Reflection group counting and q-counting

Vic Reiner
Univ. of Minnesota
reiner@math.umn.edu

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Outline

Degrees and the regular representation

Recall for a representation U of a finite group G, the character

$$\begin{array}{ccc} G & \xrightarrow{\chi_U} & \mathbb{C} \\ g & \longmapsto & \chi_U(g) := \mathrm{trace}\,(g|_U) \end{array}$$

evalues the trace of g acting on U.

In particular, its (left-)regular representation $\mathbb{C}[G]$ has

$$\chi_{\mathbb{C}[G]}(g) = \begin{cases} |G| & \text{if } g = e, \text{ the identity,} \\ 0 & \text{if } g \neq e, \end{cases}$$

Degrees and the regular representation

Corollary

Any G-representation U has its degree or dimension

$$\dim_{\mathbb{C}} U = \chi_U(e)$$

given by the character inner product

$$\begin{aligned}\dim_{\mathbb{C}} U &= \langle \chi_{\mathbb{C}[G]}, \chi_U \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g^{-1}) \cdot \chi_U(g)\end{aligned}$$

Fake degrees and the coinvariant algebra

For W a complex reflection group, the Shephard-Todd and Chevalley theorem asserted that the coinvariant algebra gives a graded version of the regular representation:

$$S/(S_+^W) \cong \mathbb{C}[W]$$

Definition

For any W-representation U, the degree

$$\dim_{\mathbb{C}} U = \langle \chi_{\mathbb{C}[G]}, \chi_U \rangle$$

has a q-analogue called the U-fake degree $f^U(q)$:

$$f^U(q) := \sum_{d \geq 0} q^d \cdot \langle \chi_{(S/(S_+^W))_d}, \chi_U \rangle$$

where $(S/(S_+^W))_d$ is the d^{th} graded component of $S/(S_+^W)$.

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The fake degree is a q-analogue of the degree

Proposition

This U-fake degree $f^U(q) = \sum_{d \geq 0} q^d \cdot \langle \chi_{(S/(S_+^W))_d}, \chi_U \rangle$

1. *lies in $\mathbb{N}[q]$,*
2. *has $f^U(1) = \dim_{\mathbb{C}} U$*

Proof.

For the 1st assertion, note $\langle \chi_{(S/(S_+^W))_d}, \chi_U \rangle$ lies in $\mathbb{N}$.

For the 2nd assertion, note

$$\begin{aligned} f^U(1) &= \sum_{d \geq 0} \langle \chi_{(S/(S_+^W))_d}, \chi_U \rangle \\ &= \langle \chi_{S/(S^W)}, \chi_U \rangle = \langle \chi_{\mathbb{C}[W]}, \chi_U \rangle = \dim_{\mathbb{C}} U. \end{aligned}$$

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$\square$

Some examples of fake degrees

Example

For a real reflection group W, both the trivial $\mathbf{1}_W$ and sign representation $\mathrm{sgn}_W = \det_W = \det_W^{-1}$ have dimension 1, so their fake degree is a power of q:

$$f^{\mathbf{1}_W}(q) = q^0 = 1$$
$$f^{\mathrm{sgn}_W}(q) = q^N$$

with $N := |\{\text{reflections}\}| = |\{\text{reflecting hyperplanes}\}|$.

Example

For $W = \mathfrak{S}_n$, one has $f^{\mathrm{sgn}_W}(q) = q^{\binom{n}{2}}$.

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For W a complex reflection group, there are often more reflections than reflecting hyperplanes:
two unitary reflections can share the same fixed hyperplane.

There is also a distinction between the two linear characters $\det_W, \det_W^{-1}$, and between their fake degrees:

$$f^{\det_W}(q) = q^{|\{\text{reflecting hyperplanes}\}|}$$

$$f^{\det_W^{-1}}(q) = q^{|\{\text{reflections}\}|}$$

We'll say more about what those exponents are in terms of degrees and codegrees next.

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$(d_1, \ldots, d_n)$ and the fake degree of V

Example

For W a real reflection group acting on V with degrees $(d_1, \ldots, d_n)$, for homogeneous invariants $f_1, \ldots, f_n$ having $S^W = \mathbb{C}[f_1, \ldots, f_n]$, a result of Solomon (1963) implies

$$f^V(q) = \sum_{i=1}^{n} q^{d_i - 1}.$$

Remark

One might also ask about the contragredient V^* of the reflection representation V. But in the real reflection group case one has $V^* \cong V$ so that $f^{V^*}(q) = f^V(q)$.

Codegrees versus degrees

On the other hand, for W a complex reflection group one need not have $V^* \cong V$, and we must pick our conventions. Suppose we let $S = \mathbb{C}[x_1, \ldots, x_n]$ be the symmetric algebra of V^*, so that $V^* = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$, and $S^W = \mathbb{C}[f_1, \ldots, f_n]$ with $\deg(f_i) = d_i$.

Then it is still true that $f^V(q) = \sum_{i=1}^n q^{d_i - 1}$ but now one has ...

Definition

$$f^{V^*}(q) = \sum_{i=1}^{n} q^{d_i^* + 1}$$

for some uniquely defined nonnegative integers $(d_1^*, \ldots, d_n^*)$ called codegrees.

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Codegrees versus degrees

Theorem

For a complex reflection group W acting on $V = \mathbb{C}^n$ with degrees d_i and codegrees d_i^*, one has

- *(Shephard-Todd 1955, Solomon 1963)*

$$\sum_{w \in W} t^{\dim(V^w)} = \prod_{i=1}^{n} (t + (d_i - 1))$$

In particular, $\sum_{i=1}^{n} (d_i - 1) = |\{\textit{reflections}\}|$.

- *(Orlik-Solomon 1980)*

$$\sum_{w \in W} \det(w) t^{\dim(V^w)} = \prod_{i=1}^{n} (t - (d_i^* + 1))$$

In particular, $\sum_{i=1}^{n} (d_i^* + 1) = |\{\textit{reflecting hyperplanes}\}|$.

Confused?

Good news: For the well-generated groups W, and hence all real reflection groups and all Shephard groups, the degrees and codegrees determine each other in a simple way.

Theorem

A complex reflection group W acting on $V = \mathbb{C}^n$ is well-generated (that is, generated by n reflections) if and only if the degrees $d_1 \leq \cdots \leq d_n$ and codegrees $d_1^* \geq \cdots \geq d_n^*$ satisfy

$$d_i^* + d_i = d_n (:= h) \text{ for } i = 1, 2, \ldots, n.$$

Proof.

Bad news: This has only been verified case-by-case! □

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Degrees, codegrees for well-generated groups

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Our previous $X(q)$ was a fake degree

Proposition

For G a complex reflection group and H any subgroup, the transitive permutation G-representation

$$U = \mathbb{C}[G/H]$$

where G left-translates $X = G/H = \{gH\}$, has fake degree

$$f^{\mathbb{C}[G/H]}(q) = X(q) = \frac{\mathrm{Hilb}(S^H, q)}{\mathrm{Hilb}(S^G, q)},$$

that is, our q-analogue of $[G : H]$ considered before.

Irreducible degrees and fake degrees

Particularly important are (fake) degrees of W-irreducibles

Example

Recall $W = \mathfrak{S}_n$ has irreducible W-representations U^λ indexed by number partitions λ of n.

Definition

A standard Young tableau of shape λ is a filling T of the Ferrers diagram of λ with the numbers $\{1, 2, \ldots, n\}$, each appearing exactly once, increasing left-to-right in rows and top-to-bottom in columns.

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Irreducible degrees for $\mathfrak{S}_n$

Example

The 5 standard Young tableaux of shape $\lambda = (3,2)$ are

$$\left\{ \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}, \quad \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}, \quad \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \end{matrix}, \quad \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix}, \quad \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix} \right\}.$$

Theorem (Young 1927)

$f^\lambda := \dim(U^\lambda)$ *counts the standard Young tableaux of shape* λ.

Theorem (Frame-Robinson-Thrall hook length formula 1954)

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

where x *runs through the cells in the Ferrers diagram of* λ*, and* $h(x)$ *denotes the hook length at* x.

Example

For $n = 5$, the partition $\lambda = 32$ has hook lengths labelled here:

$$\begin{array}{ccc} 4 & 3 & 1 \\ 2 & 1 & \end{array}$$

Hence

$$f^\lambda = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5,$$

agreeing with our count of 5 standard Young tableaux.

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Lusztig's and Stanley's fake degree formulas

Theorem (Lusztig 1979)

$$f^{U^\lambda}(q) = \sum_T q^{\mathrm{maj}(T)}$$

where the sum runs over all standard Young tableaux T of shape λ, and $\mathrm{maj}(T)$ is the sum of the entries i in T for which $i+1$ lies in a lower row of T.

Theorem (Stanley 1971)

$$f^{U^\lambda}(q) = q^{n(\lambda)} \frac{[n]!_q}{\prod_{x \in \lambda} [h(x)]_q}$$

where $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$.

Lusztig's and Stanley's fake degree formulas

Example

Here are the standard Young tableaux of shape $\lambda = 32$, highlighting the red entries that sum to $\mathrm{maj}(T)$:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}, \quad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \end{array}, \quad \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

$$\begin{aligned} f^{U^{32}}(q) &= q^3 + q^6 + q^2 + q^5 + q^4 \\ &= q^2[5]_q. \end{aligned}$$

Since $n(32) = 0 \cdot 3 + 1 \cdot 2 = 2$, Stanley's formula says

$$f^{U^{32}}(q) = q^{n(32)} \frac{[5]!_q}{[4]_q[3]_q[2]_q[1]_q[1]_q} = q^2[5]_q.$$

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$f^{U^\lambda}(q)$ is meaningful at prime powers q

Steinberg (1951) constructed some of the complex irreducible representations of $U^\lambda(q)$ of $GL_n(\mathbb{F}_q)$, particularly q-analogous to the irreducible representations U^λ of $\mathfrak{S}_n$, called unipotent representations.

Theorem (Olsson 1986)

For q the order of a finite field $\mathbb{F}_q$, the fake degree $f^{U^\lambda}(q)$ becomes the usual degree of Steinberg's unipotent $GL_n(\mathbb{F}_q)$-representation $U^\lambda(q)$.

Similar statements hold for other simple algebraic groups G over $\mathbb{F}_q$ beside $GL_n(\mathbb{F}_q)$.

$f^U(q)$ is meaningful at roots-of-unity

Here is a reformulation of Springer's isomorphism $S/(S_+^W) \cong_{W \times C} \mathbb{C}[W]$. Pick c a regular element in complex reflection group W, with ζ in $\mathbb{C}^\times$ its eigenvalue on v in V avoiding the reflecting hyperplanes, so $c(v) = \zeta \cdot v$.

Theorem

In this setting, the character value (trace) of the regular element c acting in any complex W-representation U is the evaluation of the fake-degree at $q = \zeta$:

$$\chi_U(c) = \left[f^U(q) \right]_{q=\zeta}.$$

E.g. this interprets $f^{U^\lambda}(\zeta)$ at n^{th} and $(n-1)^{st}$-roots-of-unity ζ.

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