OUTLINE:

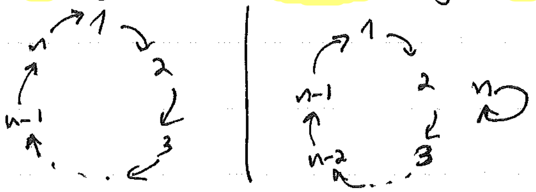
- Reflection groups
- Invariant theory
- Regular elements
- Modular analogues & generalizations
- GOOD proof of the

  $GL_n(\mathbb{F}_q)$ - PROTO - Example

Reflection groups and regular elements

generalize

Symmetric groups $\mathfrak{S}_n$ and n-cycles & $(n-1)$-cycles



Let's be general.

DEFINITION: A (pseudo-) reflection r is an element of $GL_n(k)$ of finite order whose fixed space V^r on $V = k^n$ is a hyperplane

k: a field

(= codimension 1, linear subspace)

EXAMPLES

Diagonalizable (over $\bar{k}$) reflections are $GL_n(\bar{k})$-conjugate to

$$\begin{bmatrix} \boxed{\xi} & & & 0 \\ & 1 & & \\ & & 1 \ddots & \\ 0 & & & 1 \end{bmatrix}$$

for some root-of-unity $\xi \in \bar{k}$

Non-diagonalizable reflections (called transvections) are $GL_n(k)$-conjugate to

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & & & 0 \\ & 1 & & \\ & & 1 \ddots & \\ 0 & & & 1 \end{bmatrix}$$

They only exist in characteristic $p > 0$

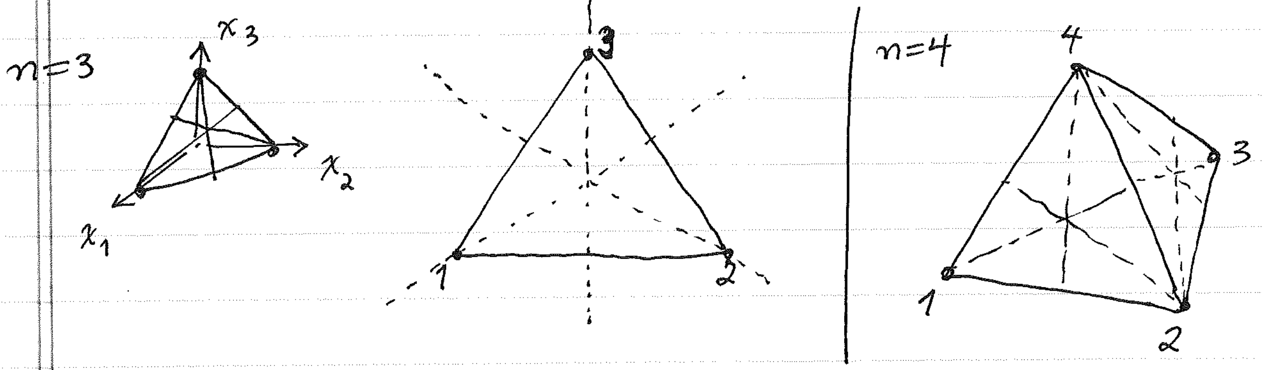
DEFINITION: A reflection group $W \leq GL_n(k)$ is a finite subgroup generated by reflections.

EXAMPLE:

$$\begin{tikzcd} \mathfrak{S}_n = \text{symmetric group on } \{1,2,--,n\} \arrow[d, hook, "\text{permuting coordinates in } k^n = V"] \\ GL_n(k) \end{tikzcd}$$

is generated by transpositions $(i,j)=r$ = reflections, with fixed hyperplane $V^r = \{x_i = x_j\}$

For $k=\mathbb{R}$, we can think of $\mathfrak{S}_n$ as **the** symmetries of a regular $(n-1)$-simplex



EXAMPLE:

$GL_n(\mathbb{F}_q)$ = (finite) general linear group

$\rotatebox{90}{$\leq$}$

$SL_n(\mathbb{F}_q)$ = (finite) special linear group

are both generated by reflections; $SL_n(\mathbb{F}_q)$ is already generated by transvections (EXERCISE!)

Invariant theory distinguishes reflection groups !

(all reflection groups)

finite groups $W \le GL_n(k)$
with $k[x_1,\dots,x_n]^W = k[f_1,\dots,f_n]$ polynomial

e.g. $GL_n(\mathbb{F}_q)$, $SL_n(\mathbb{F}_q)$

complex reflection groups $W \le GL_n(\mathbb{C})$
(u.g.g.r.'s)

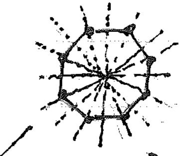
e.g. $G(de,e,n)$

real reflection groups $W \le GL_n(\mathbb{R})$
= Coxeter systems (W,S) with W finite

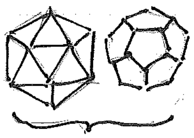
Weyl groups
= crystallographic
reflection groups in $GL_n(\mathbb{R})$

e.g. type D_n

Symmetries of regular polytopes



$I_2(m)$



H_3

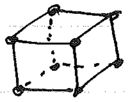
120-cell,
600-cell

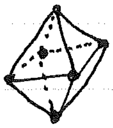
H_4

24-cell

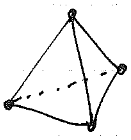
F_4

Symmetric & hyperoctahedral groups
= Types A_{n-1} & $\underbrace{B_n/C_n}$
= symmetries
of n-cubes & n-hyperoctahedra





Symmetric groups $\mathfrak{S}_n$
= Type A_{n-1}
= symmetries of
regular simplices



Invariant theory

Subgroups $W \leq GL_n(k)$
act on $\boxed{S := k[x_1, \dots, x_n]}$

via linear substitutions: for $w \in W$, $\underset{\underset{f(x_1,\dots,x_n)}{\|}}{f(\underline{x})} \in S$

$$f = f(\underline{x}) \xmapsto{\;w\;} f(w^{-1}\underline{x}) =: w(f)$$

DEFINITION: $S^W := \{ f \in S : w(f) = f \}$
= the W-invariant subring

EXAMPLE: $W = \mathfrak{S}_n$ permutes the variables in $S = k[x_1, \dots, x_n]$

and $S^W = \boxed{k[x_1, \dots, x_n]^{\mathfrak{S}_n} = \underbrace{k[e_1, e_2, \dots, e_n]}_{\text{a polynomial algebra!}}}$

where
$$e_1 = x_1 + x_2 + \dots + x_n$$
$$e_2 = x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n$$
$$\vdots$$
$$e_n = x_1x_2 \cdots x_n$$

are the elementary symmetric functions in $x_1, \dots, x_n$

(= Fundamental Theorem of Symmetric Functions)

In general, for finite $W \leq GL_n(k)$

S^W will not be a polynomial subalgebra of $S=k[x_1,\dots,x_n]$

THM (Noether)

- $S^W \hookrightarrow S$ is an integral extension of rings

and hence

S is a finitely generated S^W-module

- S^W is a finitely generated k-algebra, requiring at least n generators.

THM (Shephard-Todd, Chevalley) 1955

A finite subgroup $\boxed{W \leq GL_n(\mathbb{C})}$ acting on $S=\mathbb{C}[x_1,\dots,x_n]$ has $S^W = \mathbb{C}[f_1,f_2,\dots,f_n]$ a polynomial subalgebra

$\Leftrightarrow$ W is a reflection group

$\Leftrightarrow$ the W-representation on the coinvariant algebra

$$S/\underbrace{(S^W_+)}_{\substack{\text{ideal gen'd}\\ \text{by } W\text{-invariants of positive degree}}} := S/(f_1,f_2,\dots,f_n)$$

is isomorphic to the W-regular representation:

$$\boxed{S/(S^W_+) \cong \mathbb{C}W}$$

graded! not graded

Regular elements

Springer added an important enhancement...

DEFINITION: In a finite reflection group $W \leq GL_n(\mathbb{C})$ acting on $V = \mathbb{C}^n$, say that $c \in W$ is a regular element if it has an eigenvector $v \in V$ that avoids all the reflection hyperplanes, or equivalently, v has free W-orbit: $|W.v| = |W|$.
Call the eigenvalue ζ for which $c(v) = \zeta \cdot v$ a regular eigenvalue for c.

THEOREM (Springer 1972) For any regular element c with a regular eigenvalue ζ in a finite reflection group $W \leq GL_n(\mathbb{C})$, one has an isomorphism of $W \times C$-representations where $C = \langle c \rangle$

$$\boxed{S/(S_+^W) \quad \cong \quad \mathbb{C}W}$$

$$\| $$

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

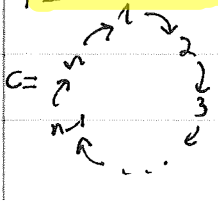
$\circlearrowright$ W by linear substitutions

$\circlearrowleft$ C by scalar substitutions: $x_i \overset{c}{\longmapsto} \zeta \cdot x_i \ \forall i$

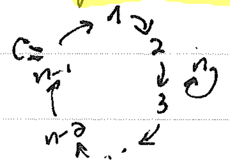
($\mathbb{C}W$) $\circlearrowright$ W by left-translations

$\circlearrowleft$ C by right-translations

EXAMPLE: Regular elements in $W = \mathfrak{S}_n$ are exactly the powers of n-cycles and powers of $(n-1)$-cycles



$v = (1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$
has $c(v) = \zeta \cdot v$
if $\zeta = e^{2\pi i/n}$



$v = (1, \omega, \omega^2, \ldots, \omega^{n-2}, 0)$
has $c(v) = \omega \cdot v$
if $\omega = e^{2\pi i/(n-1)}$

A useful equivalent rephrasing...

Recall for finite-dimensional complex W-representations U_1, U_2

$$\langle U_1, U_2\rangle_W := \frac{1}{|W|}\sum_{w\in W} \chi_{U_1}(w^{-1})\,\chi_{U_2}(w) \quad \text{where } \chi_U(w) = \mathrm{Trace}(U \xrightarrow{w} U)$$

$$= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}W}(U_1, U_2)$$

and in particular,

$$\langle U, \mathbb{C}W\rangle_W = \dim_{\mathbb{C}} U \quad \text{is the degree of } U$$

DEFINITION: For a representation U of a finite reflection group $W \leq GL_n(\mathbb{C})$, the U-fake degree polynomial is

$$f^U(q) := \sum_{i\geq 0} \langle U, \underbrace{S/(S_+^W)_i}_{\substack{i^{th}\text{ graded component of} \\ \text{coinvariant algebra } S/(S_+^W)}} \rangle_W \cdot q^i$$

$$= \mathrm{Hilb}(\mathrm{Hom}_{\mathbb{C}W}(U, S/(S_+^W)), q)$$

THEOREM (Springer 1972) For a regular element c in W, with regular eigenvalue ζ, and any W-representation U

$$\chi_U(c) = [f^U(q)]_{q=\zeta}$$

In particular, if U has a basis $\{e_x : x\in X\}$ permuted by c, that is, $c(e_x) = e_{c(x)}$ then $\left(X, \underset{\substack{\shortparallel \\ f^U(q)}}{X(q)}, \underset{\substack{\shortparallel \\ \langle c\rangle}}{C}\right)$

exhibits a CSP.

Two important examples of fake degrees

① **EXAMPLE**: $W = \mathfrak{S}_n$ has irreducible representations $\{U^\lambda\}$ indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$ with $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$

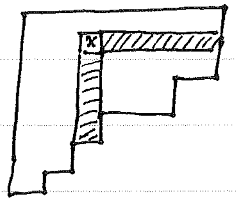
$$\dim_{\mathbb{C}} U^\lambda = \text{degree of } U^\lambda$$

$$\overset{\substack{1927 \\ \text{Young}}}{=} \#\text{of standard Young tableaux } P \text{ of shape } \lambda$$

$$\underset{\substack{\text{Frame-Robinson} \\ \text{-Thrall} \\ 1954}}{=} \boxed{\frac{n!}{\prod_{\substack{\text{cells } x \\ \text{of } \lambda}} h(x)}}$$

where $h(x)$ = hooklength at cell x of λ

e.g. $\lambda = (3,2)$ $\quad n = 3+2 = 5$ $\quad \begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}$



$$\dim_{\mathbb{C}} U^\lambda = \#\left\{ \begin{matrix} 123 \\ 45 \end{matrix}, \begin{matrix} 124 \\ 35 \end{matrix}, \begin{matrix} 125 \\ 34 \end{matrix}, \begin{matrix} 134 \\ 25 \end{matrix}, \begin{matrix} 135 \\ 24 \end{matrix} \right\}$$

$$= 5 = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} \checkmark$$

THM (Lusztig 1979) The fake degree $f^{U^\lambda}(q) = \boxed{\sum_{\substack{\text{standard} \\ \text{Young tableaux} \\ P \text{ of shape } \lambda}} q^{\text{maj}(P)}}$

where $\text{maj}(P) := \sum_{\substack{i:\, i+1 \text{ is} \\ \text{in a lower row of } P}} i$

$$\begin{matrix} 1\,2\,\textcircled{3} \\ 4\,5 \end{matrix} \quad \begin{matrix} 1\,\textcircled{2}\,\textcircled{4} \\ 3\,5 \end{matrix} \quad \begin{matrix} 1\,\textcircled{2}\,5 \\ 3\,4 \end{matrix} \quad \begin{matrix} \textcircled{1}\,3\,\textcircled{4} \\ 2\,5 \end{matrix} \quad \begin{matrix} \textcircled{1}\,\textcircled{3}\,5 \\ 2\,4 \end{matrix}$$

$$q^3 + q^6 + q^2 + q^5 + q^4 = q^2[5]_q$$

THM (Stanley 1971) The fake degree $f^{U^\lambda}(q) = \boxed{\frac{[n]!_q}{\prod_{\substack{\text{cells } x \\ \text{of } \lambda}} [h(x)]_q} \cdot q^{\sum_i (i-1)\lambda_i}}$

e.g. $f^{U^{\boxplus}}(q) = q^2 \dfrac{[5]!_q}{[4]_q[3]_2[2]_q[1]_q[1]_q} = q^2[5]_q$

EXAMPLES...

② When X carries a transitive W-action then $X \cong W/W'$ for some isotropy subgroup $W' \leq W$.

So the permutation representation U_X of W has $U_X \cong \mathbb{C}[W/W']$ with $\text{degree of } U_X = |X| = [W:W'] = |W|/|W'|$

What about the fake degree $f^{U_X}(q) = \text{Hilb}(\text{Hom}_{\mathbb{C}W}(U_X, S/(S^W_+)), q)$?

PROPOSITION: For any W-representation V,

$$\begin{array}{ccc} \text{Hom}_{\mathbb{C}W}(\mathbb{C}[W/W'], V) & \xrightarrow{\sim} & V^{W'} = \text{the } W'\text{-fixed subspace of } V \\ \varphi & \longmapsto & \varphi(eW') \end{array}$$

Hence

$$\begin{aligned} f^{U_X}(q) &= \text{Hilb}(\text{Hom}_{\mathbb{C}W}(\mathbb{C}[W/W'], S/(S^W_+)), q) \\ &= \text{Hilb}((S/(S^W_+))^{W'}, q) \\ &\qquad\quad \wr\wr \\ &= \text{Hilb}(S^{W'}/(S^W_+), q) \\ &= \frac{\text{Hilb}(S^{W'}, q)}{\text{Hilb}(S^W, q)} \end{aligned}$$

Tricky — uses averageing tricks $f \mapsto \frac{1}{|W'|}\sum_{w\in W'} w(f)$ (characteristic zero!)

Tricky — uses $S^W = \mathbb{C}[f_1, \ldots, f_n]$ and Cohen-Macaulayness of $S^{W'}$ (characteristic zero!)

COROLLARY: W a finite reflection group in $GL_n(\mathbb{C})$

W' any subgroup

c any regular element in W

Then the triple

$$\Big(\underset{W/W'}{\overset{X}{\shortparallel}} \quad , \quad \underset{\frac{Hilb(S^{W'},q)}{Hilb(S^{W},q)}}{\overset{X(q)}{\shortparallel}} \quad , \quad \underset{\langle c\rangle}{\overset{C}{\shortparallel}} \Big) \text{ exhibits a CSP.}$$

translating cosets
$wW' \xmapsto{c^d} c^d wW'$

EXAMPLE: X = k-element subsets of $\{1,2,\dots,n\}$

$= W/W'$ where $W = \mathfrak{S}_n$

$\overset{\vee|}{W}{}' = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$

e.g. $k=3$
$n=7$
$\{2,3,6\} \longleftrightarrow wW' = \left(\begin{array}{ccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2<&3<&6 & 1<&4<&5<&7 \end{array}\right) \mathfrak{S}_3 \times \mathfrak{S}_4$

$\in \mathfrak{S}_7 / \mathfrak{S}_3 \times \mathfrak{S}_4$

$$S^W = \mathbb{C}[e_1(x_1,\dots,x_n), e_2(x_1,\dots,x_n), \dots, e_n(x_1,\dots,x_n)]$$
$$\cap$$
$$S^{W'} = \mathbb{C}[e_1(x_1,\dots,x_k),\dots,e_k(x_1,\dots,x_k), e_1(x_{k+1},\dots,x_n),\dots,e_{n-k}(x_{k+1},\dots,x_n)]$$

$$\Rightarrow X(q) = \frac{Hilb(S^{W'},q)}{Hilb(S^W,q)} = \frac{\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)} \cdot \frac{1}{(1-q)(1-q^2)\cdots(1-q^{n-k})}}{\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Thus the COROLLARY implies the PROTO-Example, for both $C = \mathbb{Z}/n\mathbb{Z}$
$C = \mathbb{Z}/(n-1)\mathbb{Z}$

What about the $GL_n(\mathbb{F}_q)$-<u>analogue</u> of the PROTO-Example,
where $\mathfrak{S}_n \rightsquigarrow GL_n(\mathbb{F}_q)$?

$$k\text{-subsets} \rightsquigarrow k\text{-dimensional subspaces} \quad ?$$

$$\left.\begin{array}{r} n\text{-cycles} \\ (n-1)\text{-cycles}\end{array}\right\} \rightsquigarrow \text{Singer cycles} \quad ?$$

We need <u>positive</u> <u>characteristic</u> ("modular") <u>analogues</u> of the results of
Shephard-Todd, Chevalley, Springer.

Invariant theory for $G \leq GL_n(k) \circlearrowright S = k[x_1, \ldots, x_n]$
is <u>harder</u> when $\mathrm{char}(k) = p > 0$,
particularly when $|G| \notin k^\times$ i.e. p divides $|G|$.

- see texts by Campbell & Wehlau
Derksen & Kemper
Benson
Smith

However, we have eventually managed to find most of the modular analogues of the reflection group results that we need, <u>when $S^W = k[f_1, \ldots, f_n]$ is polynomial!</u>

Analogues of the Shephard-Todd & Chevalley results :

THEOREM (Serre 1968)

If a finite group $W \leq GL_n(k)$ acting on $S := k[x_1, \dots, x_n]$ has $S^W = k[f_1, \dots, f_n]$ polynomial

then W must be generated by reflections.

EXAMPLE :

THEOREM (Dickson 1911)

For $k = \mathbb{F}_q$ and $W = GL_n(\mathbb{F}_q)$ itself,

$S^W = \mathbb{F}_q[D_{n,n-1}, D_{n,n-2}, \dots, D_{n,1}, D_{n,0}]$ is polynomial,

with the Dickson polynomials $D_{n,i}$ of degree $q^n - q^i$ being the expansion coefficients in

$$\prod_{(c_1,\dots,c_n) \in \mathbb{F}_q^n} \left(t + (c_1x_1 + \dots + c_nx_n)\right) = t^{q^n} + D_{n,n-1} t^{q^{n-1}} + \dots + D_{n,1} t^{q^1} + D_{n,0} t^{q^0}$$

Compare this with ...

THEOREM (Fundamental Thm of Symmetric Functions)

For $k = \mathbb{C}$ or $\mathbb{Z}$ and $W = \mathfrak{S}_n$,

$S^W = k[e_1, e_2, \dots, e_{n-1}, e_n]$ is polynomial,

with the elementary symmetric functions e_{n-i} of degree $n-i$ the expansion coefficients in

$$\prod_{i=1}^{n} (t + x_i) = t^n + e_1 t^{n-1} + \dots + e_{n-1} t^1 + e_n t^0$$

NOTE: The converse to Serre's result is FALSE!

EXAMPLE: For q odd, choose a nondegenerate symplectic bilinear form on $V = \mathbb{F}_q^{2n}$

$$\begin{aligned} V \times V &\longrightarrow \mathbb{F}_q \\ (v, \; v') &\longmapsto \langle v,v' \rangle = -\langle v',v \rangle \end{aligned}$$

and define the finite symplectic group

$$Sp(\mathbb{F}_q^{2n}) := \{ g \in GL_{2n}(\mathbb{F}_q) : \langle g(v), g(v') \rangle = \langle v,v' \rangle \; \forall v \in V \}$$

THEOREM (Carlisle - Kropholler 1992)

$W = Sp(\mathbb{F}_q^{2n})$ is generated by reflections,

however S^W is not polynomial,

rather generated minimally over $\mathbb{F}_q$ by $3n-1$ generators

and having $n-1$ relations

(that form a regular sequence, that is, a complete intersection presentation):

$$S^W \cong \mathbb{F}_q[\underbrace{D_{2n,2n-1}, \dots, D_{2n,n+1}, D_{2n,n},}_{\text{half of the Dicksen polynomials}} \; \xi_1, \xi_2, \dots, \xi_{2n-1}] \Big/ (r_1, r_2, \dots, r_{n-1})$$

$$\deg(\xi_i) = q^i + 1$$

$$\deg(r_i) = q^{2n} + q^i$$

What about the coinvariant algebra $S/(S^W_+)$?

THEOREM (Mitchell 1985)

If a finite subgroup $W \leq GL_n(k)$ has S^W polynomial, then one has a Brauer-isomorphism

$$\boxed{\underset{\substack{W \text{ via linear} \\ \text{substitution}}}{\overset{}{S/(S^W_+)}}\ \cong\ \underset{\substack{W \text{ left-translating} \\ \text{i.e. the regular representation}}}{kW}}$$

What is Brauer-isomorphism of W-representations?

Either one can phrase it as

- same kW-simple composition factors, with multiplicities

or

- same Brauer character values $\chi^{\text{Brauer}}(w)$ on p-regular elements $w \in W$

THEOREM (Stanton-Webb-R. 2006)

If S^W is polynomial, and $c \in W$ is a regular element with ~~with multiplicative order [illegible], and~~ regular eigenvalue $\zeta \in \mathbb{k}^\times$, then one has a Brauer-isomorphism of $W \times C$-reps

$$\boxed{S/(S^W_+)\ \cong\ kW}$$

W via linear substitutions

$C = \langle c \rangle$ via scalar substitutions $x_i \mapsto \zeta x_i \ \forall i$

W left-translating

$C = \langle c \rangle$ right-translating

But we really needed a more flexible result...

THEOREM (Broer-Smith-Webb-R. 2011)
If S^W is polynomial, and $c \in W$ is a regular element having multiplicative order m, then for any W-representation $\mathcal{U}$ the $\mathcal{U}$-fake-degree

$$f^{\mathcal{U}}(t) := \mathrm{Hilb}(\mathrm{Hom}_{kW}(\mathcal{U}, S), t) \Big/ \mathrm{Hilb}(S^W, t)$$

gives us the Brauer character value for c:

$$\chi_{\mathcal{U}}^{\text{Brauer}}(c) = \left[f^{\mathcal{U}}(t) \right]_{t = e^{2\pi i/m}}$$

COROLLARY: If X is a finite set carrying a transitive W-action, so $X = W/W'$ for some subgroup $W' \leq W$, then the triple

$$\left(\begin{array}{ccccc} X & , & X(t) & , & C \\ \parallel & & \parallel & & \parallel \\ W/W' & & f^{\mathcal{U}_X}(t) & & \langle c \rangle \\ & & \parallel & & \text{translating cosets} \\ & & \dfrac{\mathrm{Hilb}(S^{W'}, t)}{\mathrm{Hilb}(S^W, t)} & & wW' \xmapsto{c^d} c^d wW' \end{array} \right)$$

always exhibits the CSP.

EXAMPLE

A GOOD proof of the $GL_n(\mathbb{F}_q)$-PROTO-Example ...

$$X = \{k\text{-dimensional } \mathbb{F}_q\text{-subspaces of } V = \mathbb{F}_q^n\} = Gr(k, \mathbb{F}_q^n)$$
$$= GL_n(\mathbb{F}_q)/P$$

where $P = \left\{ \begin{bmatrix} \begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \end{bmatrix} \right\}$ $\quad\underbrace{}_{k}\ \underbrace{}_{n-k}$

$$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \prod_{i=0}^{k-1} \frac{1 - t^{q^n - q^i}}{1 - t^{q^k - q^i}}$$

$$C = \underset{\substack{\| \\ \text{Singer cycle}}}{\langle c \rangle} = \mathbb{F}_{q^n}^{\times} = \{1, c, c^2, \ldots, c^{q^n - 2}\} \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$$

PROP (Stanton-Webb-R. 2006)

Regular elements in $GL_n(\mathbb{F}_q)$ (thought of as reflection group)

$= \{$ powers of Singer cycles c^d $\}$

i.e. all images of $\mathbb{F}_{q^n}^{\times} \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$ $\quad (!)$

Also, $X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \prod_{i=0}^{k-1} \frac{1 - t^{q^n - q^i}}{1 - t^{q^k - q^i}}$

$$= \frac{1}{(1-t^{q^k-q^0})\cdots(1-t^{q^k-q^{k-1}}) \cdot (1-t^{q^n-q^k})\cdots(1-t^{q^n-q^{n-1}})} \Bigg/ \frac{1}{(1-t^{q^n-q^0})\cdots(1-t^{q^n-q^{n-1}})}$$

Kuhn-Mitchell 1986
+
Dickson 1911

$$\overset{\downarrow}{=} \mathrm{Hilb}(S^P, t) \Big/ \mathrm{Hilb}(S^{GL_n(\mathbb{F}_q)}, t)$$

Thus the $GL_n(\mathbb{F}_q)$-PROTO-Example is an instance of the previous COROLLARY.

What did Kuhn & Mitchell actually prove in 1996?

THEOREM: When the parabolic subgroup

$$P := \left\{ g = \begin{array}{c} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \\ \underbrace{\quad}_{k} \ \underbrace{\quad}_{n-k} \end{array} \in GL_n(\mathbb{F}_q) \right\} \text{ acts on } S = \mathbb{F}_q[x_1, \dots, x_n]$$

one has $S^P = \mathbb{F}_q[a_0, \dots, a_{k-1} \; , \; b_0, \dots, b_{n-k-1}]$

with degrees $q^k - q^0, \dots, q^k - q^{k-1}$ $\qquad q^n - q^k, \dots, q^n - q^{n-1}$

i.e. $\deg(a_i) = q^k - q^i$ $\qquad \deg(b_j) = q^n - q^{k+j}$

In fact, $a_i = D_{k,i}(x_1, \dots, x_k)$ = Dickson polynomials in $x_1, \dots, x_k$

$$b_j = \Big[D_{n-k,j}(x_{k+1}, \dots, x_n) \Big]_{x_m \mapsto \prod_{(c_1,\dots,c_k) \in \mathbb{F}_q^k} (x_m + c_1 x_1 + \dots + c_k x_k)}$$

for $m = k+1, \dots, n$

This then implies

$$\mathrm{Hilb}(S^P, t) = \frac{1}{(1-t^{q^k-q^0}) \cdots (1-t^{q^k-q^{k-1}})} \cdot \frac{1}{(1-t^{q^n-q^k}) \cdots (1-t^{q^n-q^{n-1}})}$$

and $$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \mathrm{Hilb}(S^P, t) \Big/ \mathrm{Hilb}(S^{GL_n(\mathbb{F}_q)}, t)$$

How to prove such theorems on when S^G is polynomial?
There is an easy-to-use criterion...

THEOREM (Kemper 1996)
For a finite group $G \leq GL_n(k)$ acting on $S = k[x_1, \ldots, x_n]$
and homogeneous G-invariants $f_1, \ldots, f_n \in S^G$
of degrees $d_1, \ldots, d_n$

one has $S^G = k[f_1, \ldots, f_n]$ polynomial

$$\iff \begin{cases} f_1, \ldots, f_n \text{ are algebraically independent, and} \\ d_1 \cdots d_n = |G| \end{cases}$$

EXAMPLES:

① $k[x_1, \ldots, x_n]^{S_n} = k[e_1, \ldots, e_n]$
degrees $1, \ldots, n$
and $\boxed{|S_n| = n! = 1 \cdots n}$

② $\mathbb{F}_q[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[D_{n,n-1}, \ldots, D_{n,1}, D_{n,0}]$
degrees $q^n - q^{n-1}, \ldots, q^n - q^1, q^n - q^0$
and $\boxed{|GL_n(\mathbb{F}_q)| = (q^n - q^0)(q^n - q^1) \cdots (q^n - q^{n-1})}$

③ $\mathbb{F}_q[x_1, \ldots, x_n]^P = \mathbb{F}_q[a_0, \ldots, a_k, b_0, \ldots, b_{n-k}]$
degrees $q^k - q^0, \ldots, q^k - q^{k-1}$ $q^n - q^k, \ldots, q^n - q^{n-1}$

$$P = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\} \leq GL_n(\mathbb{F}_q)$$
$\underbrace{\quad}_{k}\underbrace{\quad}_{n-k}$

$$\boxed{|P| = (q^k - q^0) \cdots (q^k - q^{k-1}) \cdot (q^n - q^k) \cdots (q^n - q^{n-1}) = |GL_k(\mathbb{F}_q)|\,|GL_{n-k}(\mathbb{F}_q)| \cdot q^{k(n-k)}}$$

EXAMPLE: $k=2$, $n=3$

$$P = \left\{ \begin{array}{c|cc|ccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & * & * & * & * & * \\ x_2 & * & * & * & * & * \\ \hline x_3 & 0 & 0 & * & * & * \\ x_4 & 0 & 0 & * & * & * \\ x_5 & 0 & 0 & * & * & * \end{array} \right\} \leq GL_5(\mathbb{F}_q)$$

$$\circlearrowright S = \mathbb{F}_q[x_1, \dots, x_5]$$

Note that

degree $q^2 - q^0$ $\quad a_0 = D_{2,0}(x_1, x_2)$

degree $q^2 - q^1$ $\quad a_1 = D_{2,1}(x_1, x_2)$

coming from

$$\prod_{(c_1,c_2) \in \mathbb{F}_q^2} (t + c_1 x_1 + c_2 x_2) = t^{q^2} + D_{2,1} t^{q^1} + D_{2,0} t^{q^0}$$

lie in S^P because they only involve x_1, x_2

Similarly $D_{3,0}(x_3, x_4, x_5)$, $D_{3,1}(x_3, x_4, x_5)$, $D_{3,2}(x_3, x_4, x_5)$ are invariant under the subgroup

$$\left\{ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array} \right] \right\}$$

and then substituting $x_m \mapsto \prod_{(c_1,c_2) \in \mathbb{F}_q^2} (x_m + c_1 x_1 + c_2 x_2)$ for $m = 3,4,5$

makes them also invariant under

$$\left\{ \left[\begin{array}{cc|ccc} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \right\},$$

hence <u>P-invariant</u>, giving b_0, b_1, b_2

of degrees

$$\begin{array}{ccc} q^2(q^3 - q^0), & q^2(q^3 - q^1), & q^2(q^3 - q^2) \\ \shortparallel & \shortparallel & \shortparallel \\ q^5 - q^2 & q^5 - q^3 & q^5 - q^4 \end{array}$$

Note $|P| = (q^2 - q^0)(q^2 - q^1) \cdot (q^3 - q^0)(q^3 - q^1)(q^3 - q^2) \cdot q^{2 \cdot 3}$

$= \deg(a_0) \deg(a_1) \cdot \deg(b_0) \deg(b_1) \deg(b_2)$

Why are a_0, a_1, b_0, b_1, b_2 algebraically independent in $S = \mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$?

It suffices to show that

$$\mathbb{F}_q[a_0, a_1, b_0, b_1, b_2] \hookrightarrow S$$

is a (module-) <u>finite</u> , or integral extension:

$$S = \mathbb{F}_q[x_1, x_2, x_3, x_4, x_5]$$

$$\Big\uparrow \text{integral}$$

$$\mathbb{F}_q[x_1, x_2, f(x_3), f(x_4), f(x_5)] \quad \text{where } f(t) = \prod_{(c_1,c_2)\in\mathbb{F}_q^2} (t + c_1x_1 + c_2x_2) = t^{q^2} + \sum_{i<q^2} p_i(x_1,x_2)\,t^i$$

$$\Big\uparrow \text{integral}$$

$$\mathbb{F}_q[x_1, x_2, b_0, b_1, b_2] \quad \text{where } b_i := D_{3,i}\big(f(x_3), f(x_4), f(x_5)\big)$$

$$= \text{coefficients in } \prod_{(c_3,c_4,c_5)\in\mathbb{F}_q^3} \big(t + c_3 f(x_3) + c_4 f(x_4) + c_5 f(x_5)\big) = t^{q^3} + b_2 t^{q^2} + b_1 t^{q^1} + b_0 t^{q^0}$$

$$\Big\uparrow \text{integral}$$

$$\mathbb{F}_q[a_0, a_1, b_0, b_1, b_2] \quad \text{where } a_i = D_{2,i}(x_1, x_2)$$

$$= \text{coefficients in } \prod_{(c_1,c_2)\in\mathbb{F}_q^2} (t + c_1x_1 + c_2x_2) = t^{q^2} + a_1 t^{q^1} + a_0 t^{q^0}$$

A somewhat different...
EXAMPLE:

THEOREM (Stanton-Webb-R. 2006)

Let $X := \{\text{all } \underline{\text{symplectic}} \text{ forms } \langle\cdot,\cdot\rangle \text{ on } V = \mathbb{F}_q^{2n}\}$
(= nondegenerate,
anti-symmetric
$\langle y,x\rangle = -\langle x,y\rangle$)

q odd

$\cong GL_{2n}(\mathbb{F}_q) / Sp_{2n}(\mathbb{F}_q)$

with action of $C = \langle \underset{\substack{\shortparallel \\ \text{Singer cycle in } GL_{2n}(\mathbb{F}_q)}}{c} \rangle \cong \mathbb{Z}/(q^{2n}-1)\mathbb{Z}$

via $\langle\cdot,\cdot\rangle \overset{c^d}{\longmapsto} \langle\cdot,\cdot\rangle_{c^d}$ with $\langle x,y\rangle_{c^d} := \langle c^d x, c^d y\rangle$

and $X(t) := \mathrm{Hilb}(S^{Sp_{2n}(\mathbb{F}_q)},t) \Big/ \mathrm{Hilb}(S^{GL_{2n}(\mathbb{F}_q)},t)$

Carlisle & Kropholler 1992
+
Dickson 1911

$$= \frac{\prod_{i=0}^{2n-1}\left(1-t^{q^{2n}-q^i}\right)\prod_{i=1}^{n-1}\left(1-t^{q^{2n}+q^i}\right)}{\prod_{i=n}^{2n-1}\left(1-t^{q^{2n}-q^i}\right)\prod_{i=0}^{2n-1}\left(1-t^{q^i+1}\right)}$$

$$\left(= \frac{\prod_{i=0}^{2n-1}\left(1-t^{q^{2n}-q^i}\right)\prod_{i=1}^{n-1}\left(1-t^{q^{2n}+q^i}\right)}{\prod_{i=1}^{2n-1}\left(1-t^{q^i+1}\right)} \right)$$

gives a triple $(X, X(t), C)$ exhibiting a CSP.

$\underline{\text{proof}}$: Apply the COROLLARY to $X = GL_{2n}(\mathbb{F}_q) / Sp_{2n}(\mathbb{F}_q)$
$= W / W'$. ∎