(1)

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Thrall's problem (Refs: M. Schocker, J. Aust. Math Soc. 75 (2003), 9–21)
Stanley E.C. II Exer. 7.89

Fix a partition $\lambda \vdash n$,

and define $L_\lambda(x_1, x_2, \ldots) = \sum_{\substack{\sigma \in S_n \\ \text{of cycle} \\ \text{type } \lambda}} F_{\mathrm{Des}(\sigma)}$

where $F_D := \sum_{\substack{i_1 \leq \cdots \leq i_n: \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}$

~~PROB~~ THM: L_λ is a Schur positive symm. fn of $x_1, x_2, \ldots$
i.e. $L_\lambda = \sum c_\mu^\lambda \underbrace{s_\mu(x_1, x_2, \ldots)}_{\text{Schur function}}$ with $c_\mu^\lambda \in \mathbb{N}$

PROBLEM:

Interpret c_μ^λ combinatorially.

e.g. $n=3$

	σ	$\mathrm{Des}(\sigma)$	$F_{\mathrm{Des}(\sigma)}$	L_λ
$\lambda = 111$	123	$\varnothing$	h_3	$= s_{(3)}$
$\lambda = 21$	2·13	$\{1\}$	$F_{\{1\}}$	
	13·2	$\{2\}$	$+F_{\{2\}}$	$= s_{(2,1)}$
	3·2·1	$\{1,2\}$	e_3	$= s_{(1,1,1)}$
$\lambda = 3$	231	$\{1\}$	$F_{\{1\}}$	
	3·12	$\{2\}$	$+F_{\{2\}}$	$= s_{(2,1)}$

FACT: $\sum_{\lambda \vdash n} L_\lambda = (s_1)^n$

RMKS: (1) $\lambda = (n)$ has $L_{(n)} \overset{\text{Klyachko}}{=} \sum_{\substack{\text{SYT } Q \text{ of size } n: \\ \mathrm{maj}(Q) \equiv 1 \bmod}} s_{\mathrm{shape}\, Q}$

e.g. $n=3$

	123	12/3	13/2	1/2/3
maj	0	2	$\boxed{1}$	3

$\Rightarrow L_3 = s_{(2,1)}$

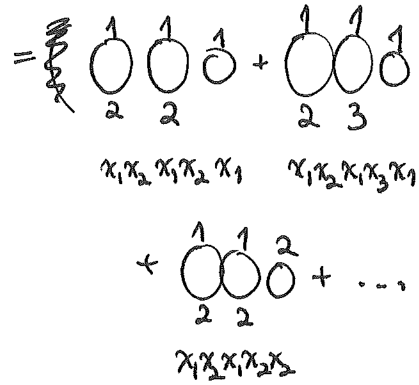
Schocker generalizes this to a +/− expression for $L_{(a^b)}$

Equivalent problems:

(a) Via Gessel's necklace bijection (see Gessel & Reutenauer)

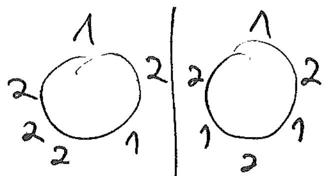
$$L_\lambda = \sum_{\substack{\text{ornaments } \omega = \\ \text{multisets of prim. necklaces} \\ \text{on } \{1,2,\ldots\} \text{ of sizes } \lambda}} x_\omega$$

e.g. $L_{2^2 1} = L_2^2 L_1$



cyclic orbits

necklaces with trivial cyclic stabilizer



primitive | not primitive

So we really want a tableau model for the $L_{(\ell^m)}$, and the can put them together

(b) As $GL(V)$-rep'ns, $V = \mathbb{C}^n$ has

$$T^\cdot(V) = \bigoplus_{d \geq 0} V^{\otimes d} = \underbrace{\mathcal{U}}_{\text{univ. env. alg.}}(Lie(V)) \cong Sym^\cdot(Lie(V))$$

$Lie(V) = \bigoplus_{d \geq 0} Lie_d(V)$

$Lie_1(V) = V$

$Lie_2(V) = [V, V]$

$\vdots$

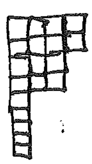
$Sym^\cdot(Lie(V)) = \bigoplus \underbrace{Sym^{m_1}(Lie_1(V)) \otimes Sym^{m_2}(Lie_2(V)) \cdots}_{Lie_\lambda(V) :=}$

$\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$

$Lie_\lambda(V) :=$ has $GL(V)$-character L_λ

$\Rightarrow L_\lambda$ Schur positive

③ $$L_{(2^m)} = h_m[e_2] \overset{\text{Littlewood}}{=} \sum_{\substack{\mu \text{ with} \\ \text{all even column} \\ \text{lengths}}} s_\mu$$



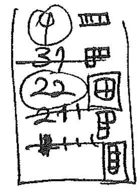
An Important coarsening...

THM (Desarmenien-Wachs, R.-Webb)

$$\text{Derange}_n := \sum_{\substack{\lambda : |\lambda| = n \\ \lambda \text{ has no 1's}}} L_\lambda = \sum_{\substack{\text{derangements} \\ \sigma \in S_n}} F_{Des(\sigma)}$$

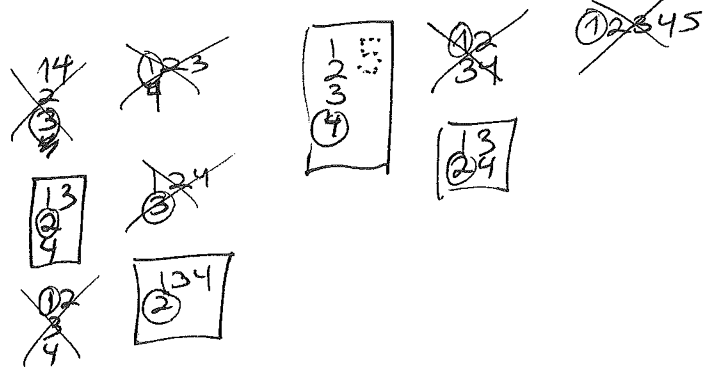
$$= \sum_{\substack{\text{std Young tab. } Q \\ \text{of size } n \\ \text{with } 1^{st} \text{ ascent even}}} s_{\text{shape } Q}$$

(3)



e.g. $\text{Derange}_4 = L_4 + L_{2^2}$

$$= s_{(2,1,1)} + s_{(3,1)} + s_{(1,1,1,1)} + s_{(2,2)}$$



PROBLEM: Find a similar (more refined) tableau model

for $\displaystyle \text{Derange}_n^k := \sum_{\substack{\lambda : |\lambda| = n \\ \lambda \text{ has no 1's} \\ \ell(\lambda) = k}} L_\lambda = \sum_{\substack{\text{derangements} \\ \sigma \in S_n \\ \text{with } k \text{ cycles}}} F_{\mathrm{Des}(\sigma)}$