C.P.S. 10/31/2014

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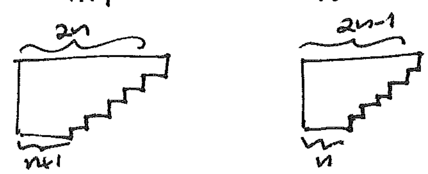
Integer partitions

Warm-up: Euler's pentagonal number theorem

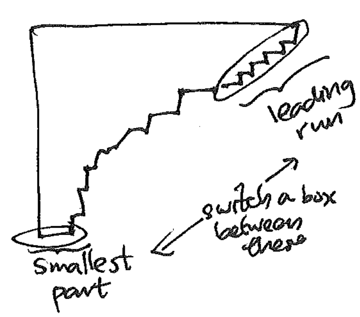
$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{n(3n+1)}{2}} + q^{\frac{n(3n-1)}{2}}\right)$$

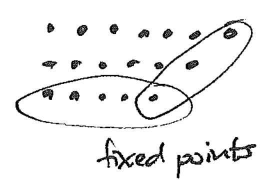
$$\sum_{\substack{\lambda \text{ w/} \\ \text{distinct} \\ \text{parts}}} q^{|\lambda|}(-1)^{\ell(\lambda)} \qquad\qquad \sum_{\substack{\mu \text{ with} \\ \text{distinct} \\ \text{parts,} \\ \text{very special}}} q^{|\mu|}(-1)^{\ell(\mu)}$$

$$\mu = \begin{matrix} 2n \\ \vdots \\ n+2 \\ n+1 \end{matrix} \quad \text{or} \quad \begin{matrix} 2n-1 \\ \vdots \\ n+1 \\ n \end{matrix}$$



Franklin's involution on distinct part λ has fixed points given by RHS





Want to do something similar for ...

THM (Rogers–Ramanujan, MacMahon, Schur)

\# partitions of N into parts with consecutive part differences ≥ 2 = \# partitions of N into parts $\equiv 1, 4 \mod 5$

e.g. $\lambda = (11, 9, 5, 3, 1)$ e.g. $\mu = (9, 9, 9, 4, 1, 1, 1, 1, 1)$

As a generating function identity:

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(1-q^1)\cdots(1-q^m)} = \frac{1}{(1-q^1)(1-q^4)(1-q^6)(1-q^9)\cdots} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}$$

We'll write down something stronger.

In preparation, Sylvester proved

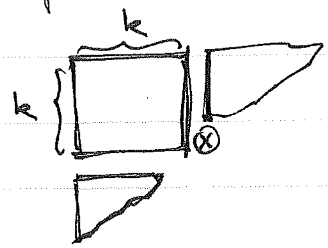
$$\prod_{k=1}^{\infty}(1+aq^k) = 1 + \sum_{k=1}^{\infty} q^{\frac{k(3k-1)}{2}} a^k \left(q^k \frac{(-aq;q)_k}{(q;q)_k} + \frac{(-aq;q)_{k-1}}{(q;q)_{k-1}} \right)$$

where $(A;q)_k = (1-A)(1-Aq)\cdots(1-Aq^{k-1})$

$$/\!/$$

$$\sum_{\substack{\lambda \text{ with} \\ \text{distinct} \\ \text{parts}}} q^{|\lambda|} a^{\ell(\lambda)}$$

On the RHS, look at the Durfee square of a partition ~~with~~ distinct parts



$$q^{k^2} a^k \begin{cases} q^k \cdot \dfrac{q^{\binom{k}{2}}}{(q;q)_k} \cdot (-aq;q)_k & \text{if } \otimes \text{ present} \\ + \dfrac{q^{\binom{k}{2}}}{(q;q)_{k-1}} \cdot (-aq;q)_{k-1} & \end{cases}$$

Key Rogers-Ramanujan mod 5 identity:

$$\prod_{k=1}^{\infty}(1+aq^k) \sum_{m=0}^{\infty} \frac{q^{m^2}(-a)^m}{(1-q)\cdots(1-q^m)} = 1 + \sum_{k=1}^{\infty} q^{\frac{k(3k-1)}{2}} a^k \times \overset{\text{THE ONLY CHANGE}}{\boxed{q^{k^2}(-a)^k}}$$

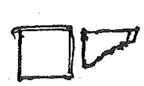
$$\times \left(q^k \frac{(-aq;q)_k}{(q;q)_k} + \frac{(-aq;q)_{k-1}}{(q;q)_{k-1}} \right)$$

Distinct(a) × Diff2$(-a)$

$\uparrow$ parts weighted by a $\qquad$ $\uparrow$ parts weighted by $-a$

PROBLEM: Find φ an involution on Distinct × ~~Diff2~~ with fixed points $= \left\{ \left(\underset{\text{distinct parts}}{\lambda}, \ k\square^{k} \right) \right\}$ (same size k Durfee square)

~~Diff2~~ replace with partitions having nothing under the Durfee square



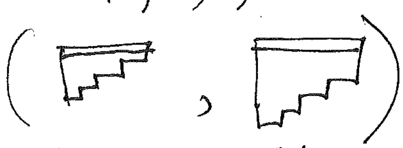
INSPIRATIONAL EXAMPLE:

$$\frac{\prod_{k=1}^{\infty}(1-q^k)}{\prod_{k=1}^{\infty}(1-q^k)} = 1$$

weighted by $(-1)^{\ell(\lambda)}$

↪ Dist × All has an involution φ whose fixed points are just $(\emptyset, \emptyset)$,

(λ, μ)



moving largest part from λ to μ or vice-versa.

Dist All

If one wants to do Rogers-Ramanujan mod $2n+3$ due to Andrews

the extra on RHS $q^{k^2}(-a)^k$ becomes $q^{k^2 n}(-a)^{kn}$

and on LHS the $\sum_{m=0}^{\infty}(\sim)$ term becomes an m-fold multisum