Reflection group counting and q-counting

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Summer School on
Algebraic and Enumerative Combinatorics
S. Miguel de Seide, Portugal
July 2-13, 2012

Outline

The Catalan numbers

Recall the Catalan number

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

counts many things
(see Stanley's "Enum. Comb. Vol. 2" Exer. 6.19).
Among them are these four:

1. Noncrossing partitions of $\{1, 2, \ldots, n\}$
2. Nonnesting partitions of $\{1, 2, \ldots, n\}$
3. Increasing parking functions of length n
4. Triangulations of a convex $(n+2)$-gon

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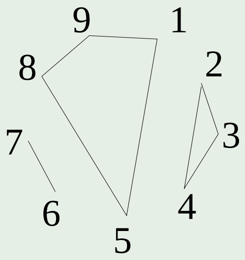
Noncrossing partitions

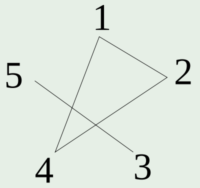
Definition

Draw $\{1, 2, \ldots, n\}$ as points around a circle, and call a set partition noncrossing if the convex hulls of its blocks are disjoint.

Example

1589|234|67 is noncrossing, while 124|35 is crossing.





The poset $NC(n)$ and Narayana numbers

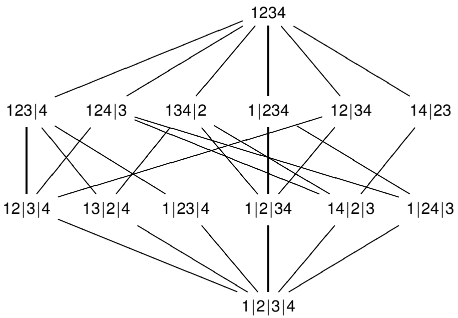
Theorem (Kreweras 1972)

The poset $NC(n)$ of all noncrossing partitions of $\{1, 2, \ldots, n\}$ inside the partition lattice Π_n has the Narayana numbers

$$\mathrm{Nar}(n, k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

as rank numbers.

The noncrossing partition poset $NC(4)$



$\mathrm{Nar}(4,1) = \frac{1}{4}\binom{4}{1}\binom{4}{0} = 1$

$\mathrm{Nar}(4,2) = \frac{1}{4}\binom{4}{2}\binom{4}{1} = 6$

$\mathrm{Nar}(4,3) = \frac{1}{4}\binom{4}{3}\binom{4}{2} = 6$

$\mathrm{Nar}(4,4) = \frac{1}{4}\binom{4}{4}\binom{4}{3} = 1$

Nonnesting partitions

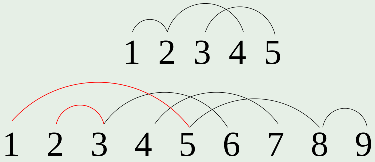
Plot $\{1, 2, \ldots, n\}$ along the x-axis, and depict set partitions by semicircular arcs in the upper half-plane, connecting i, j in the same block if no other k with $i < k < j$ is in that block.

Definition

Say the set partition is nonnesting if no pair of arcs nest.

Example

$124|35$ is nonnesting,
while $1589|234|67$ is nesting as arc 15 nests arc 23.



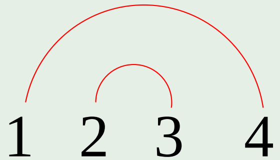
Narayana numbers and nonnesting partitions

Narayana numbers $\mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ also count nonnesting set partitions with k blocks, or $n-k$ arcs.

Example

$$\mathrm{Nar}(4,2) = \frac{1}{4}\binom{4}{2}\binom{4}{2-1} = 6$$

as 1 one of the $7 = S(4,2)$ partitions of $\{1,2,3,4\}$ is nesting:



Increasing parking functions

Definition

An increasing parking function of length n is a weakly increasing sequence $(a_1 \leq \ldots \leq a_n)$ with a_i in $\{1, 2, \ldots, i\}$.

Definition

A parking function is sequence $(b_1, \ldots, b_n)$ whose weakly increasing rearrangement is an increasing parking function.

Theorem (Konheim and Weiss 1966)

There are $(n+1)^{n-1}$ *parking functions of length* n

By definition parking functions have an $\mathfrak{S}_n$-action on positions

$$w(b_1, \ldots, b_n) = (b_{w(1)}, \ldots, b_{w(n)})$$

and increasing parking functions represent the $\mathfrak{S}_n$-orbits.

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Parking functions of length $n = 3$

Example

The $(3+1)^{3-1} = 16$ parking functions of length 3, grouped into the $C_3 = \frac{1}{4}\binom{6}{3} = 5$ different $\mathfrak{S}_3$-orbits, with increasing parking function representative shown leftmost:

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

Narayana numbers and increasing parking functions

The Narayana number $N(n, k)$ also counts increasing parking functions by their number of distinct values.

Example

The $C_4 = \frac{1}{5}\binom{8}{4} = 14$ increasing parking functions of length 4, grouped by number of distinct values:

increasing parking function	k	N(4,k)
1111	1	1
1112, 1113, 1114 1122, 1222, 1133	2	6
1123, 1124, 1134 1223, 1224, 1233	3	6
1234	4	1

(Or Dyck paths $(0, 0) \to (2n, 0)$ counted by number of peaks.)

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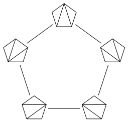
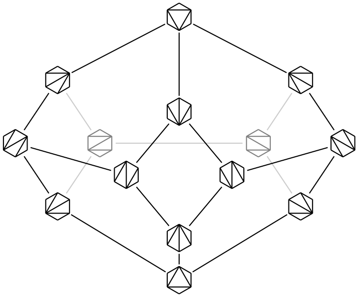
Triangulations of an $(n+2)$-gon

There are $C_3 = 5$ for a convex $(3+2)$-gon,



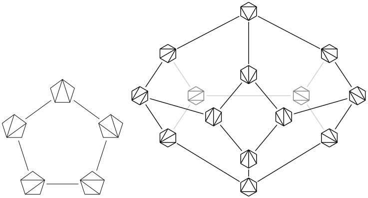
and $C_4 = 14$ for a convex $(4+2)$-gon



Triangulations and the associahedron

Theorem (Stasheff 1963, Milnor 1963, Haiman 1984, Lee 1989, Gelfand-Kapranov-Zelevinksy 1989)

Triangulations of a convex $(n+2)$-label the vertices of an $(n-1)$-dimensional convex polytope: the associahedron.



What about faces of higher dimension than the vertices?

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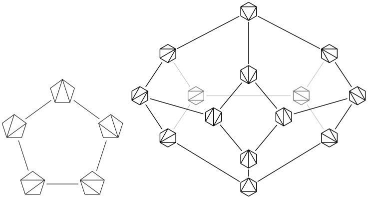


What about faces of higher dimension than the vertices?

Kirkman-Cayley numbers

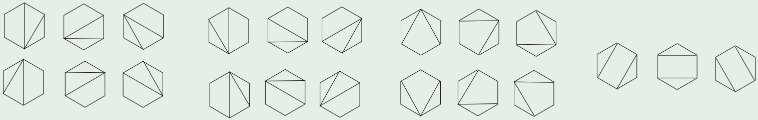
Theorem (Kirkman 1857, Cayley 1890)

$$\mathrm{Kirk}(n,k) := \frac{1}{k+1}\binom{n+k+1}{k}\binom{n-1}{k}$$

count dissections of the $(n+2)$-gon using k diagonals.

Example

$\mathrm{Kirk}(4,2) = \frac{1}{2+1}\binom{4+2+1}{2}\binom{4-1}{2} = \frac{1}{3}\binom{7}{2}\binom{3}{2} = 21$

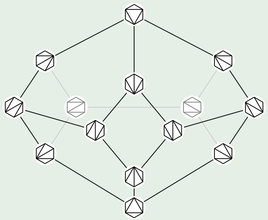


Counting faces of associahedra

$\mathrm{Kirk}(n,k)$ counts $(n-1-k)$-dim'l faces of the associahedron.

Example

k	$\mathrm{Kirk}(4,k) = \frac{1}{k+1}\binom{4+k+1}{k}\binom{4-1}{k}$	
3	14	vertices
2	21	edges
1	9	2-faces
0	1	the 3-face



Kirkman is to Narayana as f-vector is to h-vector

The relation between Kirkman and Narayana numbers is the (invertible) relation of the f-vector $(f_0, \ldots, f_n)$ of a simple n-dimensional polytope to its h-vector $(h_0, \ldots, h_n)$:

$$\sum_{i=0}^{n} f_i t^i = \sum_{i=0}^{n} h_i (t+1)^{n-i}.$$

Example

The 3-dimensional associahedron has f-vector $(14, 21, 9, 1)$, and h-vector $(1, 6, 6, 1)$.

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & & 1 & & 9 & & \\
 & & 1 & & 8 & & 21 & \\
 & 1 & & 7 & & 13 & & 14 \\
\hline
(1, & & 6, & & 6, & & 1) &
\end{array}$$

Reflection group Catalan objects

It turns out that one can at least generalize

noncrossing partitions	to	well-generated reflection groups
nonnesting partitions	to	Weyl groups
increasing parking functions	to	Weyl groups
triangulations	to	real reflection groups.

These give generalizations of the parking function, Catalan, Kirkman, Narayana numbers, and for most of them also q-analogues.

Nevertheless, many mysteries about them remain.

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Noncrossing partitions as interval in absolute order

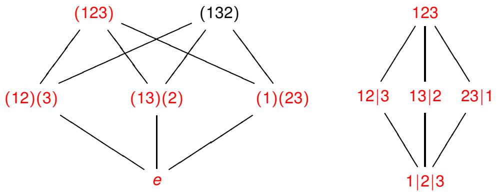
Let c be an n-cycle $(1, 2, \ldots, n)$ in $W = \mathfrak{S}_n$.

Biane (2002) observed that the map

$$(W, <) \longrightarrow \Pi_n$$

sending w to its cycle partition restricts to an isomorphism

$$[e, c] \to NC(n)$$



Noncrossing partitions as interval in absolute order

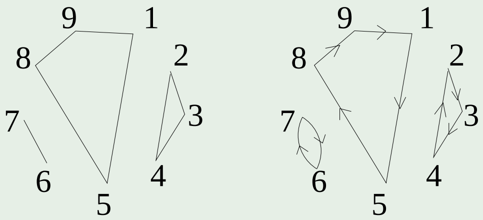
Theorem (Biane 2002)

A permutation w in $\mathfrak{S}_n$ lies in the absolute order interval $[e, c]$ if and only if the cycles of w are noncrossing and oriented clockwise when we draw $\{1, 2, \ldots, n\}$ clockwise around a circle.

Proof.

See the exercises. □

Example



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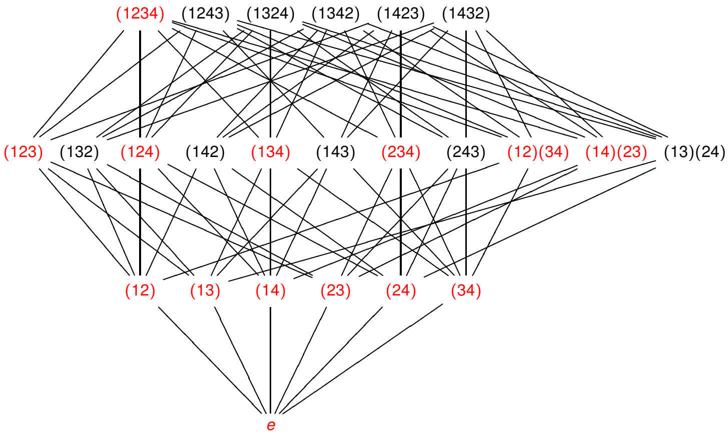

Coxeter elements for well-generated groups

Who plays the role of $c = (1, 2, \ldots, n)$ for more general W?

Definition

For W any complex reflection group, define the Coxeter number

$$h := \frac{1}{2}\left(\#\{\text{reflections}\} + \#\{\text{reflecting hyperplanes}\}\right).$$

Coxeter elements for well-generated groups

For W well-generated the largest d_n of the degrees $(d_1 \leq \cdots \leq d_n)$ has $d_n = h$,

A theorem of Lehrer and Michel (2003) implies existence of a regular element c of order h with eigenvalue $\zeta = e^{\frac{2\pi i}{h}}$.

Definition

Call such an element c a Coxeter element for c.

Example (Coxeter 1948)

For real reflection groups W with simple reflections $S = \{s_1, \ldots, s_n\}$, the product $c = s_1 s_2 \cdots s_n$ is always a Coxeter element in the above sense.

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Noncrossing partitions for well-generated groups

Definition (Bessis 2003, 2006)

For W a well-generated complex reflection group, define the poset of noncrossing partitions $NC(W)$ to be the interval $[e, c]$ in the absolute order $(W, <)$

Theorem (Bessis 2006)

The W-noncrossing partition poset $NC(W)$

- *is ranked with $\mathrm{rank}(w) = n - \dim(V^w)$,*
- *is self-dual with antiautomorphism $w \mapsto w^{-1}c$,*
- *is a lattice, and*
- *has cardinality given by the W-Catalan number*

$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{h + d_i}{d_i} = \frac{1}{|W|} \prod_{i=1}^{n} (h + d_i).$$

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Noncrossing partitions for well-generated groups

The first two properties (ranked, self-dual) are easy to prove uniformly, and the self-duality $w \mapsto w^{-1}c$ generalizes Kreweras complementation on $NC(n)$.

The last two properties (lattice, cardinality $\mathrm{Cat}(W)$) have only case-by-case proofs currently.

The lattice property has uniform proofs for real reflection groups, due to Brady and Watt (2005) and to Reading (2005).

Problem

Prove $|NC(W)| = \mathrm{Cat}(W)$ uniformly for

- *well-generated groups,*
- *or even just for real reflection groups,*
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Narayana numbers for well-generated groups

Rank numbers of $NC(W)$ generalize Narayana numbers.

Example

For the hyperoctahedral group $W = \mathfrak{S}_n^{\pm}$,
with degrees $(d_1, \ldots, d_n) = (2, 4, \ldots, 2n)$, one finds that

- $\text{Cat}(W) = \binom{2n}{n}$,
- $NC(W)$ is the subposet of centrally symmetric noncrossing partitions inside $NC(2n)$,
- there are $\binom{n}{k}^2$ elements in $NC(W)$ of rank k, so these are the W-Narayana numbers.

(Note that $\binom{2n}{n} = \sum_k \binom{n}{k}^2$.)

Narayana numbers for well-generated groups

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Nonnesting partitions for Weyl groups

Recall we said nonnesting partitions generalize to Weyl groups W (=crystallographic real reflection groups)

Such groups preserve a lattice, and have choices of root systems Φ as a W-stable collection of normal vectors $\pm\alpha$ to all the reflecting hyperplanes.

One can always split Φ into positive and negative roots

$$\Phi = \Phi^+ \sqcup (-\Phi^+)$$

by fixing a fundamental chamber C_0 in $V = \mathbb{R}^n$ cut out by the hyperplanes, and saying Φ^+ are roots pairing positively with C_0.

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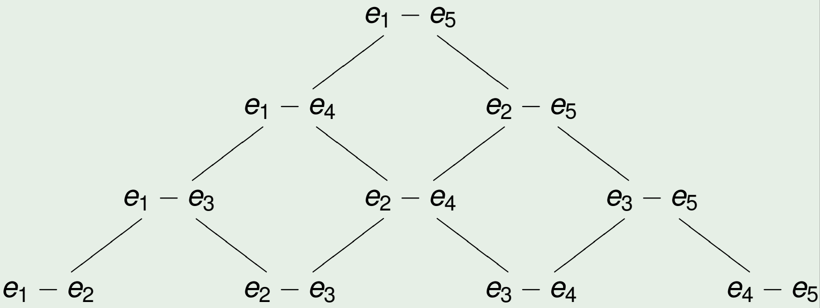
Nonnesting partitions for Weyl groups

Definition

The root order on Φ_+ says that $\alpha < \beta$ if $\beta - \alpha$ is a nonnegative combination of roots in Φ_+.

Example

For $W = \mathfrak{S}_5$, the root order on $\Phi_+ = \{e_i - e_j : 1 \leq i < j \leq 5\}$ is



Nonnesting partitions for Weyl groups

Postnikov (1996) observed nonnesting partitions of $\{1, 2, \ldots, n\}$ biject with antichains in the poset Φ_+ for $\mathfrak{S}_n$:
to each arc $i < j$ associate the root $e_i - e_j$.

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$124|35$ is nonnesting, corresponding to antichain $\{e_1 - e_2, e_2 - e_4, e_3 - e_5\}$:

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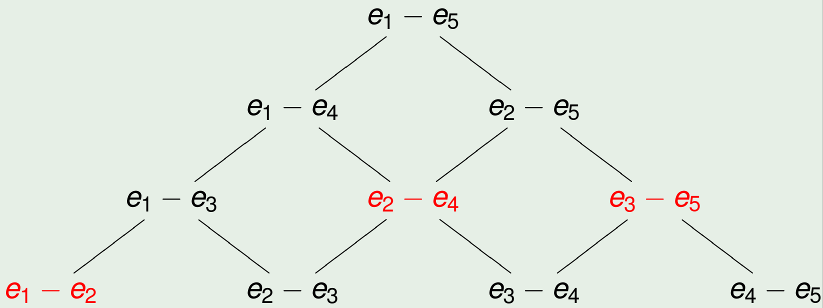

Nonnesting partitions for Weyl groups

Definition (Postnikov)

For any Weyl group W with a choice of root system Φ and positive roots Φ_+, call an antichain in the poset Φ_+ a nonnesting partition for W.

Let Q be the root lattice $\mathbb{Z}$-spanned by Φ.

Theorem (Shi 1986, Cellini-Papi 2002)

Antichains in the poset Φ_+ also parametrize the W-orbits $W \backslash Q/(h+1)Q$ when W acts on $Q/(h+1)Q$.

Theorem (Haiman 1993)

The $(h+1)^n$ elements of $Q/(h+1)Q$ fall into $\mathrm{Cat}(W)$ many W-orbits $W \backslash Q/(h+1)Q$.

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Parking functions for Weyl groups

Haiman also pointed out for $W = \mathfrak{S}_n$ how the root lattice Q can be identified W-equivariantly with $\mathbb{Z}^n/\mathbb{Z}\mathbf{1} \cong \mathbb{Z}^{n-1}$ where $\mathbf{1} = (1, 1, \ldots, 1)$.

Then parking functions of length n give representatives for the $(n+1)^{n-1}$ different cosets $Q/(h+1)Q = Q/(n+1)Q$.

Thus

- $Q/(h+1)Q$ generalizes parking functions, and
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Parking, increasing parking functions for Weyl groups

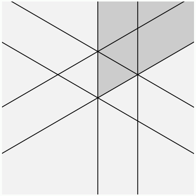
Shi and Cellini-Papi also biject parking functions and increasing parking functions with the $(h+1)^n$ chambers cut out by the

$$\text{Shi arrangement } \{(\alpha, x) = 0, 1 : \alpha \in \Phi_+\}$$

and the subset of $\mathrm{Cat}(W)$ many chambers that lie within the dominant cone where $(\alpha, x) > 0$ for all α in Φ_+.

Example

The Shi, dominant Shi chambers for $W = \mathfrak{S}_3$:



Here $h^n = 4^{(3-1)} = 16$ and $\mathrm{Cat}(W) = \frac{1}{4}\binom{6}{3} = 5$.

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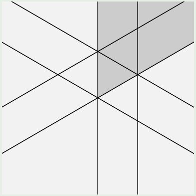

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It has been checked case-by-case that the W-Narayana numbers defined earlier (=rank numbers of $NC(W)$) also count

- the nonnesting partitions or antichains $A \subset \Phi_+$ for which the intersection subspace

$$X_A := \bigcap_{\alpha \in A} H_\alpha$$

in $\mathcal{L}_W$ has a given dimension, and

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More refined: Kreweras numbers

Theorem (Kreweras 1972)

The number of noncrossing partitions of $\{1, 2, \ldots, n\}$ for which the cycle size partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ has m_i parts of size i is

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Recall taking the cycle size partition λ of a set partition is mapping an intersection subspaces to its W-orbit:

$$\begin{array}{rcl} \mathcal{L}_W & \longrightarrow & W \backslash \mathcal{L}_W \\ X & \longmapsto & W.X \end{array}$$

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Generalization of Kreweras numbers

The case-by-case check of the Narayana number coincidence actually showed for each W-orbit $W.X$ in $W\backslash\mathcal{L}_W$ that the following W-Kreweras numbers coincide:

- number of w in $NC(W) = [e, c]$ with V^w in $W.X$
- number of antichains $A \subset \Phi_+$ having the subspace $X_A := \bigcap_{\alpha \in A} H_\alpha$ in $W.X$, or equivalently,
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Kreweras numbers have a product formula

For Weyl groups W one even has a product formula.

Theorem (Sommers-Trapa 1997, Broer 1998, Douglass 1999)

The number of antichains $A \subset \Phi_+$ with $X_A = \bigcap_{\alpha \in A} H_\alpha$ in $W.X$ is

$$\frac{1}{[N_W(W_X) : W_X]} \prod_{i=1}^{\ell} (h + 1 - e_i^X)$$

where e_i^X are integers called the Orlik-Solomon exponents of the restriction $\mathcal{A}|X$ to X of the reflection arrangement $\mathcal{A}$.

The Orlik-Solomon exponents are the roots of the restricted arrangement's characteristic polynomial

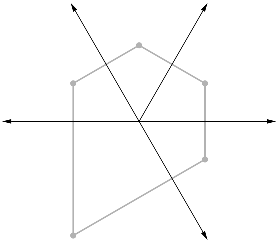
$$\sum_{Y \in \mathcal{L}_{\mathcal{A}|X}} \mu(\hat{0}, Y) t^{\dim(Y)} = \prod_{i=1}^{\ell} (t - e_i^X).$$

Triangulations, clusters and Cambrian fans

We won't do justice to this topic!

In Fomin and Zelevinsky's theory of cluster algebras, a special role is played by those of finite type, which have a classfication parallels that of Weyl groups.

To each such Weyl group and finite type cluster algebra one associates the cluster fan, Δ_W, a complete simplicial fan in $V = \mathbb{R}^n$.



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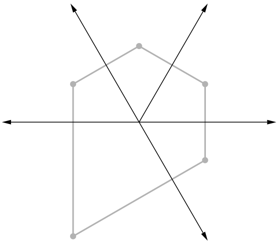

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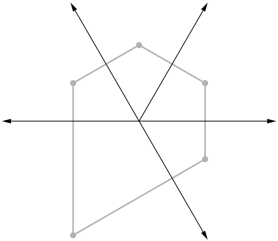


Triangulations, clusters and Cambrian fans

Example

The cluster algebra corresponding to $W = \mathfrak{S}_n$ is isomorphic to the coordinate ring of the Grassmannian $G(2, \mathbb{C}^{n+2})$.

It is the subalgebra of $\mathbb{C}[a_{ij}]_{i \leq 2, j \leq n+2}$ generated by 2×2 minors

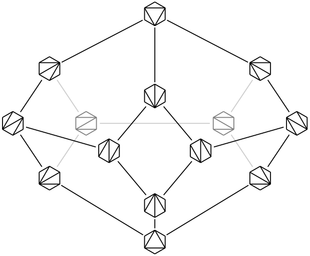
$$\Delta_{i,j} = \det \begin{bmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{bmatrix}$$

of a $2 \times (n+2)$-matrix of indeterminates

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n+2} \\ a_{21} & a_{22} & \cdots & a_{2,n+2} \end{bmatrix}$$

The type A cluster fan

The minors Δ_{ij} are the cluster variables, and they biject with the diagonals ij in the $(n+2)$-gon.
Certain $(2n-3)$-element subsets of the minors Δ_{ij} are called clusters. In this case, clusters biject with triangulations of the $2n$-gon, thought of as the diagonals present in the triangulation (including the n outside diagonals $\{12, 23, \ldots\}$).



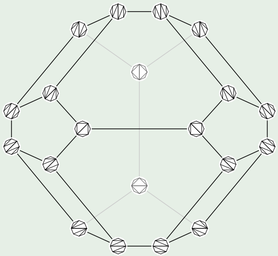
Triangulations, clusters and Cambrian fans

Theorem (Chapoton, Fomin, and Zelevinsky 2002)

A finite type cluster fan is the normal fan of a convex polytope.

Example

For $W = \mathfrak{S}_n^{\pm}$, it is the Bott-Taubes/cyclohedron/type B associahedron considered by Bott and Taubes, Simion. Vertices are centrally symmetric $2n$-gon triangulations.



Triangulations, clusters and Cambrian fans

Theorem (Reading 2006)

For real reflection groups, one can define a Cambrian fan, coarsening the reflection arrangement fan, combinatorially isomorphic to the cluster fan for Weyl groups.

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Catalan, Kirkman, Narayana in W-associahedra

Reading also developed theories of c-sortable elements, and shard intersection order, explaining uniformly the following.

Theorem (Reading 2005)

For real reflection groups W, the W-associahedron (resp. Cambrian fan) has

- *vertices (resp. top dimensional cones) bijecting with $NC(W)$, hence counted by $\mathrm{Cat}(W)$), and*
- *the f-vector to h-vector map sends its face numbers, the W-Kirkman numbers, into the rank numbers of $NC(W)$, the W-Narayana numbers.*

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q-parking functions, q-Catalan, q-Kirkman

Where to find natural q-analogues of the

- $(h+1)^n$ many W-parking functions $Q/(h+1)Q$,
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Homogeneous systems of parameters again

A starting point was found by Haiman for $W = \mathfrak{S}_n$, and later by others for real reflection groups in work on finite-dimensional representations of rational Cherednik algebras.

Theorem (Berest-Etingof-Ginzburg 2003, Gordon 2003)

For a real reflection group W acting on V and on $S = \mathrm{Sym}(V^*) = \mathbb{C}[x_1, \ldots, x_n]$, there always exists

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- *with all θ_i homogeneous of degree $h + 1$,*
- *whose linear span $\mathbb{C}\theta_1 + \cdots \mathbb{C}\theta_n$ carries the representation $V^*(\cong V)$ inside S_{h+1}.*

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A starting point was found by Haiman for $W = \mathfrak{S}_n$, and later by others for real reflection groups in work on finite-dimensional representations of rational Cherednik algebras.

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For a real reflection group W acting on V and on $S = \mathrm{Sym}(V^*) = \mathbb{C}[x_1, \ldots, x_n]$, there always exists

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h.s.o.p.'s for $\mathfrak{S}_n$ and $\mathfrak{S}_n^{\pm}$

Example

For the hyperoctahedral groups $\mathfrak{S}_n^{\pm}$, one has $h = 2n$, and one can take $\Theta = (x_1^{2n+1}, \ldots, x_n^{2n+1})$.

But in general, these Θ are not so easy to construct!
One seems to need rational Cherednik theory or other insight.

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For the symmetric groups $\mathfrak{S}_n$, one has $h = n$, and one can take

$$\theta_i = \text{ coefficient of } t^{n+1} \text{ in } \frac{\prod_{j=1}^{n}(1 - x_j t)^{\frac{n+1}{n}}}{(1 - x_i t)}$$

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Graded parking spaces

Θ a system of parameters means the quotient $S/(\Theta)$ is a finite-dimensional $\mathbb{C}$-vector space.

Cohen-Macaulayness further implies S is a free module over $\mathbb{C}[\Theta] := \mathbb{C}[\theta_1, \ldots, \theta_n]$.

Definition

Call the quotient

$$S/(\Theta) = S/(\theta_1, \ldots, \theta_n)$$

the graded parking space for the real reflection group W.

Graded parking spaces

Theorem (Haiman 1994, BEG 2003, Gordon 2003)

The graded parking space is isomorphic as W-representation to the W-permutation representation on $Q/(h+1)Q$, with

$$\mathrm{Hilb}(S/(\Theta), q) = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(\mathbb{C}[\Theta], q)} = \frac{1/(1-q)^n}{1/(1-q^{h+1})^n} = [h+1]_q^n.$$

the q-parking function number for W.

Its W-fixed subspace as a graded vector space has

$$\mathrm{Hilb}((S/(\Theta))^W, q) = \mathrm{Cat}(W, q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

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Mysteries of the q-Catalan number for W

Sadly, this theory gives the only uniform proof known that

$$\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

lies in $\mathbb{N}[q]$, for real reflection groups, or even for Weyl groups.

Problem

Is there a simple statistic $\mathrm{stat}(-)$ on any W-Catalan objects

- *$NC(W)$,*
- *$W\backslash Q/(h+1)Q$ or antichains in Φ_+, or dominant Shi chambers,*
- *W-clusters, for which*

$$\mathrm{Cat}(W,q) = \sum_x q^{\mathrm{stat}(x)}?$$

q-Catalan in the well-generated case

Work of Gordon and Griffeth (2009) shows that for well-generated W

$$\mathrm{Cat}(W,q) = \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

still lies in $\mathbb{N}[q]$, but their proof relies on some uniformly-stated facts about bases for the Hecke algebras $\mathcal{H}_W$ that have only been checked case-by-case.

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CSP's for the q-Catalan

One has CSP triples $(X, X(q), C)$ for various of the W-Catalan objects X and $X(q) = \mathrm{Cat}(W, q)$, with different cyclic actions C.

And sadly, none have been proven in a truly uniform fashion. In each case, some aspect of the proofs have relied on a fact checked case-by-case.

The noncrossing partition CSP

Recall the noncrossing partitions $NC(W) = [e, c]$ have an antiautomorphism $w \mapsto w^{-1}c$, the Kreweras complementation.

Doing it twice gives the conjugation automorphism

$$w \longmapsto (w^{-1}c)^{-1}c = c^{-1}wc$$

Theorem (R.-Stanton-White 2004, Bessis-R. 2007)

One has a CSP triple $(X, X(q), C)$ where $X = NC(W)$ and $X(q) = \mathrm{Cat}(W, q)$ with $C = \mathbb{Z}/h\mathbb{Z} = \langle c \rangle$ acting via conjugation.

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The noncrossing partition CSP

Bessis-R. also suggested a generalization involving q-Kreweras numbers, which was proven and generalized even further in work of Krattenthaler and Müller (2010), for all well-generated groups.

Unfortunately this is all checked case-by-case.

The nonnesting partition CSP

For any poset P, one has simple bijections between its

- order ideals (=sets closed under going downward in P)
- order filters (=sets closed under going upward in P)
- antichains

Specifically, complementation $I \leftrightarrow P \setminus I$ sends order ideals to order filters, and the maximal (resp. minimal) elements of an order ideal (resp. order filter) give an antichain which uniquely determines it.

Duchet, Brouwer-Schrijver, Deza-Fukuda, Cameron-FonDerFlaass, Panyushev action

This leads to an interesting cyclic action on the antichains, considered first for Boolean algebras by Duchet, then for posets by other authors, and more recently by Panyushev for the positive root poset Φ_+ for a Weyl group W.

Definition

Given an antichain A in a poset P, it generates an ideal

$$P_{\leq A} := \{p \in P : p \leq a \text{ for some } a \in A\}$$

with complementary filter $P \setminus P_{\leq A}$, and then antichain

$$\Psi(A) := \{ \text{ minimal elements of } P \setminus P_{\leq A} \}.$$

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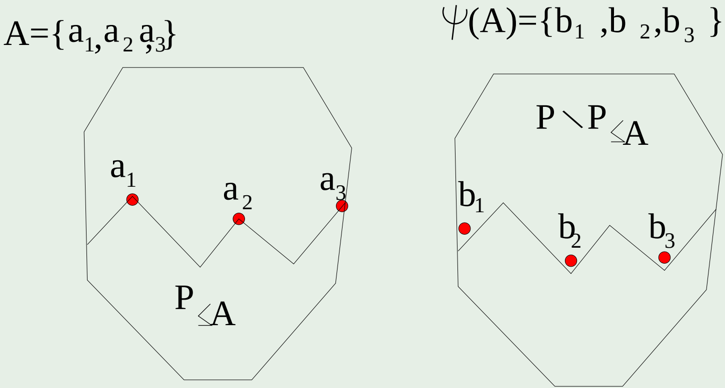
The Ψ action on antichains

Example



Deza and Fukuda's example

Example (Deza and Fukuda 1990)

For a matroid on ground set E,
within the Boolean algebra $P := 2^E$,

- the bases $\mathcal{B}$ form an antichain, with
- the independent sets $\mathcal{I}$ equal to $P_{\leq \mathcal{B}}$,
- the dependent sets $\mathcal{D}$ equal to $P \setminus P_{\leq \mathcal{B}}$, and
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The nonnesting partition CSP

Panyushev (2009) conjectured that for $P = \Phi_+$ this Ψ operation on antichains had order $2h$.
Bessis-R. conjectured that it actually gave a CSP.

Theorem (Armstrong, Thomas, Stump 2011)

One has a CSP triple $(X, X(q), C)$ where X is the antichains in Φ_+, and $X(q) = \mathrm{Cat}(W, q)$ with $C = \mathbb{Z}/2h\mathbb{Z} = \langle\Psi\rangle$.

In fact, there is a C-equivariant bijection from this X to the set $NC(W)$ with $C = \mathbb{Z}/2h\mathbb{Z}$ acting via the Kreweras antiautomorphism $w \mapsto w^{-1}c$, giving another CSP with same $X(q) = \mathrm{Cat}(W, q)$.

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Triangulations give a CSP

Theorem (R.-Stanton-White 2004)

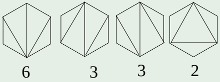
One has a CSP triple $(X, X(q), C)$ in which

- *X is the triangulations of an $(n+2)$-gon,*
- *$X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ is the q-Catalan,*
- *$C = \langle c \rangle = \mathbb{Z}/(n+2)\mathbb{Z}$ having c act by $\frac{2\pi}{n+2}$ rotation.*

Triangulations give a CSP

Example

For $n = 4$ there are four C-orbits of 6-gon triangulations:



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$$= [7]_q(1 - q + q^2)(1 + q^4)$$
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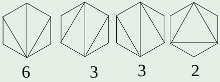

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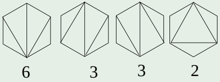

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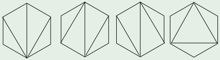

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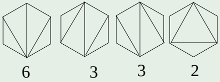

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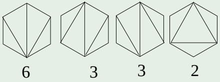

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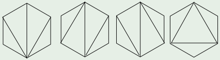

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The cluster/Cambrian fan CSP

More generally, Fomin and Zelevinsky's clusters in a cluster algebra of finite type carry a natural cyclic action $C = \mathbb{Z}/(h+2)\mathbb{Z}$, generated by the deformed Coxeter element τ. Similarly, one has such an action on the top dimensional cones in the Cambrian fan for real reflection groups.

Theorem (Eu and Fu 2008)

In this context, one has a CSP triple $(X, X(q), C)$ where X is the set of clusters or top-dimensional cones in the Cambrian fan, with $C = \mathbb{Z}/(h+2)\mathbb{Z}$ as above, and $X(q) = \mathrm{Cat}(W, q)$

Proven case-by-case.

The q-Kirkman numbers

What about dissections of the $(n+2)$-gon?

Theorem (R.-Stanton-White 2004)

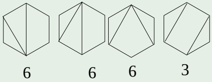
One has a CSP triple $(X, X(q), C)$ in which

- *X is the dissections of an $(n+2)$-gon with k diagonals,*
- $X(q) = \mathrm{Kirk}(n,k,q) = \frac{1}{[k+1]_q} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$
- *$C = \langle c \rangle = \mathbb{Z}/(n+2)\mathbb{Z}$ having c act by $\frac{2\pi}{n+2}$ rotation.*

The q-Kirkman numbers

Example

For $n = 4$ and $k = 2$, there are four C-orbits of dissections:



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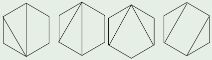

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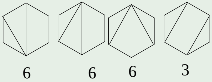

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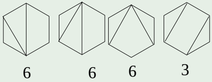

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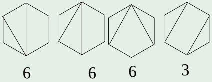

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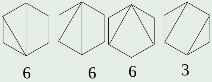

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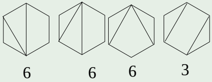

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Eu and Fu were able to prove analogous CSPs for some of the other real reflection groups, where X were faces in the cluster complex or cones in the Cambrian fans of a fixed dimension, using $W - q$-Kirkman numbers defined case-by-case ad hoc.

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W-Kirkman numbers as irreducible multiplicities

An (imperfect) remedy comes from the following observations.

Theorem (Steinberg 1968(?))

For a complex reflection group W acting irreducibly on $V = \mathbb{C}^n$, the exterior powers $\wedge^k V$ for $k = 0, 1, 2, \ldots, n$ are also irreducible W-representations.

Theorem (Armstrong-R.-Rhoades 2012)

For a real reflection group W, the W-Kirkman number counting k-dimensional faces in the W-associahedron is the same as the multiplicity of the W-irreducible $\wedge^k V$ in the parking function W-permutation representation on $Q/(h+1)Q$.

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It suggests the following.

Definition

For real reflection groups W define the q-Kirkman number

$$\mathrm{Kirk}(W,k,q) := \sum_{d \geq 0} q^d \cdot \langle \wedge^k V, S/(\Theta)_d \rangle_W.$$

This is imperfect as it only coincides with the ad hoc q-Kirkman numbers used by Eu and Fu for $W = \mathfrak{S}_n$ and $W = \mathfrak{S}_n^{\pm}$. In fact, in some other types, they seem not to give the desired CSP!

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A parking space conjecture

There is a conjecture that would explain at least these:

- why $NC(W)$ (and clusters) are counted by $\mathrm{Cat}(W)$,
- why $X = NC(W)$ and $X(q) = \mathrm{Cat}(W, q)$ has a CSP for the conjugation action of the Coxeter element, and
- why Kirkman numbers give multiplicities of $\wedge^k V$ in $Q/(h+1)Q$.

A parking space conjecture

Given a real reflection group W and Θ an h.s.o.p. of degree $h+1$ that carries the (dual) reflection representation V^*, assume that one has picked the coordinates $x_1, \ldots, x_n$ so that

$$\begin{array}{rcl} V^* & \longrightarrow & \mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n \\ x_i & \longmapsto & \theta_i \end{array}$$

defines a W-equivariant isomorphism.

Let V^{Θ} be the subset of V which is the zero locus of the ideal $(\theta_1 - x_1, \ldots, \theta_n - x_n)$.

Alternatively, this zero locus can be thought as the fixed points for the map

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Conjecture (Armstrong-R.-Rhoades 2012)

1. *The locus Z contains $(h+1)^n$ distinct points of V.*
2. *As $W \times C$-permutation representation it is a direct sum*

$$\bigoplus_{X \in NC(W)} \mathbb{C}[W/W_X]$$

where (u, c^d) in $W \times C$ sends $wW_X \longmapsto uwc^{-d}W_{c^d X}$.

Etingof has shown that the first assertion holds when Θ is the h.s.o.p. that comes from rational Cherednik algebra theory. The second assertion is open, even for such h.s.o.p.'s.

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