Cluster Monomials in Graph LP Algebras

2023 Twin Cities REU in Combinatorics & Algebra

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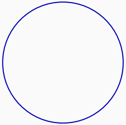
Background

Context



Cluster Algebras

Fomin and Zelevinsky (2002)

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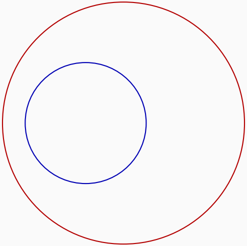
Laurent Phenomenon (LP) Algebras

Lam and Pylyavskyy (2016a)



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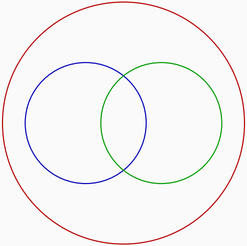

Cluster Algebras

Fomin and Zelevinsky (2002)

Graph LP Algebras

Lam and Pylyavskyy (2016b)

Preliminaries

Notation on drawings of directed graphs

Let Γ be a directed graph with vertex set $[n] = \{1, \ldots, n\}$.

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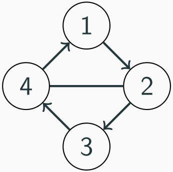
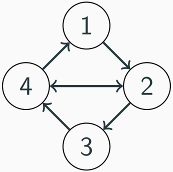
If an edge is **bidirected**, we draw it without arrows.





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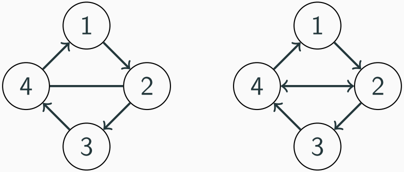
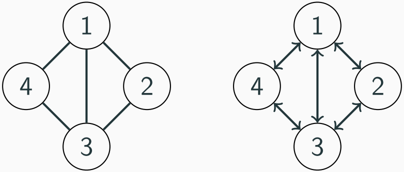
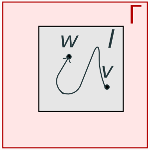


If all edges of Γ are bidirected, we may say Γ is **undirected**.



Strongly Connectedness

A non-empty subset $I \subset [n]$ is **strongly connected** if for all $v, w \in I$, there is some directed path $v \to w$ within I.



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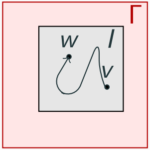
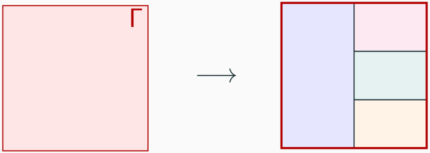
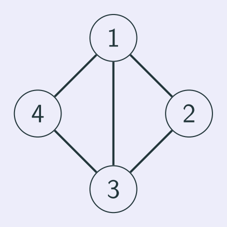


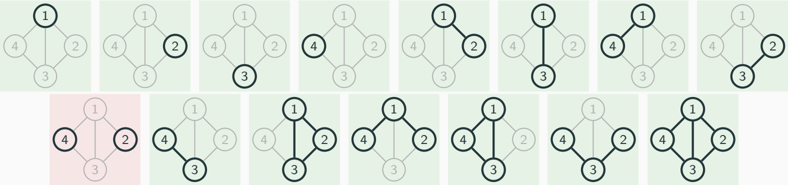
A directed graph is partitioned into **strongly connected components**, maximally strongly connected subsets of vertices.



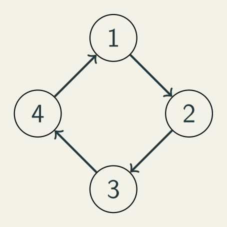
Strongly Connectedness (Examples 1/2)



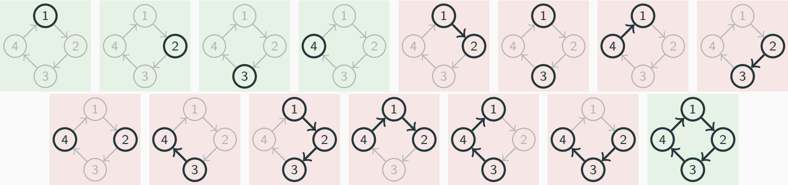
All subsets are strongly connected except 24.



Strongly Connectedness (Example 2/2)



The strongly connected subsets are 1, 2, 3, 4, and 1234.



Nested Collections

A collection $\mathcal{S}$ of subsets is **nested** if:

(Condition 1) For all $I, J \in \mathcal{S}$,

$$I \cap J = \varnothing \quad \text{or} \quad I \subset J \quad \text{or} \quad I \subset J.$$

(Condition 2) For any disjoint sets $I_1, I_2, \ldots, I_i \in \mathcal{S}$, these sets are the strongly connected components of their union.

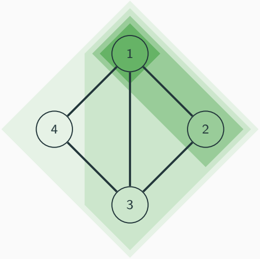
Remark Condition 2 implies that $I \in \mathcal{S}$ must be strongly connected.

Nested Collections (Example 1/4)

A collection is **nested** if:

(1) $I \cap J = \varnothing$ or $I \subset J$ or $I \subset J$.

(2) Disjoint sets are the connected components of their union.



Claim $\{1, 12, 123, 1234\}$ is nested.

Nested Collections (Example 2/4)

A collection is **nested** if:

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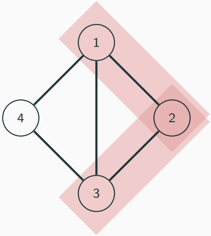

Claim $\{12, 23\}$ is NOT nested.

Nested Collections (Example 3/4)

A collection is **nested** if:

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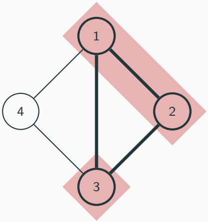
(2) Disjoint sets are the connected components of their union.



Claim $\{12, 3\}$ is NOT nested, since the strongly connected components of $12 \cup 3$ are NOT 12 and 3.

Nested Collections (Example 4/4)

A collection is **nested** if:

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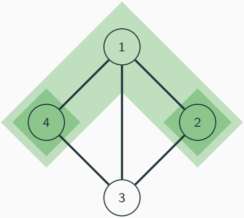
(2) Disjoint sets are the connected components of their union.



Claim $\{2, 4, 124\}$ is nested.

Acyclic Functions

We consider functions $f\colon I \subset [n] \to [n]$ such that, for all $v \in I$,

$$f(v) = v \qquad \text{or} \qquad (v, f(v)) \text{ is an edge of } \Gamma.$$

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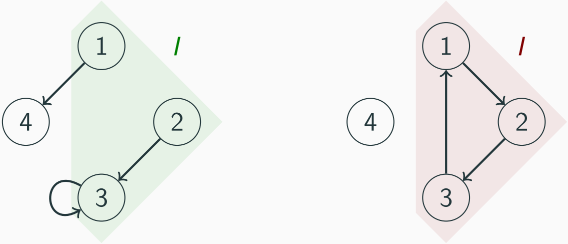
An **acyclic function** on I is a function f with no cycles (loops are allowed).



Weight of Function

The **weight** of a function f is

$$w(f) = \prod_{i \in I} \tilde{X}_{f(i)}, \qquad \text{where } \tilde{X}_{f(i)} = \begin{cases} X_{f(i)} & \text{if } f(i) \neq i \\ A_{f(i)} & \text{if } f(i) = i. \end{cases}$$

Multiply X_j for an edge $i \to j$, and multiply A_i for a loop $i \to i$.

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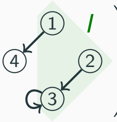
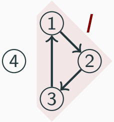
$w\Big($ 

$\Big) = X_4 X_3 A_3,$ $\qquad w\Big($ 

$\Big) = X_2 X_3 X_1.$

Remark For cycles, $w(f) = \prod_{i \in I} X_i$.

Variables Y_I

Given a subset I, we define

$$Y_I = \frac{1}{\prod_{i \in I} X_i} \cdot \sum_{\substack{\text{acyclic} \\ f : I \to [n]}} w(f)$$

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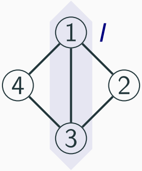

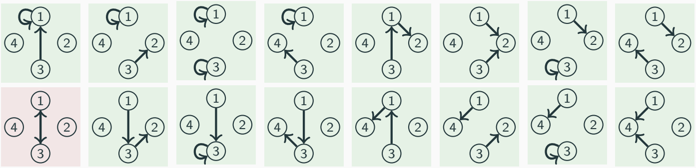

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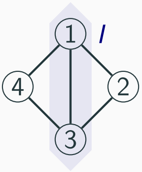
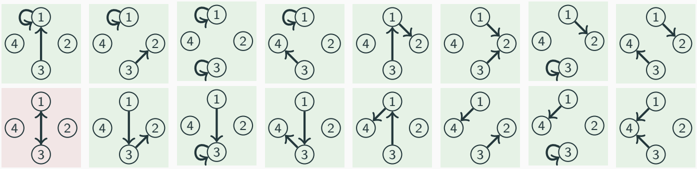




$$Y_{13} = \frac{(A_1 + X_2 + X_3 + X_4)(X_1 + X_2 + A_3 + X_4) - X_1 X_3}{X_1 X_3}$$

After calculations, we have

$$Y_{13} = Y_1 Y_3 - 1.$$

Graph LP Algebra (Lam and Pylyavskyy (2016b))

Given a directed graph Γ, its associated graph LP algebra $\mathcal{A}_\Gamma$ can be described as the algebra generated by

$$\{X_1, \ldots, X_n\} \cup \{\, Y_I \mid I \text{ is strongly connected}\},$$

with coefficient ring $R = \mathbb{Z}[A_1, \ldots, A_n]$.

Structure of Clusters

Theorem (Lam and Pylyavskyy (2016b))

The graph LP algebra $\mathcal{A}_\Gamma$ has:

- *cluster variables*

$$\{X_1, \ldots, X_n\} \cup \{Y_I \mid I \text{ is strongly connected}\},$$

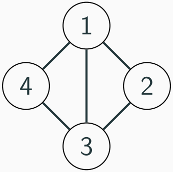
- *clusters of the form*

$$\{X_{i_1}, \ldots, X_{i_k}\} \cup \{Y_I \mid I \in \mathcal{S}\}$$

where $\mathcal{S}$ is some maximal nested collection on $\Gamma \setminus \{i_1, i_2 \ldots i_k\}$.
The Y-variables are supported by the nested collection $\mathcal{S}$.

A **cluster monomial** is a monomial with variables from the same cluster.

Example of Cluster Monomials



Nested set	Cluster	Cluster Monomial
$\varnothing$	$\{X_1, X_2, X_3, X_4\}$	$X_1^2 X_2$
$2, 4$	$\{X_1, X_3, Y_2, Y_4\}$	$X_3 Y_2 Y_4^3$
$1, 12, 123, 1234$	$\{Y_1, Y_{12}, Y_{123}, Y_{1234}\}$	$Y_1 Y_{123}^5 Y_{1234}$

A monomial of only Y-variables is a cluster monomial if it is supported by a nested collection $\mathcal{S}$.

Primary Conjecture (Lam and Pylyavskyy (2016b))

Recall that $\mathcal{A}_\Gamma$ is generated by the cluster variables, and that
a **cluster monomial** is a monomial with variables from the same cluster.

Conjecture (Lam and Pylyavskyy (2016b))

(1) *Cluster monomials are a linear* **basis** *for* $\mathcal{A}_\Gamma$ *over* $R = \mathbb{Z}[A_1, \ldots, A_n]$.

- **(1a)** *Cluster monomials* **span** $\mathcal{A}_\Gamma$.
- **(1b)** *Cluster monomials are* **linarly independent**.

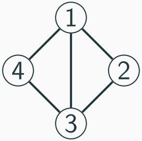
(2) *Any monomial in the cluster variables of* $\mathcal{A}_\Gamma$ *can be expressed as a R-linear combination of cluster monomials with* **positive coefficients**.

We prove **(1a)** and make progress towards **(2)**.

Example of Positivity

Recall that, in a previous example, we computed

$$Y_{13} = \frac{(A_1+X_2+X_3+X_4)(X_1+X_2+A_3+X_4)-X_1X_3}{X_1X_3} = Y_1Y_3 - 1.$$



Note that Y_1Y_3 is a monomial in the cluster variables,
but Y_1Y_3 is NOT a cluster monomial.

Still,

$$Y_1Y_3 = Y_{13} + 1,$$

so Y_1Y_3 is a positive linear combination of cluster monomials.

Progress

Main Quest

The monomials in the cluster variables of $\mathcal{A}_\Gamma$ which are more challenging to decompose are monomials consisting only of Y-variables.

Very Hard Question How to decompose

$$Y_{I_1} Y_{I_2} \cdots Y_{I_k}?$$

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Very Hard Question How to decompose

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Hard Question How to decompose

$$Y_I Y_J?$$

Remark When Γ is undirected, it is enough to answer the second.

Example of Techniques: Disjoint Case

Theorem (D., T., W. (2023))

Let $I \cap J = \varnothing$. *Then,*

$$Y_I Y_J = \sum_{\mathcal{C}} Y_{(I \cup J) \setminus \mathcal{C}},$$

where $\mathcal{C}$ *ranges over families of disjoint cycles which are not in* I *nor in* J.

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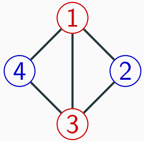
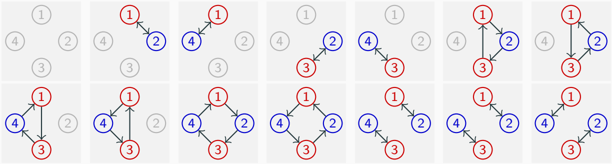
Example





Then, using the Theorem,

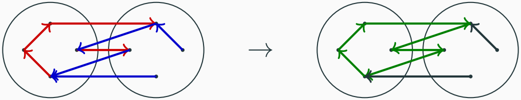
$$Y_{13} Y_{24} = Y_{1234} + Y_{34} + Y_{23} + Y_{14} + Y_{12} + 2Y_4 + 2Y_2 + 4.$$

Sketch for the Disjoint Case

$$Y_I Y_J = \sum_{\mathcal{C}} Y_{(I \cup J) \setminus \mathcal{C}}$$

We associate a pair of acyclic functions (f_I, f_J) with their union.

However, the union may have some disjoint cycles.



By adding “correction terms’ for these cycles, we get the identity.

Cluster Monomials Span $\mathcal{A}_\Gamma$

Theorem (D., T., W. (2023))

For all directed graphs Γ, cluster monomials span $\mathcal{A}_\Gamma$ over R.

Idea We prove the identity

$$\underbrace{\left(\sum_{\mathcal{C} \text{ in } I} Y_{I\setminus\mathcal{C}}\right)}_{\text{any function in } I}\underbrace{\left(\sum_{\mathcal{C} \text{ in } J} Y_{J\setminus\mathcal{C}}\right)}_{\text{any function in } J} = \underbrace{\left(\sum_{\mathcal{C} \text{ in } I\cup J} Y_{I\cup J\setminus\mathcal{C}}\right)}_{\text{any function in } I\cup J}\underbrace{\left(\sum_{\mathcal{C} \text{ in } I\cap J} Y_{I\cap J\setminus\mathcal{C}}\right)}_{\text{any function in } I\cap J}$$

and isolate the term $Y_I Y_J$ that appears in LHS.

Positivity for Trees

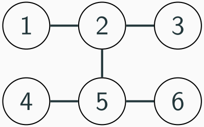
Theorem (D., T., W. (2023))

If Γ is an undirected tree or cycle, then $\mathcal{A}_\Gamma$ satisfies positivity.

Sketch It suffices to have a formula for $Y_I Y_J$. For trees, we have

$$Y_I Y_J = \sum_{\mathcal{P}} Y_{I \cup J \setminus \mathcal{P}} Y_{I \cap J \setminus \mathcal{P}}$$

where $\mathcal{P}$ ranges over families of disjoint paths "traveling from I to J".



$$Y_{1245} Y_{2356} = Y_{123456} Y_{25} + Y_{456} Y_5 + Y_{123} Y_2 + Y_{43} + Y_{16} + 1$$

Further Work

- We can decompose $Y_I Y_J$ when $|I \cap J| = 0$, 1, or 2.
  How can we do it in general?
  - $Y_I Y_J$: undirected case.
  - $Y_{I_!} Y_{I_2} \cdots Y_{I_k}$: directed case.
- If not in general, how can we do it for special graphs, for example
  - planar graphs,
  - graphs with small maximum degree, etc.
- Prove that cluster monomials are linearly independent, hence form a basis.

Acknowledgements

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- Zeus was supported by Haverford College funding.

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