\ c_k - 1 & c'_{k-1} = c_{k-1} \text{ and } d_{k-1} \neq p-1 \\ c_k - 1 & c'_{k-1} = c_{k-1} - 1 \end{cases}$$

In case (2), since the lth coefficient of $b_i + 1$ is $b_{il} + 1$, applying 2 to $x_i x^{\mathbf{b}}$ we have

$$c'_{l+1} = \left\lfloor \frac{c'_l + 1 + \sum_{s=1}^n b_{s,l}}{p} \right\rfloor$$

Since c'_l is either c_l or $c_l - 1$ from case (1), by 3:

$$c'_{l+1} = \begin{cases} c_{l+1} + 1 & c'_l = c_l \text{ and } d_l = p-1 \\ c_{l+1} & c'_l = c_l \text{ and } d_l \neq p-1 \\ c_{l+1} & c'_l = c_l - 1 \end{cases}$$

In case (3) since $k > l+1$, by induction c'_{k-1} is either $c_{k-1}+1$ or c_{k-1}. As $k-1 > l$, we know the $(k-1)$th coefficient of $b_i + 1$ is $b_{i,k-1}$, so applying 2 to $x_i x^{\mathbf{b}}$, we get:

$$c'_k = \left\lfloor \frac{c'_{k-1} + \sum_{s=1}^n b_{s,k-1}}{p} \right\rfloor$$

So by 3

$$c'_k = \begin{cases} c_k + 1 & c'_{k-1} = c_{k-1} + 1 \text{ and } d_{k-1} = p-1 \\ c_k & c'_{k-1} = c_{k-1} + 1 \text{ and } d_{k-1} \neq p-1 \\ c_k & c'_{k-1} = c_{k-1} \end{cases}$$

□

As the first application of Lemma 2.17, we look at the new carry c'_{M+1} when $d = p^M - 1$.

Corollary 2.18. *Let $d = p^M - 1$. Then $c'_{M+1} = 0$ if and only if $x^{\mathbf{b}} = x_i^d$ for some variable x_i and we choose to multiply $x^{\mathbf{b}}$ by x_i. Otherwise, $c'_{M+1} = 1$.*

Proof. Since $d = p^{M+1} - 1$, we know $d_{k-1} = p-1$ for $1 \leq k \leq M+1$, so by Lemma 2.17 we see that the new carry $c'_{M+1} = c_{M+1} = 0$ if and only if cases (2) and (3) are both empty. This means that $c'_{M+1} = 0$ if and only if $l = M+1$, or equivalently, $x^{\mathbf{b}}$ is the dth power of a single variable x_i and we choose the multiply $x^{\mathbf{b}}$ by the same x_i. Otherwise, $c'_{M+1} = c_{M+1} + 1 = 1$. □

Remark 2.19. Suppose $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$ has carry pattern $(c_1, \cdots, c_M)$. For a variable x_i, let l be the smallest integer such that $b_{il} \neq p-1$. By Lemma 2.17, the carry pattern of $x_i x^{\mathbf{b}}$ can be viewed as a path in the following directed graph.

$$\begin{tikzcd}
& c_1 \arrow[r] \arrow[ddr] & c_2 \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & c_l \arrow[r] \arrow[ddr] & c_{l+1}+1 \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & c_M+1 \arrow[r] \arrow[ddr] & c_{M+1}+1 \\
0 \arrow[ur] \arrow[dr] & & & & & & & & \\
& c_1-1 \arrow[r] & c_2-1 \arrow[r] & \cdots \arrow[r] & c_l-1 \arrow[r] & c_{l+1} \arrow[r] & \cdots \arrow[r] & c_M \arrow[r] & c_{M+1}
\end{tikzcd}$$

Let j be the smallest integer such that $d_j \neq p-1$. Define the downward edge of d to be the edge from the j-th vertex in the upper row down to the $j+1$-th vertex in the lower row. Then the carry pattern of $x_i x^{\mathbf{b}}$ will be determined by the unique path containing the j-th downward edge of d. By convention, $c_0 = c'_0 = 0$ and $c_{M+1} = 0$.

Note that the value of l determines how early we get $+1$'s in the top row and avoid -1's in the bottom row, while the value of j determines the position of the downward edge. Thus l and j determine the new carry pattern. In particular, let $(c'_1, \cdots, c'_M, c'_{M+1})$ be the new carry pattern, then

$$c'_k = \begin{cases} c_k + 1 & l < k \leq j \\ c_k & (k \leq l \text{ and } k \leq j) \text{ or } (k > l \text{ and } k > j) \\ c_{k-1} - 1 & j < k \leq l \end{cases}$$

Corollary 2.20. *Let $x^{\mathbf{b}}, x^{\mathbf{b}'}$ be any two monomials in degree d with carry pattern c. Let $b_i, b_{i'}$ be the exponents on some variable from $x^{\mathbf{b}}, x^{\mathbf{b}'}$ respectively. Let l be the smallest integer where $b_{il} \neq p-1$, and l' be the smallest integer where $b_{i'l'} \neq p-1$. Without loss of generality, let $l \leq l'$. Then $c(x_i x^{\mathbf{b}}) \leq c(x_{i'} x^{\mathbf{b}'})$, with equality if and only if $l = l'$.*

Proof. Since $x^{\mathbf{b}}$ and $x^{\mathbf{b}'}$ have the same degree, the value of j, the smallest integer such that $d_j \neq p-1$, is fixed. So when $l = l'$, by Remark 2.19, the carry patterns of $x_i x^{\mathbf{b}}$ and $x_{i'} x^{\mathbf{b}'}$ are the same.

It remains to show that $c(x_i x^{\mathbf{b}}) < c(x_{i'} x^{\mathbf{b}'})$ when $l+1 = l'$. So suppose $l+1 = l'$. By Remark 2.19, the carry patterns of $x_i x^{\mathbf{b}}$ and $x_{i'} x^{\mathbf{b}'}$ only differ in the $(l+1)$-th carry. In particular:

$$c(x_i x^{\mathbf{b}})_{l+1} = \begin{cases} c_{l+1} + 1 & l+1 \leq j \\ c_{l+1} & l+1 > j \end{cases}$$

$$c(x_{i'} x^{\mathbf{b}'})_{l+1} = \begin{cases} c_{l+1} & l+1 \leq j \\ c_{l+1} - 1 & l+1 > j \end{cases}$$

Since the value of j is fixed, $c(x_i x^{\mathbf{b}}) < c(x_{i'} x^{\mathbf{b}'})$. ☐

Corollary 2.21. *Fix a carry pattern c with degree d. Then, the set of carry patterns of $x_i x^{\mathbf{b}}$ for all $1 \leq i \leq n$ and $x^{\mathbf{b}} \in T_c$ is a totally ordered set.*

Proof. Since Corollary 2.20 holds for any two monomials with the same carry pattern, we get a total order. ☐

3. Decomposition of stable ideals

Recall that an ideal I in $S = k[x_1, \dots, x_n]$ is called $\mathrm{GL}_n(k)$-stable if $Af \in I$ for all $A \in \mathrm{GL}_n(k)$, $f \in I$.

In what follows, for $\mathrm{GL}_n(k)$-stable ideals we suppress the prefix "$\mathrm{GL}_n(k)$" and call them "stable ideals", and they are ideals in S unless otherwise stated.

To determine if a given ideal is stable, it suffices to consider the action of $\mathrm{GL}_n(k)$ on the generators of I. Suppose $I = \langle g_1, g_2, \dots, g_\ell \rangle$ for g_i in S, and that for any $A \in \mathrm{GL}_n(k)$, Ag_i is in I. Let f_i be in S so that $f = \sum_i f_i g_i$ is in I. Then $Af = \sum_i (Af_i)(Ag_i)$ is in I because Af is a polynomial combination of elements in I.

Example 3.1. Let $S = k[x, y]$, where $\mathrm{char}(k) = 3$, and let $I = \langle x^3, y^3 \rangle$. Consider the action of an arbitrary element $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ on the generators of I:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^3 = (ax+cy)^3 = a^3x^3 + 3a^2cx^2y + 3ac^2xy^2 + c^3y^3 = a^3x^3 + c^3y^3$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} y^3 = (bx+dy)^3 = b^3x^3 + 3b^2dx^2y + 3bd^2xy^2 + d^3y^3 = b^3x^3 + d^3y^3$$

Let f be in I, where $f = f_1x^3 + f_2y^3$ such that $f_1, f_2 \in S$. Then for any $A \in \mathrm{GL}_2(k)$

$$\begin{aligned} Af &= (Af_1)(Ax^3) + (Af_2)(Ay^3) \\ &= ((Af_1)a^3 + (Af_2)b^3)x^3 + (A(f_1)c^3 + A(f_2)d^3)y^3 \end{aligned}$$

Therefore, $Af \in I$, and I is $\mathrm{GL}_2(k)$-stable. If instead $\mathrm{char}(k) = 3$, I is not stable:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^3 = (ax+cy)^3 = a^3x^3 + 3a^2cx^2y + 3ac^2xy^2 + c^3y^3$$

The monomials x^2y and xy^2 are not in I, so I is not $\mathrm{GL}_2(k)$-stable.

Example 3.2. Let $S = k[x, y]$ where $\mathrm{char}(k) = 2$. Let $I = \langle x^2, y^2, x^5, x^4y, xy^4, y^5 \rangle$. To determine if I is $\mathrm{GL}_2(k)$-stable, consider the action of an arbitrary matrix on the generators of I. For any ideal, it suffices to check the action on half of the generators because variables can be permuted using permutation matrices, which are contained in $\mathrm{GL}_n(k)$. Conversely, the generators of stable ideals must be symmetric in all variables.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^2 = (ax+cy)^2 = a^2x^2 + c^2y^2$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^5 = (ax+cy)^5 = a^5x^5 + 5a^4cx^4y + 5ac^4xy^4 + c^5y^5$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^4y = (a^4x^4 + c^4y^4)(bx+dy) = a^4bx^5 + a^4dx^4y + bc^4xy^4 + c^4dy^5$$

Because it suffices to check the action on the ideal generators for stability, one simple way to obtain stable ideals is to take a $\mathrm{GL}_n(k)$-submodule of S_d, and form the ideal generated by the elements of that submodule. We focus our study on ideals generated in this manner.

Definition 3.3. Let B be an order-closed subset of $C(d)$, and let $T_{B,d}$ be the corresponding $\mathrm{GL}_n(k)$-submodule of S_d. Denote by $I_{B,d}$ the ideal generated by the elements of $T_{B,d}$ (equivalently the ideal generated by the generators of $T_{B,d}$).

For $T_{B,d}$ a $\mathrm{GL}_n(k)$-submodule of S_d, the corresponding ideal $I_{B,d}$ is a stable ideal since the S-generators of $I_{B,d}$ are the $\mathrm{GL}_n(k)$-generators of $T_{B,d}$. The action of $\mathrm{GL}_n(k)$ takes a generator of $T_{B,d}$ to a linear combination of generators of $T_{B,d}$, from which it follows that $I_{B,d}$ is a stable ideal.

Form other stable ideals out of these stable ideals by taking sums. The sum of finitely many stable ideals is a stable ideal (since stable ideals are $\mathrm{GL}_n(k)$-submodules of S).

Lemma 3.4 (Containment of stable ideals)**.** *Let I, J be stable ideals. By Theorem 2.14, we have for each $d \geq 0$, that $I_d = T_{B_d,d}$ and $J_d = T_{B'_d,d}$ for $B_d, B'_d \subseteq C(d)$.*

Then $I \subseteq J$ if and only if for every $d \geq 0$, B_d is a subset of B'_d.

Proof. If $I \subseteq J$, then for each $d \geq 0$, $I_d = T_{B_d,d}$ is a subset of $J_d = T_{B'_d,d}$. By Theorem 2.14, $B_d \subseteq B'_d$ for each $d \geq 0$. Conversely, if $B_d \subseteq B'_d$ for each $d \geq 0$, then again by Theorem 2.14 we have $I_d = T_{B_d,d} \subseteq T_{B'_d,d} = J_d$ for each $d \geq 0$ as needed. □

Below is an algorithm which decomposes any stable ideal into an irredundant sum of ideals generated by submodules of S_d for various d.

Algorithm 1: Decomposes stable ideals into irredundant sum of stable ideals, each generated in a single degree.

```
Data: stable ideal I
Result: set of ideals I_{B_d,d} for which I = ∑_d I_{B_d,d}
integer D ← smallest degree d for which I_d is nonzero;
set B_D ← {c(m) | m a generator of I_D};
set L ← {I_{B_D,D}};
integer d ← D + 1;
while I_d ≠ S_d do
    set H ← {c(x_i m) | m a generator of I_{d−1}, 1 ≤ i ≤ n};
    set G ← {c(m) | m a generator of I_d};
    if H ≠ G then
        D ← d;
        B_D ← G;
        L ← L ∪ {I_{B_d,d}};
    end
    d ← d + 1;
end
return L;
```

We use the algorithm to prove Theorem 1.2.

Proof (of Theorem 1.2). Let I be a (nontrivial) stable ideal. We set D to be the smallest d for which $I_d \neq 0$, and set B_D to be the carry patterns of the monomials generating I_D. By Lemma 3.4, $I_{B_D,D} \subseteq I$. Set L to be the set containing $I_{B_D,D}$.

Let $d = D + 1$ and consider the $\mathrm{GL}_n(k)$-submodule I_d.

If $I_d = S_d$, stop and return the list L. (We verify that this algorithm terminates later.) If not, then do the following: Set H to be the set of carry patterns of monomials $x_i m$ for m a $\mathrm{GL}_n(k)$-generator of I_{d-1}, and $1 \leq i \leq n$. These monomials $x_i m$ are the degree d monomials appearing in the degree d homogeneous component of $I_{B_D,D}$. Set G to be the set of carry patterns of monomials m for m a $\mathrm{GL}_n(k)$-generator of I_d. Certainly $H \subseteq G$. If $H \neq G$, then append to the set L the ideal $I_{G,d}$. Increment d by one and repeat this paragraph for I_d.

Every monomial in S_{nD} is divisible by some x_i^D by the pigeonhole principle. Thus I_{nD} contains all monomials of degree nD, since I_D contains x_j^D for each $1 \leq j \leq n$. As $I_{nD} = S_{nD}$, the algorithm must terminate.

Observe that at each stage of the algorithm, the sum of the elements in L is contained in I by Lemma 3.4. It follows that the sum of the elements of the L that is returned is contained in I.

Let $I' = \sum_d I_{B_d,d}$ be the sum of the elements of L. We show that I is contained in I'. Let $\ell \geq 0$. Then I_ℓ is contained in I'_ℓ if and only if the carry patterns of monomials in I_ℓ appear as carry patterns of monomials in I'_ℓ. The algorithm ensures that this is the case, since we are always comparing the set H of carry patterns of monomials of $I_{\ell-1}$ multiplied with variables x_i with the set G of carry patterns of monomials in I_ℓ, at each degree ℓ.

When $H = G$, all the monomials of I_ℓ appear in I'_ℓ. No ideal is added to the set L in this case. If H is a proper subset of G, then there are monomials in I_ℓ which are not obtained by taking products of monomials in $I_{\ell-1}$ with variables x_i. So in this case we add in the ideal $I_{G,\ell}$ generated by monomials whose carry pattern is in G to the list L. In either case, $I_\ell = I'_\ell$.

If the ideal $I_{G,\ell}$ were not added to L in the second case above, then there would be a monomial in I_ℓ which did not appear in I'_ℓ. In this sense the algorithm produces an irredundant (or minimal) list of ideals L, as removing any of the elements of L would cause the equality $I = I'$ to be false.

Since the equality $I_\ell = I'_\ell$ is true for each $\ell \geq 0$, $I = I'$. $\square$

This theorem has some downsides. Using this theorem, we cannot characterize stable ideals without without computing and comparing many carry patterns and taking products of many monomials, as in the algorithm above.

Example 3.5. Let $S = k[x, y]$ where $\mathrm{char}(k) = 2$. Consider the ideal $I = \langle x^5, x^4y, xy^4, y^5\rangle$. Since $I = I_{(0,0),5}$, the decomposition of I using Algorithm 1 should only be itself. We check that the algorithm gives this result.

Assign $D = 5$, $B_D = \{(0, 0)\}$, and $L = I_{(0,0),5}$.

Multiply each monomial in $I_5 = T_{(0,0),5}$ by x or y to obtain the monomials $\{x^6, x^5y, x^4y^2, x^2y^4, xy^5, y^6\}$. Then $H = \{(0, 0), (1, 0)\}$. Since G, the set of carry patterns of monomials in I_6, is the same as H, no ideal is added to L.

Repeat, multiplying the monomials in I_6 by x and y. The result list of monomials is the list of all monomials in S_7, so the algorithm terminates. Hence the decomposition of I is $I_{(0,0),5}$ as expected.

Example 3.6. Let $S = k[x, y]$ where $\mathrm{char}(k) = 2$. Consider the ideal $I = \langle x^8, x^9, x^8y, x^5y^4, x^4y^5, xy^8, y^9, y^8\rangle$. We will decompose I into a sum of stable ideals using Algorithm 1.

Set $D = 8$ and $B_D = \{(0, 0, 0)\}$, so $L = I_{(0,0,0),8}$.

Multiply the monomials in I_8 by x and y. The set of carry patterns of the resulting set of monomials is H. Compare this with G, the set of carry patterns from monomials in I_9. Since $H \neq G$, assign $B = G$ and add the ideal $I_{B,9}$ to L.

$$
\begin{aligned}
I_8 &= (\langle x^8, y^8\rangle)_8 = T_{(0,0,0),8} \\
\{x, y\}I_8 &= (\langle x^9, x^8y, xy^8, y^9\rangle)_9 = T_{H,9} \\
H &= \{(0, 0, 0)\} \text{ and } G = \{(0, 0, 0), (0, 0, 1)\} \longrightarrow B = G \\
I_{B,9} &= \langle x^9, x^8y, x^5y^4, x^4y^5, xy^8, y^9\rangle \\
L &= I_{(0,0,0),8} + I_{B,9}
\end{aligned}
$$

Multiply I_9 by elements in $\{x, y\}$ and call the resulting set of carry patterns H. The degree ten piece of I has no monomials of higher carry pattern so $H = G$, and no ideal is added to L.

$$\begin{aligned}\{x, y\}I_9 &= (\langle x^{10}, x^9y, x^8y^2, x^6y^4, x^4y^6, x^2y^8, xy^9.y^{10}\rangle)_{10} = T_{H_2,10}\\ H &= \{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\} = G\\ L &= I_{(0,0,0),8} + I_{B,9}\end{aligned}$$

Multiplying I_{10} by elements in $\{x, y\}$ results in S_{11}, so the algorithm terminates, and the decomposition is

$$I = I_{(0,0,0),8} + I_{B',9}$$

4. Free resolutions of $\mathrm{GL}_2(k)$-stable ideals

For a graded S-module M, denote by $M(-j)$ the graded S-module whose graded pieces are $[M(-j)]_i = M_{i-j}$; that is, $M(-j)$ is the graded module which is the same as M, but each graded piece of M has been shifted up j in degree. We will mostly suppress this notation in this work.

Definition 4.1. [3] A *free resolution* of a finitely generated S-module M is an exact sequence of S-module homomorphisms

$$\mathbf{F}: \quad 0 \leftarrow M \xleftarrow{\partial_0} F_0 \xleftarrow{\partial_1} F_1 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{\partial_i} F_i \leftarrow \cdots$$

for which each F_i is a free S-module.

- When M is graded, we say $\mathbf{F}$ is a *graded free resolution* if each F_i is graded and each map ∂_i has degree 0.
- Say $\mathbf{F}$ is *minimal* if $\partial_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i$ for all $i \geq 0$. This implies that no units will appear in the matrices for each ∂_i.
- Say $\mathbf{F}$ is a *finite resolution* if its *length*, $\max\{i \mid F_i \neq 0\}$, is finite. Otherwise $\mathbf{F}$ is an *infinite resolution*.
- Let G be a group. If S, M are G-representations and if in $\mathbf{F}$ the maps ∂_i are all G-equivariant maps, then $\mathbf{F}$ is an *equivariant free resolution* of M.

Theorem 4.2 (Hilbert's Syzygy Theorem [3])**.** *The minimal graded free resolution of a finitely generated graded S-module is finite and has length at most n.*

Definition 4.3. The *projective dimension* of an S-module M, denoted $\mathrm{pd}_S(M)$ or just $\mathrm{pd}(M)$, is the minimum length of all of its finite free resolutions.

Remark 4.4. Theorem 7.5 in [3] asserts the existence of a unique minimal free resolution of a finitely generated graded S-module M up to an isomorphism of complexes. So to find the projective dimension of M, it suffices to find the length of a minimal free resolution of M.

Definition 4.5. For M an S-module, an *M-regular sequence* is a sequence $(f_1, \ldots, f_\ell)$ of elements in $\mathbf{m} := \oplus_{d\geq 1} S_d$ such that f_i is a non-zero divisor on $M/(f_1, \ldots, f_{i-1})M$ for $i = 1, \ldots, \ell$.

Definition 4.6. For M an S-module, we define the *depth* of M, denoted $\mathrm{depth}_S(M)$ or just $\mathrm{depth}(M)$, is the maximum of the lengths of all M-regular sequences.

Theorem 4.7 (Auslander-Buchsbaum formula [3])**.** *For M a finitely generated graded S-module, we have*

$$\mathrm{pd}(M) = n - \mathrm{depth}(M).$$

We study the minimal free resolutions of S/I (equivalently, minimal free resolutions of I) for stable ideals I.

Proposition 4.8. *Let I be a stable ideal. Then* $\mathrm{depth}(S/I) = 0$.

Proof. By Theorem 1.2, there is a d' large enough so that $I_{d'} = S_{d'}$. Take d' to be the smallest d' for which $I_{d'} = S_{d'}$. [From the discussion preceding the theorem, we have that $d' \leq nD$, where D is the smallest integer for which I_D is not 0.] It follows that any element f of $\mathbf{m}$ is a zero divisor (i.e. is not a regular element) on S/I:

Let $f \in \mathbf{m}$, so that the homogeneous components of f each have degree at least 1. Observe the S-module $(S/I)_{d'-1}$ is nonzero because d' was chosen minimally above. Then for any nonzero

$\overline{g} \in (S/I)_{d'-1}$, we have $f\overline{g} = 0$ in S/I, since each homogeneous component of $f\overline{g}$ has degree at least d'. □

Corollary 4.9. *The length of a minimal free resolution of S/I for a stable ideal I is n, the number of indeterminates x_i in S.*

Proof. The Auslander-Buchsbaum formula or Hilbert's Syzygy theorem combined with the previous proposition yields the result. □

We show that Frobenius powers of powers of the maximal ideal $\langle x, y\rangle$ are the only ideals with syzygies in one degree.

Definition 4.10. Let $I = \langle f_1, \dots, f_m\rangle$ be an ideal of S. Then the p^e-th *Frobenius power* of I is the ideal $I^{[p^e]} = \{f^{p^e} \mid f \in I\}$. Since k has characteristic p, it follows that $I^{[p^e]} = \langle f_1^{p^e}, \dots, f_m^{p^e}\rangle$.

Proposition 4.11. *A stable ideal I in $S = k[x, y]$ has syzygies of the same degree if and only if I is of the form $(\langle x, y\rangle^m)^{[p^e]}$.*

Proof. Let $I = (\langle x, y\rangle^m)^{[p^e]} = \langle x^{p^e a} y^{p^e(m-a)} \mid 0 \le a \le m\rangle$. The monomial $x^{p^e a} y^{p^e(m-a)}$ is greater than the monomial $x^{p^e(a-1)} y^{p^e(m-a+1)}$ in lexicographical order (for $0 \le a \le m-1$), and there are no monomials $x^{\mathbf{b}} \in I$ such that $x^{p^e a} y^{p^e(m-a)} \ge x^{\mathbf{b}} \ge x^{p^e(a-1)} y^{p^e(m-a+1)}$. Then the syzygy for the monomials $x^{p^e a} y^{p^e(m-a)}$ and $x^{p^e(a-1)} y^{p^e(m-a+1)}$ is of degree p^e.

Conversely, suppose I has syzygies in a single degree c. Then $I = \langle x^m, x^{m-c}y^c, \dots, x^c y^{m-c}, y^m\rangle$ for some m, and it follows that c divides m. We show that c is a power of p.

Suppose that c is not a power of p. Then m is not a power of p for any e. Then let p^e be the smallest power of p dividing m with $e \ge 0$. Then $m - p^e > 0$, so the monomial $x^{p^e} y^{m-p^e}$ has carry pattern $(0, \dots, 0)$. But x^m is a monomial in I with carry pattern $(0, \dots, 0)$ and I is a stable ideal, so $x^{p^e} y^{m-p^e} \in I$ also. This is a contradiction since c is not a power of p. □

smallest $\mathrm{GL}_n(k)$-stable ideals in a given degree

We first note that the smallest $\mathrm{GL}_n(k)$-stable ideals, in terms of number of generators, for a given degree are exactly $I_{(0,\dots,0;d)}$. The rest of this section studies ideals of this form.

Lemma 4.12. *For $d = \sum_i d_i p^i$, a monomial $x^a y^{d-a}$ is carry pattern $(0, \dots, 0)$ iff $a = \sum_i a_i p^i$ where each $0 \le a_i \le d_i$.*

Proof. We first prove that $x^a y^{d-a}$ is a monomial of carry pattern $(0, ..., 0)$.
We have that the p-adic expansion of a is $\sum_i a_i p^i$ by construction. Also, as each $0 \le a_i \le d_i$, the expansion of $d - a$ is $\sum_i (d_i - a_i)p^i$. Thus when we find the carry pattern of $x^a y^{d-a}$ examining every power i in the expansion we receive the sum $d_i - a_i + a_i = d_i < p$. And thus there is no carry. As this is the case with every position i, the carry pattern is $(0, .., 0)$, as desired.

We now examine the converse of the above statement, which we prove by contrapositive. We consider some $x^a y^{d-a}$ where a has expansion $\sum_i a_i p^i$ such that some $d_i < a_i \le p - 1$. Assume that j is the first position in which $a_i > d_i$. Note that the expansion of $d - a$ at position j becomes $(d_j + p) - a_j$ in accordance with subtraction in base p. Hence [the sum $(d - a)_j + a_j$ is greater than p, so $c_{j+1} > 0$.] when we find the carry pattern of $x^a y^{d-a}$, in position j we have some sum of the form $(d_j - a_j + p) + (a_j) = p + d_j > p$, corresponding to a carry in this position. Hence $x^a y^{d-a}$ is not of carry $(0, ..., 0)$. □

A sequence $d_r = d_s = 0$ with $r = s$ (of length 1) is allowed.

Blocks are substrings of the base p expansion of d and always have the form

$$(p - 1, \dots, p - 1, a - 1).$$

Note that $a - 1$ may be zero, in particular when $d_M = p - 1$.

Definition 4.13. The *starting position* of a block $(d_r, \dots, d_s)$ of d is r. Alternatively, we say the block $(d_r, \dots, d_s)$ *starts* at r.

Definition 4.14. Let d be a natural number with $(d_0, \dots, d_M)$ its base p expansion. For any subsequence $(d_r, \dots, d_s)$ of d, its *content* is $\sum_{i=0}^{s-r} d_{r+i}p^i$, denoted by $|(d_r, \dots, d_s)|$.

For a block b, denote its content by $|b|$. A block with content 0 is called a *zero block*, otherwise it is called a *nonzero block*.

The content of b is the value corresponding to the tuple b viewed as a base p expansion of a natural number.

Remark 4.15. Alternatively, a nonzero block of d is a tuple $(d_r, \dots, d_s)$ of maximal length with $0 \le r \le s \le M+1$, such that $\sum_{i=r}^{s} d_i p^i = p^r(ap^{s-r} - 1)$ for some $1 \le a \le p-1$.

Proposition 4.16. *For d a natural number with $(d_0, \dots, d_M)$ its base p expansion, the number of nonzero blocks of d is*

$$|\{i \in \mathbb{Z}_{\ge 0} \mid 0 < d_i < p-1\}| + |\{i \in \mathbb{Z}_{\ge 0} \mid d_i = 0 \text{ and } d_{i-1} = p-1\}|,$$

where $d_i = 0$ for $i > M$.

Proof. Let d be as above. Every block of d is of the form $(p-1, \dots, p-1, a-1)$ for some a between 1 and $p-1$. It follows that every block either terminates with a digit less than or equal to $p-2$. If a block terminates with zero and is not preceded by a $p-1$ in the base p expansion of d, then it is a zero block.

Every d_i between 1 and $p-2$ is the terminating digit for a nonzero block. The only other nonzero blocks are those that terminate with 0 and are preceded by $p-1$. ☐

Example 4.17. Let $p = 5$ and $d = (2, 0, 1, 4, 2, 2)$. (So $d = 8027$ in base 10.) There are five blocks: $(2), (0), (1), (4, 2)$, and (2), which agrees with Proposition 4.16:

$$|\{i \in \mathbb{Z}_{\ge 0} \mid 0 < d_i < p-1\}| = |\{0, 2, 4, 5\}|$$
$$|\{i \in \mathbb{Z}_{\ge 0} \mid d_i = 0 \text{ and } d_{i-1} = p-1\}| = |\varnothing|$$

The number of nonzero blocks is $|\{0, 2, 4, 5\}| = 4$.

We can also write d as in Remark 4.15:

$$d = 5^0(3 \cdot 5^0 - 1) + 5^2(2 \cdot 5^0 - 1) + 5^3(3 \cdot 5^1 - 1) + 5^5(3 \cdot 5^0 - 1)$$

Example 4.18. Let $p = 3$ and $d = (1, 0, 2, 2)$. (So $d = 73$ in base 10.) There are three blocks: (1), (0), and $(2, 2, 0)$, which agrees with Proposition 4.16: $|\{0\}| + |\{5\}| = 2$.

We are now ready to proof Theorem 1.4.

Proof (of Theorem 1.4). (1)

Let $A = \{a \in \mathbb{N} \mid a \le d,\, a_k \le d_k \forall k\} \subseteq \mathbb{N}$. A is a totally ordered by the order induced from $\mathbb{N}$. By Lemma 4.12, A is the set of possible exponents on x for the minimal generators of I. By construction of A, the degrees of syzygies are the differences between consecutive terms in A, so it suffices to show that the differences between consecutive terms in A are of the form.

We consider consecutive numbers within the set A. For d with base p expansion $(d_0, ..., d_M)$, we consider some exponent $a \in A$ given by $(a_0, ..., a_M)$. We note that, by Lemma 4.12, digits must consist only of $0 \le a_j \le d_j$. Let i be the first index such that $a_i < d_i$. We now take the smallest exponent greater than a in A. This is given by $a' = (0, \dots, 0, a_i + 1, a_{i+1} \dots a_M)$. Notice that the difference is $a' - a = p^i - \sum_{j<i} d_j p^j = p^i - |(d_0, \dots, d_{i-1})|$.

We note two things:

First, if $d_j = 0$, we must always have $a_j = 0$, as such there is never a case such that i corresponds to $d_i = 0$, as this would imply that we have an exponent outside our set A. Thus, when considering syzygy degrees, we may skip any digit i in which $d_i = 0$. Thus, zero blocks do not correspond to a unique degree of syzygy.

Second, if we take i to be some value in which $d_{i-1} = p-1$, we have that the sum

$$p^i - |(d_0, \dots, d_{i-1})| = p^i - (p-1)p^{i-1} - |(d_0, \dots, d_{i-2})| = p^{i-1} - |(d_0, \dots, d_{i-2})|.$$

Hence if $d_{i-1} = p-1$, the degree of the syzygy calculated at indices i and $i-1$ are the same, and it suffices to only consider the starting position of each nonzero block. Indeed, in a substring of d given by $(d_e, ..., d_f)$ where $d_j = p-1$ for all $e \leq j < f$, it is sufficient to consider only the staring position of this block calculating degrees of the syzygy.

These two statements show that indeed, $p^i - \sum_{j<i} d_j p^j$ is unique only when i corresponds to the start of a new nonzero block. Also, the sum $\sum_{j<i} d_j p^j$ is equivalent to $p^i - |(d_0, \ldots, d_{i-1})|$, by construction of the content of blocks. That is to say that the distinct syzygy degrees are always of the form $p^i - |(d_0, \ldots, d_{i-1})|$ for $i \in \mathcal{I}$.

We note that each syzygy degree must occur, as for such an i, the exponents given by $(a_0, ..., a_M)$, and $(0, 0, ..., a_i + 1, a_{i+1}, ..., a_M)$ do indeed occur, as guaranteed by Lemma 4.12.

(2)

By (1), every syzygy degree is of the form $p^i - |(d_0, \ldots, d_{i-1})|$ for a block starting at i, and as such the map from blocks to syzygy degrees is surjective. We must now show that the map is injective, that is each $p^i - |(d_0, \ldots, d_{i-1})|$ is unique.

We wish to show that each $p^i - |(d_0, \ldots, d_{i-1})|$ is distinct. We assume that some

$$p^i - |(d_0, \ldots, d_{i-1})| = p^k - |(d_0, \ldots, d_{k-1})|.$$

Without loss of generality we may assume that $i > k$. We have the following:

$$p^i - p^k = |(d_0, \ldots, d_{i-1})| - |(d_0, \ldots, d_{k-1})|$$

Expressing both sides in base p, we get:

$$\sum_{j=k}^{i-1} (p-1)p^j = \sum_{j=k}^{i-1} d_j p^j$$

By the uniqueness of base p expansions, each d_j must thus be equal to $p-1$, meaning the expansion of d from k through $i-1$ are all $p-1$, so digits k, and i are in the same block. Hence each $p^i - |(d_0, \ldots, d_{i-1})|$ is unique, and thus the map from nonzero blocks to syzygy degrees is injective, as desired.

(3)

First, we show that a syzygy degree of the form $p^i - |(d_0, \ldots, d_{i-1})|$ occurs between two consecutive monomials if and only if the larger exponent a is given by $(0, ..., 0, a_i, ..., a_s, a_{s+1}, ..., a_M)$, where i and s correspond to the start and end position of the block, and the first nonzero entry of a occurs within the block. By the same argument on consecutive exponents in (1), and the second observation, such a pair of monomials will have syzygy degree of $p^i - |(d_0, \ldots, d_{i-1})|$. And by the bijection shown in (2), these are the only such pairs of monomials.

We now wish to count how many of such monomials occur. Fix every a_j such that $j > s$. Now consider the function

$$\phi : (a_i, \ldots, a_s) \mapsto |(a_i, \ldots, a_s)|$$

By the uniqueness of p-expansion, ϕ is injective. Since there exists some $i \leq j \leq s$ such that $a_j \neq 0$, the image of ϕ is the set $\{1, 2, \ldots, |(d_i, \ldots, d_s)|\}$. Hence for every set of fixed $a_{s+1}, \ldots, a_M$, the number of possible exponents is $|(d_i, \ldots, d_s)| = |b_i|$.

Finally, we find how many total possible monomials with the desired syzygy degrees exist. As the values a_j, such that $j > s$, do not affect the syzygy degree of monomials described we must consider every possible permutation. By Lemma 4.12 the possible values are $0 \leq a_j \leq d_j$, so the number of options are $d_j + 1$ for each a_j. Hence the number of syzygies with desired degree are $(d_{s+1} + 1)(d_{s+2} + 1)...(d_M + 1)(|b_i|) = |b_i| \prod_{j>i}(d_j + 1)$ as desired.

(4)

As every exponent in a monomial in $I_{(0,\dots,0)}$ is given by Lemma 4.12 we must simply use principle of counting on the possible values given by the lemma. For every d_j, the possible values of the digit for an exponent a are $0 \leq a_j \leq d_j$, giving $d_j + 1$ choices. By principle of counting the total number of generators is thus $\prod_j (d_j + 1)$.

□

Corollary 4.19. *Let $\alpha = \prod_{i=0}^{M}(d_i+1)$, $\beta_j = |b_{i_j}| \prod_{k>i_j}(d_k+1)$, and $\delta_{i_j} = p^{i_j} - |(d_0, \dots, d_{i_j-1})|$, where ℓ is the number of distinct degrees of syzygies. Then we have the following minimal free resolution:*

$$0 \leftarrow S/I \leftarrow S \xleftarrow{F} S(-d)^{\alpha} \xleftarrow{G} \bigoplus_{1 \leq j \leq \ell} S(-d - \delta_{i_j})^{\beta_j} \leftarrow 0. \tag{4}$$

Example 4.20. Consider again the set-up of 4.17, where $p = 5$ and $d = (2, 0, 1, 4, 2, 2)$. Since there are four nonzero blocks, there are four distinct degrees of syzygies. We find the degree of these syzygies and the multiplicity of each.

First, the set of starting positions for the nonzero blocks is $\mathcal{I} = \{0, 2, 4, 5\}$. The content of each nonzero block is $|(2)| = 2$, $|(1)| = 1$, $|(4, 2)| = 14$, and $|(2)| = 2$. By Theorem 1.4, we have the following degrees of syzygies and the number of each degree of syzygy:

starting position of block	degree of syzygy	multiplicity
$i_1 = 0$	$5^0 - 0 = 1$	$2(2 \cdot 5 \cdot 3 \cdot 3) = 180$
$i_2 = 2$	$5^2 - 2 \cdot 5^0 = 23$	$1(5 \cdot 3 \cdot 3) = 45$
$i_3 = 3$	$5^3 - 1 \cdot 5^2 - 2 \cdot 5^0 = 98$	$14(3) = 42$
$i_4 = 5$	$5^5 - 14 \cdot 5^3 - 1 \cdot 5^2 - 2 \cdot 5^0 = 1348$	2

And so the number of minimal generators of I is $3 \cdot 1 \cdot 2 \cdot 5 \cdot 3 \cdot 3 = 270$

We also have the minimal free resolution:

$$0 \leftarrow S/I \leftarrow S \xleftarrow{F} S(-8027)^{\alpha} \xleftarrow{G} \bigoplus_{1 \leq j \leq \ell} S(-8027 - \delta_{i_j})^{\beta_j} \leftarrow 0. \tag{5}$$

where $\alpha = 270$, $\beta_j \in \{180, 45, 42, 2\}$, and $\delta_{i_j} \in \{1, 23, 98, 1348\}$.

5. Future work

Within Section 2 we described an algorithm, Lemma 2.17, to determine carry patterns of monomials in degree greater than d in a stable ideal given the carry patterns of monomials in degree d in the stable ideal. We seek to find a criterion that more explicitly finds the carry patterns of monomials of degree higher than d.

In Section 4 we determined the form of minimal free resolutions of stable ideals of $S = k[x, y]$ generated by degree d monomials with carry pattern $(0, \dots, 0)$. We know how to determine the form of minimal free resolutions of stable ideals generated by degree d monomials with carry pattern less than or equal to $c \in C(d)$ also, but this result is not included in this report (the proofs need polishing). We seek to find a method of determining the form of minimal free resolutions of stable ideals in more than three variables generated in a single degree by monomials with carry pattern less than $c \in C(d)$.

We also showed that the Frobenius power of a power of the maximal ideal $\langle x_1, \dots, x_n \rangle$ has syzygies of a single degree. We suspect that ideals formed by taking products of these Frobenius powers of powers of the maximal ideal $\langle x_1, \dots, x_n \rangle$ will have syzygies of more than one degree also.

Finally, we seek to capture the above results about submodules of S_d for $d \geq 0$ and stable ideals of S in a Macaulay2 package.

References

[1] Stephen Doty. "Submodules of symmetric powers of the natural module for GL_n". In: Invariant theory (Denton, TX, 1986). Vol. 88. Contemp. Math. Amer. Math. Soc., Providence, RI, 1989, pp. 185–191. ISBN: 0-8218-5094-6. DOI: 10.1090/conm/088/999991. URL: https://doi.org/10.1090/conm/088/999991.

[2] Eknath Ghate and Ravitheja Vangala. "The monomial lattice in modular symmetric power representations". In: Algebr. Represent. Theory 25.1 (2022), pp. 121–185. ISSN: 1386-923X,1572-9079. DOI: 10.1007/s10468-020-10013-x. URL: https://doi.org/10.1007/s10468-020-10013-x.

[3] Irena Peeva. Graded Syzygies. Algebra and Applications. Springer London, 2013. ISBN: 978-1-4471-2616-4.

[4] Jerzy Weyman. Cohomology of Vector Bundles and Syzygies. Cambridge Tracts in Mathematics. Cambridge University Press, 2003. ISBN: 9780511546556.

Metropolitan State University of Denver
Email address: bcattell@msudenver.edu

Binghamton University
Email address: edelarg1@binghamton.edu

Carleton College
Email address: gangulya@carleton.edu

University of Califonia, Berkeley
Email address: seanguan@berekely.edu

University of Florida
Email address: sivakumars@ufl.edu