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Moments of orthogonal polynomials

measure $d\mu(x)$

$$\mu_n = \int_{-\infty}^{\infty} x^n \, d\mu(x) < \infty \quad \text{for } n=0,1,2,\dots$$

Laguerre polynomials $L_n^\alpha(x)$

$$d\mu(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} dx \quad \text{for } \alpha > 0$$

$$\mu_n = (\alpha+1)(\alpha+2)\cdots(\alpha+n)$$

Explicit formula

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\middle| x\right)$$

3-term recurrence for monic $\{p_n(x)\}$

$$p_{n+1}(x) = (x-b_n)p_n(x) + \lambda_n p_{n-1}(x)$$

$$b_n = 2n+\alpha+1$$

$$\lambda_n = n(n+\alpha)$$

GENERAL FACT: $\mu_n = \sum_{P \text{ Motzkin path } (0,0)\to(n,0)} wt(P)$

Interested in associated Laguerre polynomials:

Replace $n \to n+c$ in the 3-term recurrence

FACTS: (0) They have an explicit measure $\mu_{\alpha,c}$ which is known but complicated.

(1) There is a double sum formula for them:

$$\sum_k (-1)^{n-k} x^k \, {}_3F_2\left(\because \middle| 1\right)$$

Replace $\alpha, c \longrightarrow X, Y$ where $X = \alpha + c$

$Y = c+1$

Let $b_n = 2n + X + Y$

$\lambda_n = (n+X)(n+Y-1)$

FACTS on $\mu_n(X,Y)$

① $$\mu_n = \sum_{\pi \in S_n} (X+1)^{RLmin(\pi)} (X+Y)^{pivot(\pi)} Y^{LRmax(\pi)-pivot(\pi)}$$

$$= \sum_{\pi \in S_{n+1}} X^{RLmin(\pi)-1} Y^{LRmax(\pi)-pivot(\pi)}$$

$$\mu_n = \sum_{T \in \underbrace{PT}_{\text{permutation tableaux}}{}_{n+1}} X^{urr(T)-1} Y^{urc(T)}$$

② $$\sum_{n=0}^{\infty} \mu_n(X,Y) t^n = \frac{{}_2F_0(1+X, Y \mid t)}{{}_2F_0(X, Y-1 \mid t)}$$

③ Recurrence relation for $\mu_n(X,Y)$

$$\sum_{k=0}^{n} \mu_k(X,Y) \frac{(Y-1)_{N-k}(X)_{N-k}}{(N-k)!} = \frac{(Y)_N (X+1)_N}{N!}$$

④ PROBLEM: Find any explicit formula for $\mu_n(X,Y)$!

(e.g. a sum or double sum,

but not with $n!$ or $(n+1)!$ terms)