Reflection group counting and q-counting

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Algebraic and Enumerative Combinatorics
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Outline

The twelve-fold way again

balls N	boxes X	any f	injective f	surjective f
dist.	dist.	x^n	$(x)(x-1)(x-2)\cdots(x-(n-1))$	x! S(n,x)
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$S(n,1) + S(n,2) + \cdots + S(n,x)$	1 if $n \leq x$ 0 else	S(n,x)
indist.	indist.	$p_1(n) + p_2(n) + \cdots + p_x(n)$	1 if $n \leq x$ 0 else	$p_x(n)$

Set partitions, number partitions, compositions

Let's begin our reflection group generalizations with

1. Stirling numbers $S(n, k)$ of the 2nd kind,
2. Stirling numbers $s(n, k)$ of the 1st kind,
3. Signless Stirling numbers $c(n, k)$ of the 1st kind,
4. Composition numbers 2^{n-1},
5. Partition numbers $p(n)$.

Stirling numbers of the 2nd kind

Stirling numbers of the 2nd kind S(n,k) count set partitions of $\{1, 2, \ldots, n\}$ with k blocks.

These are rank numbers of the lattice Π_n of set partitions partially ordered via refinement:

$S(4, 1) = 1$

$S(4, 2) = 7$

$S(4, 3) = 6$

$S(4, 4) = 1$

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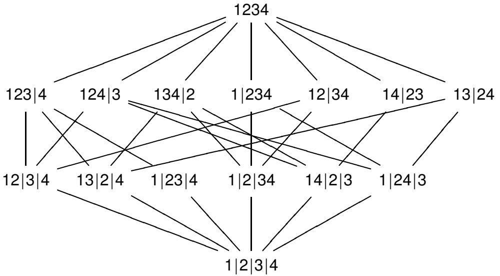

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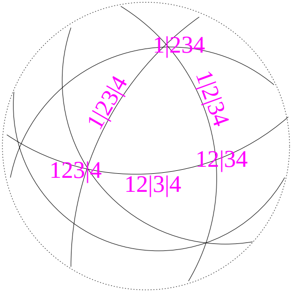
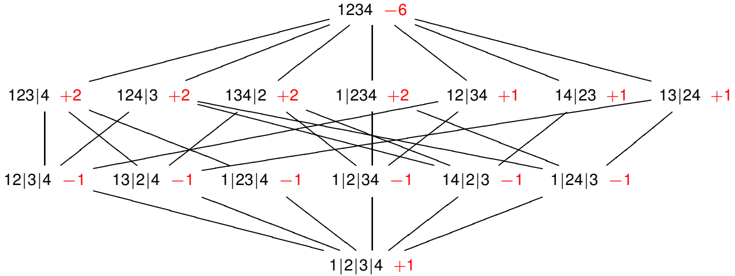
Partition lattice generalizes to intersection lattice

This generalizes for any complex reflection group W to the lattice $\mathcal{L}_W$ of all intersection subspaces of the reflecting hyperplanes, ordered via reverse inclusion, a geometric lattice.



The Stirling numbers of the 1st kind s(n,k) are rank sums of Möbius function values $\mu(\hat{0}, x)$ in the partition lattice Π_n:



$s(4,1) = -6$

$s(4,2) = +11$

$s(4,3) = -6$

$s(4,4) = +1$

A theorem of Orlik and Solomon

Theorem (Orlik-Solomon 1980)

The intersection lattice $\mathcal{L}_W$ for a real reflection group W with degrees $d_1, \ldots, d_n$ has

$$\sum_{X \in \mathcal{L}_W} \mu(\hat{0}, X) x^{\dim X} = \prod_{i=1}^{n} (x - (d_i - 1)) .$$

For any complex reflection group W, the same holds replacing

- *the exponents $d_i - 1$ with*
- *the coexponents $d_i^* + 1$ to be explained later.*

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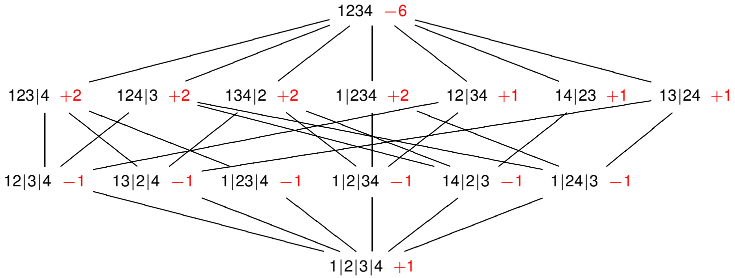
Example: $W = \mathfrak{S}_n$

Example

$$\sum_{k=1}^{n} s(n, n-k) x^k = (x-0)(x-1)(x-2)\cdots(x-(n-1))$$

$$+1x^4 - 6x^3 + 11x^2 - 6x^1 = (x-0)(x-1)(x-2)(x-3) \text{ for } n = 4.$$



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Signless Stirling numbers of the 1st kind

Definition

Recall the signless Stirling number of the 1st kind $c(n,k) = |s(n,k)|$ counts permutations w in $\mathfrak{S}_n$ with k cycles.

One has

$$\sum_{k=1}^{n} c(n,k) x^k = (x+0)(x+1)(x+2)\cdots(x+n-1).$$

Theorem (Shephard-Todd 1955, Solomon 1963)

For any complex reflection group W with degrees $(d_1, \ldots, d_n)$,

$$\sum_{w \in W} x^{\dim V^w} = \prod_{i=1}^{n} (x + (d_i - 1)).$$

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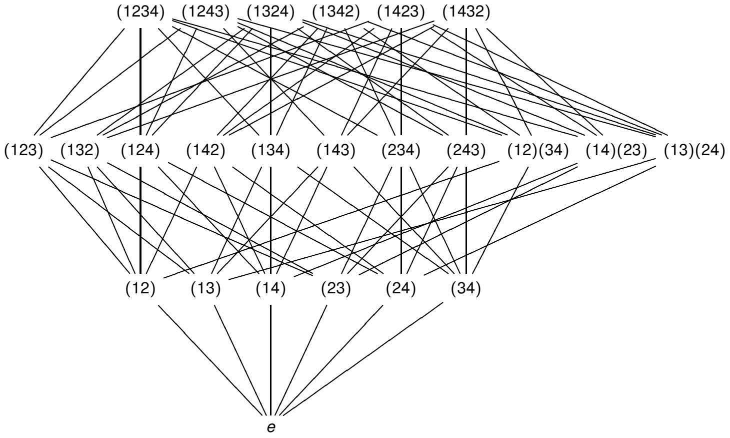
Signless Stirling numbers of the 1st kind

There is another ranked poset relevant here:



$c(4,1) = 6$

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$s(4,4) = 1$

The absolute length and absolute order

Who was this ranked poset having $c(n,k)$ as rank numbers?

Definition

In a complex reflection group W with set of reflections T, the absolute or reflection length is

$$\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ with } t_i \in T\}.$$

WARNING! For real reflection groups W,
this is NOT the usual Coxeter group length $\ell(w) := \ell_S(w)$!

Example

For $W = \mathfrak{S}_n$, where T is the set of transpositions $t_{ij} = (i,j)$, and S is the subset of adjacent transpositions $s_i = (i, i+1)$, one has

$$\ell_S(w) = \#\{\text{ inversions of } w\}$$
$$\ell_T(w) = n - \#\{\text{ cycles of } w\} = n - 1 - dim(V^w)$$

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Definition

Define the absolute order $<$ on W by $u < w$ if

$$\ell_T(u) + \ell_T(u^{-1}w) = \ell_T(w)$$

i.e., when factoring $w = u \cdot v$ one has $\ell_T(u) + \ell_T(v) = \ell_T(w)$.

Theorem (Carter 1972, Brady-Watt 2002)

For real reflection groups W acting on $V = \mathbb{R}^n$, the absolute order $(W, <)$ is a ranked poset with $\mathrm{rank}(w) = n - \dim V^w$.

Thus the (co-)rank generating function for $(W, <)$ is

$$\prod_{i=1}^{n} (x + (d_i - 1)).$$

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Mapping absolute order to the intersection lattice

There is also a natural order-preserving and rank-preserving poset map

$$\begin{array}{rcl} (W,<) & \longrightarrow & \mathcal{L}_W \\ w & \longmapsto & V^w \end{array}$$

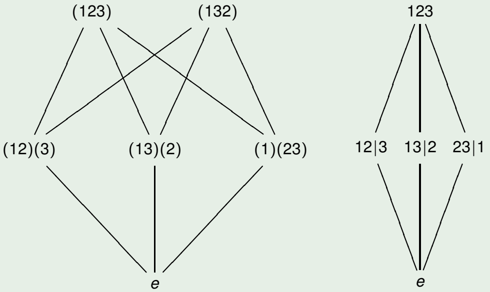
Theorem (Orlik-Solomon 1980)

This map $w \mapsto V^w$ surjects $(W,<) \twoheadrightarrow \mathcal{L}_W$ for any complex reflection group W.

Mapping absolute order to the intersection lattice

Example

For $W = \mathfrak{S}_n$, this map $w \mapsto V^w$ sends a permutation w to the partition π of $\{1, 2, \ldots, n\}$ whose blocks are the cycles of w.



Set partitions mod $\mathfrak{S}_n$ are number partitions

$W = \mathfrak{S}_n$ acts on the set partitions Π_n,
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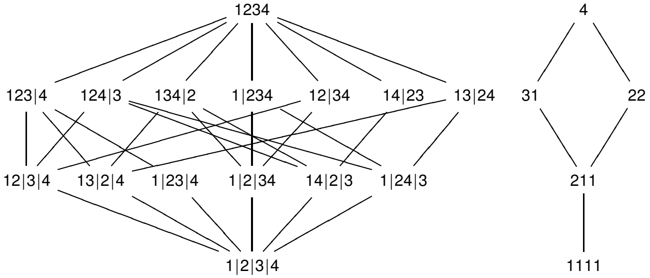

$p_1(4) = 1$

$p_2(4) = 2$

$p_3(4) = 1$

$p_4(4) = 1$

The intersection lattice and its W-orbits

This corresponds to the quotient map of posets

$$\begin{array}{rcl} \mathcal{L}_W & \longrightarrow & W\backslash\mathcal{L}_W \\ X & \longmapsto & W.X \end{array}$$

where $W.X$ is the W-orbit of the hyperplane intersection X

Thus $p_k(n)$ correspond to the rank numbers of $W\backslash\mathcal{L}_W$.

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Compositions to set partitions to partitions

The refinement poset on ordered compositions of n

$$\alpha = (\alpha_1, \ldots, \alpha_\ell)$$

is isomorphic to the Boolean algebra $2^{\{1,2,\ldots,n-1\}}$.

It naturally embeds into the lattice of set partitions Π_n:

$$\{1, 2, \ldots, \alpha_1\} | \{\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2\} | \cdots$$

Example

$\alpha = (2, 4, 1, 2)$ is sent to the partition $12|3456|7|89$

One can then map the set partition to the number partitions, forgetting the order in the composition.

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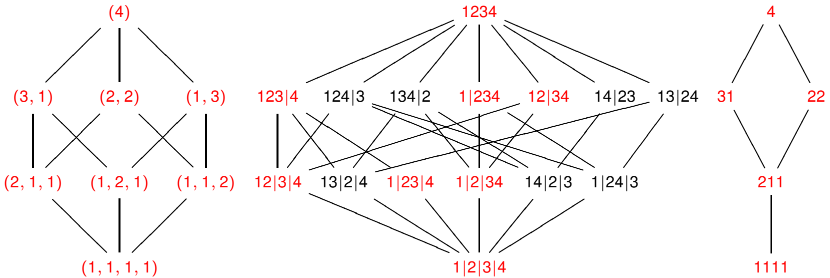

Compositions are subsets of simple reflections

The composition poset $2^{\{1,2,\ldots,n-1\}}$ generalizes for a real reflection groups W, with simple reflections S, to the Boolean algebra 2^S.

Mapping compositions $\to$ set partitions $\to$ partitions corresponds to

$$\begin{array}{ccccc} 2^S & \longrightarrow & \mathcal{L}_W & \longrightarrow & W\backslash\mathcal{L}_W \\ J & \longmapsto & V^J := \bigcap_{s\in J} V^s & & \\ & & X & \longmapsto & W.X \end{array}$$

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Ordered set partitions

The remaining entry in the last column here

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Ordered set partitions

Definition

An ordered set partition of $\{1, 2, \ldots, n\}$ is a set partition $\pi = (B_1, \ldots, B_\ell)$ with a linear ordering among the blocks B_i,

Example

$(\{2, 5, 6\}, \{1, 4\}, \{3, 7\})$ and $(\{3, 7\}, \{1, 4\}, \{2, 5, 6\})$ are different ordered set partitions of $\{1, 2, 3, 4, 5, 6, 7\}$.

There are $k!S(n, k)$ ordered set partitions of $\{1, 2, \ldots, n\}$ with k blocks. These are the rank numbers for the refinement poset on ordered set partitions.

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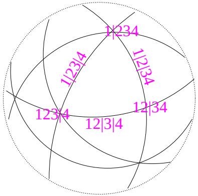
The geometry of ordered set partitions

They label the cones cut out by the reflecting hyperplanes.

Example

$(\{2,5,6\},\{1,4\},\{3,7\}) \leftrightarrow \{x_2 = x_5 = x_6 \leq x_1 = x_4 \leq x_3 = x_7\}$

$(\{3,7\},\{1,4\},\{2,5,6\}) \leftrightarrow \{x_3 = x_7 \geq x_1 = x_4 \geq x_2 = x_5 = x_6\}$



Denote by Σ_W the poset of all such cones ordered via inclusion.

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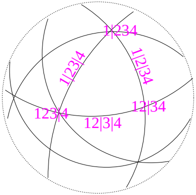

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