C.P.S. 2/20/2015

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Volumes of combinatorial polytopes
(and Ehrhart quasi-polynomials)

$$i(P,k) = \# kP \cap \mathbb{Z}^n$$

P a lattice polytope $\Rightarrow$ $i(P,k)$ is polynomial in k

$$\sum_{k\geq 0} i(P,k)x^k = \frac{h_0^* + h_1^* x + \dots + h_n^* x^n}{(1-x)^{n+1}}$$

$$\sum_j h_j^* = NV(P)$$

normalized volume of P

Q1: What is the volume of the Birkhoff polytope

B_n = convex hull of $n\times n$ permutation matrices

CONJ: $P(M)$ = the matroid base polytope for a matroid M

Then • the h^*-vector of $P(M)$ is unimodal (but not log-concave)

• the Ehrhart polynomial of all matroid polytopes has only positive coefficients

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For a semisimple Lie $\mathfrak{g}$ w/ highest weight reps V^λ

define $V^\lambda \otimes V^\mu = \sum_\nu \underbrace{C_{\lambda\mu}^\nu}_{\text{Clebsch-Gordan coeffs}} V^\nu$

FACT: $C_{\lambda\mu}^\nu$ = # integer points in a polytope P of a certain dimension d with vertices in $\mathbb{Q}^d$, not $\mathbb{Z}^d$

$$\Rightarrow \lim_{n\to\infty} C_{n\lambda, n\mu}^{n\nu} = \frac{\mathrm{vol}(P)}{d!}$$

THM: In types A, B, C, D

$$C^{n\nu}_{n\lambda, n\mu} \text{ is a quasi-polynomial } \underline{\text{mod } 2}$$

CONJ: Both polynomials mod 2 have nonnegative coefficients.